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# 복합적층판의 고유진동수에 대한 형상비의 영향
# The Influence of the Aspect Ratio on the Natural Frequency of the Composite Laminated Plates

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**ABSTRACT**

Theories for advanced composite structures are too difficult for such design engineers for construction and some simple but accurate enough methods are necessary. The senior author has reported that some laminate orientations have decreasing values of $D_{16}$, $B_{16}$, $D_{26}$ and $B_{26}$ stiffnesses as the ply number increases. For such plates, the fiber orientations given above behave as specially orthotropic plates and simple formulas developed by the senior author. Most of the bridge and building slabs on girders have large aspect ratios. For such cases further simplification is possible by neglecting the effect of the longitudinal moment terms(Mx) on the relevant partial differential equations of equilibrium. In this paper. the influence of the aspect ratio on the natural frequency of the composite laminated plates is studied and it is concluded that the method used is sufficiently accurate for engineering purposes.

**요 지**

건설관련 설계기술자들에게는 첨단 복합신소재 구조에 대한 이론이 너무 어려워서 간단하면서도 쉽게 적용할 수 있는 정확한 방법을 필요로 하고 있다. 몇 가지 섬유 배향각을 가진 적층판은 층수가 증가하면 $D_{16}$, $B_{16}$, $D_{26}$ 및 $B_{26}$ 강성이 감소하게 되어 특별직교이방성 판처럼 거동함을 밝히고, 간단한 공식들을 개발하여 발표한 바 있다. 대부분의 교량이나 건물의 상판은 형상비가 큰 경우가 많은데, 이런 구조물의 평형방정식에 대한 종방향모멘트항(Mx)의 영향은 매우 작아서, 더욱 간단한 해석이 가능하다. 본 논문에서는 복합적층판의 고유진동수에 대한 형상비의 영향을 연구하였으며 이 방법을 사용하면 충분히 정확한 값을 쉽게 산출할 수 있다.

**Key Words:** finite difference method(유한차분법), composite laminates (복합적층판), beam theory(보이론)

## 1. INTRODUCTION

The future of material industry will depend on if and when the conventional construction materials are replaced by advanced composite materials.

If composite materials are used for construction, the quantity is huge : in tons, not in kilos or pounds. Composite materials can be used economically and efficiently in broad civil engineering applications when standards and processes for analysis, design, fabrication, construction and quality control are established.

The problem of deteriorating infrastructures is very serious all over the world. The U.S. Civil Engineering Research Foundation (CERF) report, “High-Performance Construction Material and System : An Essential Program for America and its Infrastructure", published in April, 1993, in collaboration with several organizations, cities, U.S. Department of Transportation figures as follows :

(1) 230,000 of the nations(U.S.A.) 575,000 bridges are structurally deficient or obsolete.

(2) 143,000 of these bridges are more than 50 years old and unsuitable for current or projected traffic.

(3) Traffic delays alone will cost the country 50 billion dollars per year in lost work time and fuel by the year 2005.

Steel girders become rusty. The reinforcing bars embedded in concrete beams or slabs are subject to corrosion caused by electro- chemical action. Underground fuel tanks are under similar condition. In 1979, the U.S. Bureau of Standards (NIST) study showed that yearly loss caused by corrosion related damages mounted to 82 billion dollars, about 4.9% of GNP. About 32 billion dollars could

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be saved if existing technologies were used to prevent such losses.

These figures are in the United States of America, where various federal, state, and other agencies are doing their best in maintaining such structures in good condition. The issue of deteriorating and damaged infrastructures and lifelines has become a critically important subject in the United States as well as Japan and Europe. The problem in developing nations, where degree of construction quality control and maintenance are in question must be much more profound (Kim 1995).

The advanced composite materials can be effectively used for repairing such structures. Because of the advantages of these materials, such repair job can fulfill two purposes :

(1) Repair of existing damage caused by corrosion, impact, earthquake, and others.
(2) Reinforcing the structure against anticipated future situation which will require increasing the load beyond the design parameters used for this structure.

Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the dependable methods is to evaluate the in-situ stiffness of the structure by means of obtaining the natural frequency. By comparing the in-situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

The reinforced concrete slab can be assumed as a $[0^0/90^0/0^0]$r type specially orthotropic plate as a close approximation, assuming that the influence of $B_{16,}B_{26}$, $D_{16}$ and $D_{26}$ stiffness are negligible. Many of the bridge and building floor systems, including the girders and cross-beams, also behave as similar specially orthotropic plates. Such plates are subject to the concentrated mass/masses in the form of traffic loads, or the test equipments such as the accelerator in addition to their own masses.

Analysis of such problems is usually very difficult. The most of the design engineers for construction has academic background of bachelors degree. Theories for advanced composite structures are too difficult for such engineers and some simple but accurate enough methods are necessary.

The senior author has reported that some laminate orientations such as $[\alpha,\beta]_{r,}$ $[\alpha,\beta,\gamma]_{r,}$ $[\alpha,\beta,\beta,\alpha,\alpha,\beta]_{r,}$ and $[\alpha,\beta,\beta,\gamma,\alpha,\alpha,\beta]_{r,}$ with $\alpha=-\beta$, and $\gamma=0^o$, or $90^o$ and with increasing r, have decreasing values of $B_{16,}B_{26,}D_{16}$ and $D_{26}$ stiffness. Most of the civil and architectural structures are large in sizes and the numbers of laminae are large, even though the thickness to length ratios are small enough to allow to neglect the transverse shear deformation effects in stress analysis. For such plates, the fiber orientations given above behave as specially orthotropic plates and simple formulas developed by the reference (Kim 1995, Han & Kim 2001, 2003) can be used. Most of the bridge and building slabs on girders have large aspect ratios. For such cases further simplification is possible by neglecting the effect of the longitudinal moment terms (Mx) on the relevant partial differential equations of equilibrium. In this paper, the result of the study on the subject problem is presented. Even with such assumption, the specially orthotropic plate with boundary conditions other than Navier or Levy solution types, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical method for eigenvalue problems are also very much involved in seeking such a solution (Timosenko 1989, Ashton 1970, Whitney 1970, Pagano 1970).

The method of vibration analysis used is the one developed by Kim. He developed and reported in 1974, a simple but exact method of calculating the natural frequency of beam and tower structures with irregular cross-sections and attached mass/masses. Since 1989, this method has been extended to two-dimensional problems with several types of given conditions and has been reported at several international conferences. This method uses the deflection influence surfaces. The finite difference method is used for this purpose, in this paper.

## 2. METHOD OF ANALYSIS

The equilibrium equation for the specially orthotropic plate is :

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=q(x,y) \quad (1)$$

where $D_1=D_{11}, D_2=D_{22}, D_3=D_{12}+2D_{66}$

The assumptions needed for this equation are :

(1) The transverse shear deformation is neglected.
(2) Specially orthotropic layers are arranged so that no coupling terms exist, i.e, $B_{ij}=0, (\ \ )_{16}=(\ \ )_{26}=0$.
(3) No temperature or hydrothermal terms exist.

The purpose of this paper is to demonstrate, to the practicing engineers, how to apply this equation to the slab systems made of plate girders and cross-beams.

In case of an orthotropic plate with boundary conditions other than Navier or Levy solution type, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical methods for eigenvalue problems are also very much involved in seeking such a solution. Finite difference method (F.D.M) is used in this paper. The resulting linear algebraic equations can be used for any cases with minor modifications at the boundaries, and so on.

The problem of deteriorating infrastructures is very serious all over the world. Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the dependable methods is to evaluate the in-situ stiffness of the structure by means of obtaining the natural frequency. By comparing the in- situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

The basic concept of the Rayleigh method, the most popular analytical method for vibration analysis of a single degree of freedom system, is the principle of conservation of energy ; the energy in a free vibrating system must remain constant if no damping forces act to absorb it. In case of a beam, which has an infinite number of degree of freedom, it is necessary to assume a shape function in order to reduce the beam to a single degree of freedom system. [14] The frequency of vibration can be found by equating the maximum strain energy developed during the motion to the maximum kinetic energy. This method, however, yields the solution either equal to or larger than the real one. Recall that Rayleigh's quotient $\geq 1$. For a complex beam, assuming a correct shape function is not possible. In such cases, the solution obtained is larger than the real one.

Design engineers need to calculate the natural frequencies of such element but obtaining exact solution to such problems is very much difficult. Pretlove reported a method of analysis of beams with attached masses using the concept of effective mass. This method, however, is useful only for certain simple types of beams. Such problems can be easily solved by presented method.

A simple but exact method of calculating the natural frequency corresponding to the first mode of vibration of beam and tower structures with irregular cross-sections and attached mass/masses was developed and reported. This method consists of determining the deflected mode shape of the member due to the inertia force under resonance condition. Beginning with initially “guessed” mode shape, “exact” mode shape is obtained by the process similar to iteration. Recently, this method was extended to two dimensional problems including composite laminates, and has been applied to composite plates with various boundary conditions with/without shear deformation effects and reported at several international conferences including the Eighth Structures Congress(1990) and Fourth Materials Congress(1996) of American Society of Civil Engineers.

This method is used for vibration analysis in this paper.

A natural frequency of a structure is the frequency under which the deflected mode shape corresponding to this frequency begins to diverge under the resonance condition. From the deflection caused by the free vibration, the force required to make this deflection can be found, and from this force, resulting deflection can be obtained. If the mode shape as determined by the series of this process is sufficiently accurate, then the relative deflections (maximum) of both the converged and the previous one should remain unchanged under the inertia force related with this natural frequency. Vibration of a structure is a harmonic motion and the amplitude may contain a part expressed by a trigonometric function. Considering only the first mode as a start, the deflection shape of a structural member can be expressed as

$$w = W(x, y) F(t) = W(x, y) \sin \omega t \tag{2}$$

where

W : maximum amplitude
ω : circular frequency of vibration
$t$ : time

By Newton's second law, the dynamic force of the vibrating mass, m, is

$$F = m \frac{\partial^2 w}{\partial t^2} \tag{3}$$

Substituting (2) into this,

$$F = -m (\omega)^2 W \sin \omega t \tag{4}$$

In this expression, ω and $W$ are unknowns. In order to obtain the natural circular frequency, ω, the following process is taken.

The magnitudes of the maximum deflection at a certain number of points are arbitrarily given as

$$w(i, j)(1) = W(i, j)(1) \tag{5}$$

where (i, j) denotes the point under consideration. This is absolutely arbitrary but educated guessing is good for accelerating convergence.

The dynamic force corresponding to this (maximum) amplitude is

$$F(i,j)(1)=m(i,j)\{\omega(i,j)(1)\}2w(i,j)(1) \quad (6)$$

The “new” deflection caused by this force is a function of $F$ and can be expressed as

$$w(i,j)(2)=f\{m(k,l)\{\omega(i,j)(1)\}2\ w(k,l)(1)\}= \sum^{k,l} \triangle(i,j,k,l)\{m(k,l)\{\omega(i,j)(1)\}2\ w(k,l)(1)\} \quad (7)$$

where △ is the deflection influence surface. The relative (maximum) deflections at each point under consideration of a structural member under resonance condition, w(i,j)(1) and w(i,j)(2), have to remain unchanged and the following condition has to be held :

$$w(i,j)(1)\ /\ w(i,j)(2)=1. \quad (8)$$

From this equation, w(i,j)(1) at each point of (i,j) can be obtained, but they are not equal in most cases. Since the natural frequency of a structural member has to be equal at all points of the member, i.e., w(i,j) should be equal for all (i,j), this step is repeated until sufficient equal magnitude of w(i,j) is obtained at all (i,j) points.

However, in most cases, the difference between the maximum and the minimum values of w(i,j) obtained by the first cycle of calculation is sufficiently negligible for engineering purposes. The accuracy can be improved by simply taking the average of the maximum and the minimum, or by taking the value of w(i,j) where the deflection is the maximum. For the second cycle, w(i,j)(2) in the absolute numerics of w(i,j)(2) can be used for convenience.

$$w(i,j)(3)=f\{m(i,j)\ [\ \omega(i,j)(2)\ ]^2 w(i,j)(2)\} \quad (9)$$

In case of a structural member with irregular section including composite one, and non-uniformly distributed mass, regardless of the boundary conditions, it is convenient to consider the member as divided by finite number of elements [9]. The accuracy of the result is proportional to the accuracy of the deflection calculation.

For practical design purposes, it is desirable to simplify the vibration analysis procedure. One of the methods is to neglect the weight of the structural element. The effect of neglecting the weight (thus mass) of the plate is studied as follow. If a weightless plate is acted upon by a concentrated load, $P=N\cdot q\cdot a\cdot b$, the critical circular frequency of this plate is

$$\omega_n=\sqrt{\frac{g}{\delta_{st}}} \quad (10)$$

where $\delta_{st}$ is the static deflection.

Similar result can be obtained by the use of Eqs. (7) and (8).

$$[\ \omega(i,j)\ ]^2=\frac{1}{[\ \triangle(i,j,i,j)\cdot\frac{P(i,j)}{g}\ ]} \quad (11)$$

where,

$$P(i,j)=N\cdot q\cdot a\cdot b \quad (12)$$

In case of the plate with more than one concentrated loads,

$$[\omega(i,j)\ ]^2=\frac{1}{[\ \sum^{k,l}\triangle(i,j,k,l)\cdot\frac{P(k,l)}{g}\ ]} \quad (13)$$

If we consider the mass of the plate as well as the concentrated loads,

$$\begin{aligned} w(i,j)(1) &= w(i,j)(2) \\ &= \{\sum^{k,l}\triangle\ (i,j,k,l)\cdot m(k,l)\cdot w(k,l)(1) \\ &\quad + \sum^{m,n}\triangle(i,j,m,n)\cdot\frac{P(m,n)}{g}\cdot w(m,n)(1)\} \\ &\quad \times[\ \omega(i,j)(1)\ ]^2 \end{aligned} \quad (14)$$

where (m,n) is the location of the concentrated loads. The effect of neglecting the weight of the plate can be found by simply comparing Eqs. (13) and (14).

The method used in this paper requires the deflection influence surfaces. F. D. M. is applied to the governing equation of the specially orthotropic plates.

The number of the pivotal points required in the case of the order of error △2, where △ is the mesh size, is five for the central differences of the fourth order single derivative terms. This makes the procedure at the boundaries complicated. In order to solve such problem, the three simultaneous partial differential equations of equilibrium with three dependent variables, w, Mx, and My, are used instead of Eq. (1) for the bending of the specially orthotropic plate.

$$D_1\frac{\partial^2 Mx}{\partial x^2} - 4D_{66}\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^2 My}{\partial y^2}$$
$$= -q(x,y) + kw(x,y) \quad (15)$$

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} \quad (16)$$

$$M_y = -D_{12}\frac{\partial^2 w}{\partial x^2} - D_{22}\frac{\partial^2 w}{\partial y^2} \quad (17)$$

If F.D.M. is applied to these equations, the resulting matrix equation is very large in sizes, but the tridiagonal matrix calculation scheme used by Kim is very efficient to solve such equations.

In order to confirm the accuracy of the F.D.M., [A/B/A]r type laminate with aspect ratio of a/b=1m/1m=1 is considered.

For simplicity, it is assumed that $A=0^0$, $B=90^0$ and r=1. Since one of the few efficient analytical solutions of the specially orthotropic plate is Navier solution, and this is good for the case of the four edges simple supported, F.D.M. is used to solve this problem and the result is compared with the Navier solution.

Calculation is carried out with different mesh sizes and the maximum errors at the center of the plate are as follows.

10 x 10 case : 0.140 %
20 x 20 case : 0.035 %

The error is less than 1%. This is smaller than the predicted theoretical errors ;

If F.D.M. is applied to these equations, the resulting matrix equation is very large in sizes, but the tridiagonal matrix calculation scheme used by Kim. is very efficient to solve such equations. Since one of the few efficient analytical solutions of the specially orthotropic plate is Navier solution, and this is good for the case of the four edges simple supported, F.D.M is used to solve this problem. The result is satisfactory as expected.

By neglecting the Mx terms, the sizes of the matrices needed to solve the resulting linear equations are reduced to two thirds of the "non-modified" equations (4).

## 3. NUMERICAL EXAMINATION

### 3.1 Structure under Consideration

The plate considered is as shown in Fig. 1,

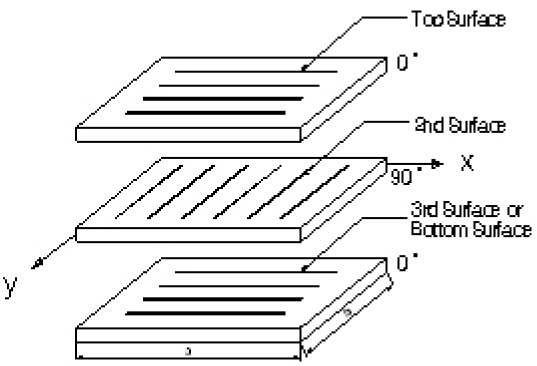


Fig. 1 Configuration of [a/b/a]r Laminated Plate

The material properties are :

$E_1$ = 67.36 GPa, $E_2$ = 8.12 GPa,
$G_{12}$ = 3.0217 GPa
$\nu_{12}$ = 0.272, $\nu_{21}$ = 0.0328, r = 1
Ply thickness = 0.005 m
Orientation : $[0^o/\ 90^o\ /\ 0^o]$r

The stiffnesses are :

$D_{11}$ = 2929, $D_{22}$ = 18492, $D_{12}$ = 627
a = nb, n = an integer 1 ~ 5
and $D_{66}$=849, b=3 m
Loading : q = 286.65 N/$m^2$

Boundary condition of plate is as shown in Fig. 2.

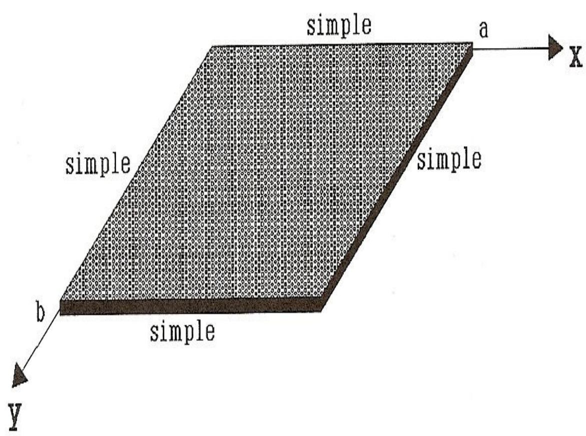


Fig. 2 Boundary Condition of Plate

### 3.2 Numerical Results

In order to study the influence of Mx on the equilibrium equations, three cases are considered :

Case A : $w$ , Mx and My are considered.

Case B : $w$ and My are considered, Mx is neglected.

Case C : Beam with unit width.

F.D.M. is used to obtain w, Mx, and My, and obtain the natural frequency. The result is as shown in Fig 3.

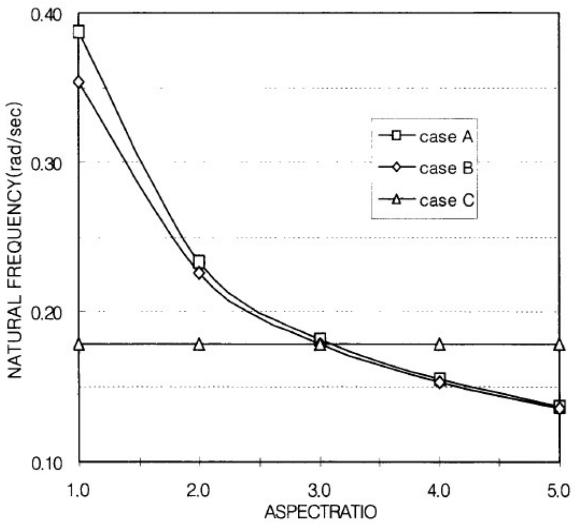


Fig. 3 Natural Frequency of Case A, Case B, Case C

## 4. CONCLUSION

Most of the bridge and building slabs have plate aspect ratios larger than 2. For such cases, design analysis becomes much simpler if influence of the longitudinal moment (Mx) terms on the relevant differential equations of equilibrium can be neglected. The result of the study on this subject is presented in this paper.

The result of numerical examination is quite promising. It is concluded that, for all boundary conditions, neglecting Mx terms is acceptable if the aspect ratio (a/b) is equal to or larger than 3. This conclusion gives good guide line for design of bridge and building slabs on main girders, for which the aspect ratio is larger than, at least, three.

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