A POLAR, THE CLASS AND PLANE JACOBIAN CONJECTURE

Dosang Joe

ABSTRACT. Let P be a Jacobian polynomial such as $\deg P = \deg_y P$. Suppose the Jacobian polynomial P satisfies the intersection condition satisfying $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = \deg P - 1$, we can prove that the Keller map which has P as one of coordinate polynomial always has its inverse.

1. Introduction

Let $P, Q \in \mathbb{C}[x, y]$ be a pair of polynomial of two variables. (P, Q) is called a Jacobian pair if the Jacobian of two polynomials,

$$[P, Q] = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial Q}{\partial x}\frac{\partial P}{\partial y}$$

is non-zero constant. Let $F(x, y) = (P(x, y), Q(x, y))$ be a map from $\mathbb{C}^2$ to $\mathbb{C}^2$ induced by a Jacobian pair, which is usually called Keller map. The Jacobian conjecture in dimension two is about whether every Keller map has the global inverse over $\mathbb{C}^2$ or not. There have been extensive studies and various partial results for this conjecture, see more detailed information in the van den Essen's book [6]. It has been known that the conjecture is true if P is a rational polynomial or has a one point at infinity by the earlier work of S. S. Abhyankar and M. Razar [1, 12]. A main purpose of this paper is to prove the following theorem.

Theorem. *Let P be a Jacobian polynomial of degree n and $\deg_y P = \deg P = n$. Suppose $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = n-1$, then any Keller map defined by a pair of Jacobian polynomials (P, Q) is invertible.*

Let P be a Jacobian polynomial of degree n and $\deg_y P = \deg P = n$. Suppose $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = n-1$, then we can show that the affine curve defined by the polynomial P has genus zero and has one point at infinity, which will be proved in the Proposition 4. Moreover this condition enables us to show that

generic fiber $P = c$, $c \in \mathbb{C}$ is a rational affine curve. By the work of M. Razar, N. V. Chau [12, 4], we can show that the Keller map (P, Q) is always invertible. The main theorem will be proved in Theorem 7 in the Section 2. Hence the Jacabian conjecture in dimension two can be fixed by proving the dimension condition $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = n - 1$. Later, we will discuss a sufficient condition for the plane Jacobian conjecture, which is so called *equi-dimension* conjecture such as $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle xP_x + yP_y, P_y \rangle$. There is no enough evidence for this conjecture should be true, however this *equi-dimension* conjecture is equivalent to the Jacobian conjecture in dimension two for the Jacobian polynomial P. The good things for the *equi-dimension* conjecture is that there might be a pure algebraic approach by investigating the monomial ideal structures of given two ideals using the Gröbner basis theory. Examples and some evidence for the conjecture is discussed in the Section 3.

Given plane curve C in the projective plane, the notion of the dual curve in dual space is well known. The degree of the dual curve, the *class* of the curve, can be used to compute the genus of the curve combining with the multiplicities of all infinitely near singular point of C. The plane curve C defined by $P(x, y)$ in $\mathbb{C}^2$, the intersection number of C and the $\tilde{C}$ defined by P_y in the affine space $\mathbb{C}^2$ is summing up to the class of the curve. The intersection number, $e = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle$ is in general strictly less than the class of the curve C. The intersection number, e in the affine piece has the minimum value, i.e., $\deg P - 1 \leq e = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle$ among all possible smooth affine curve which is monic in y.

2. Proof of Theorem

2.1. A polar and the class of projective plane curve

In this section, we will discuss what the intersection number $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle$ is meant to be. It eventually has some information of the class of given smooth plane curve defined by P. The class of a projective curve is defined to be the degree of its dual curve. Let C be a projective curve in $\mathbb{C}P^2$. The definition of the dual curve C^* is the set of the tangents of C. Unless C contains lines as component, C^* is in fact an algebraic curve in $\mathbb{C}P^{2*}$. There is a way to compute the class of given curve C by using the intersection number of the curve and its polar.

Definition 1. Let C be a plane curve of degree n defined by a homogeneous equation $F(x, y, z) = 0$. Let $p = (x_0 : y_0 : z_0) \in \mathbb{C}P^2$ be a point for which the homogeneous polynomial

$$D_p^1 F(x, y, z) = x_0 \frac{\partial F}{\partial x} + y_0 \frac{\partial F}{\partial y} + z_0 \frac{\partial F}{\partial z} = x_0 F_x + y_0 F_y + z_0 F_z$$

does not vanish identically. Then the curve of degree $n - 1$ with equation $D_p^1 F(x, y, z) = 0$ is called the first polar of C relative to the point p.

Let p_i be the singular points of C and let v_{ij} be the multiplicities of the infinitely near singular points of (C, p_i). Finally, let v_k be the orders of the singular branches of C. Let n' be the class of the plane curve C of degree n. The following formulae holds for the class n'.

$$(1) \qquad n' = n(n-1) - \sum v_{ij}(v_{ij} - 1) - \sum (v_q^k - 1).$$

It is a part of the generalization of the Plücker formulae found by Weierstrass and M. Noether [3]. The proof of the formula (1) is reduced to the determination of the intersection of C with $\tilde{C}$, a first polar of C relative to p. Let $\{L_c\}$ be the associated family of lines passing through the fixed point $p \in \mathbb{C}P^2$. The main idea to proving equation (1), is to show that the difference of the intersection numbers,

$$v_q(C, \tilde{C}) - v_q(C, L_{c_q})$$

depends only on the singularity of the curve C. Moreover the difference of the intersection number can be computed by successive blowing up process at the singular point $q \in C$, which is

$$(2) \qquad v_q(C, \tilde{C}) - v_q(C, L_{c_q}) = \sum v_j(v_j - 1) - \rho,$$

where ρ is the number of branches of C_{t_0} at q and the sum is of the multiplicities v_j of all infinitely near singularities of (C, q).

Note that the *class* equation (1) is derived from the above equation. In particular, if the point q is a smooth point of C_{t_0}, the formula (2) reads that

$$(3) \qquad v_q(C, \tilde{C}) = v_q(C, L_{c_q}) - 1 = v_q(C, L_{c_q}) - v_q(C).$$

Moreover suppose $q \in C$ is a singular point and $v_k(q)$ is the order of singular branch of C at q, we have the following equation from equations $(1), (2)$.

$$v_q(C, \tilde{C}) - \sum v_j(v_j - 1) - \sum (v_k(q) - 1) \quad = \quad v_q(C, L_{c_q}) - \sum v_k(q)$$
$$(4) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \quad v_q(C, L_{c_q}) - v_q(C).$$

The following lemma is followed from equations $(3), (4)$.

Lemma 2. *Let C be a plane curve which does not contain a line component and let $p \notin C$ be a point in $\mathbb{C}P^2$. Let $\{L_c\}$ be the family of lines passing through the given point p. The class of the curve C, which is the degree of the dual curve C^*, n' is computed in terms of the intersection number of C and L_c as follows.*

$$n' = \sum_{q \in C} v_q(C, L_{c_q}) - v_q(C),$$

where $v_q(C, L_{c_q})$ is the intersection number of C and L_c at the point q and v_q is the multiplicity of C at q.

Note that the line L_{c_q} is uniquely determined by the point $q \in C$ and the summation in the equation is in fact finite because $v_q(C, L_{c_q}) = v_q(C)$ for generic point $q \in C$ as long as C does not contain a line component.

Let $p = (0 : 1 : 0) \notin C$. Suppose $F(X, Y, Z)$ be the defining equation for C. Then the first polar of C relative to has the equation $\frac{\partial F}{\partial Y}(X, Y, Z) = 0$. If we introduce the affine coordinate $x = \frac{X}{Z}, y = \frac{Y}{Z}$, then the pencil of lines through p (with exception of line at infinity, $Z = 0$) becomes the family of parallel lines

$$x = c, \quad c \in \mathbb{C}.$$

The curve C has the affine equation

$$f(x, y) = 0, \qquad \text{where } f(x, y) = F(x, y, 1).$$

The affine equation of the polar $\tilde{C}$ reads

$$\frac{\partial f}{\partial y} = f_y(x, y) = 0.$$

Lemma 3. *Let C be the projective plane curve without a line component defined by $F(x, y, z)$. Let $f(x, y) = F(x, y, 1) \in \mathbb{C}[x, y]$ be the polynomial of two variables. Suppose $f(x, y)$ defines a smooth affine plane curve in $\mathbb{C}^2$. Then the class of the C, n' is computed as follows.*

$$n' = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle f, f_y \rangle + (v_\infty - \sum v_j),$$

where $v_\infty = \deg f$ is the sum of intersection multiplicities of $C \cap L_\infty (Z = 0)$ and v_j is the multiplicity of C at the point $q_j \in C \cap L_\infty$.

Proof. Since the affine equation $f(x, y)$ define a smooth plane curve in $\mathbb{C}^2$, a polar $\tilde{C}$ defined by $f_y = \frac{\partial f}{\partial y}$ intersect the curve C at a smooth point $q \in \mathbb{C}^2$. Let $\{x_c\}$ be the family of line passing through $p = (0 : 1 : 0) \in \mathbb{C}P^2$, then we have

$$v_q(C, \tilde{C}) = v_q(C, x_{c_q}) - 1, \quad q \in \mathbb{C}^2.$$

Summing up the intersection multiplicities over the points in the affine plane $\mathbb{C}^2$, we have

$$\sum_{q \in \mathbb{C}^2} v_q(C, x_{c_q}) - 1 = \sum_{q \in \mathbb{C}^2} v_q(C, \tilde{C}) = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle f, f_y \rangle.$$

Let $q_1, q_2, \ldots, q_k$ be the intersection of C and $\tilde{C}$ which lies at the line at infinity, $Z = 0$. From equation (4), we have

$$v_{q_i}(C, Z) - v_{q_i}(C) = v_{q_i}(C, \tilde{C}) - \sum v_{q_i}(v_{q_i} - 1) - \sum (v_k(q_i) - 1)$$

and they are all summed up to the equation.

$$\sum_{q \in L_\infty} v_{q_i}(C, Z) - v_{q_i}(C) = \deg f - \sum v_{q_i} = v_\infty - \sum v_i.$$

The equation is followed from the Lemma 2, so we are done. $\square$

Example 1 (Class of polynomial curve). Let f be a polynomial of two variables, which has parametric representation by $x = x(t), y = y(t)$, where $x, y \in \mathbb{C}[t]$. Moreover, let n, m be the degrees of x, y respectively. The dual curve C^* has the parametrization $p = p(t), q = q(t)$, where $p(t)x + q(t)y = 1$ is the affine equation of the tangent line to C at $(x(t), y(t))$, which has the form [8]

$$p(t) = \frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad q(t) = \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}.$$

The class of the curve C, which is the degree of the dual curve, is same as $\deg x + \deg y - 1 = n + m - 1$. Assume $n > m$, the polynomial curve C has a unique irreducible point at $p = (1 : 0 : 0)$ and has the multiplicity $n - m$ at p.

2.2. Genus formula for plane curve

Let C be an irreducible projective plane curve of degree n. Then the genus of the curve C is computed as follows.

$$(5) \qquad 2g - 2 = n(n - 3) - \sum v_{ij}(v_{ij} - 1),$$

where the summation involves the multiplicities v_{ij} of all infinitely near singular points of (C, p_i). Equation (3) is followed by a computation of degree of canonical divisor, adjoint linear system. For more detail, we refer to the book [3].

Proposition 4. *Let $f(x, y) \in \mathbb{C}[x, y]$ be an irreducible polynomial which defines a smooth affine curve in $\mathbb{C}^2$. Let C be the projective plane curve defined by $F(x, y, z)$ which is the homogeneous equation of degree n such that $f(x, y) = F(x, y, 1)$. Suppose $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle f, f_y \rangle = n - 1$, then C defines a curve of genus zero and has at most one point at infinity.*

Proof. By the assumption of the polynomial of f, the point $p = (0 : 1 : 0)$ is not on the intersection of C and the line at infinity, $L(z = 0)$. From the equation 2, we have

$$n' = n(n - 1) - \sum v_{ij}(v_{ij} - 1) - \sum (v_k - 1)$$

which is equal to the following equation by the Lemma 3.

$$n' = \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle f, f_y \rangle + (v_\infty - \sum v_k).$$

Note that

$$v_\infty = \sum_{p \in C \cap L_\infty} v_p(C, L_\infty) = \deg C = n, \quad L_\infty := \{z = 0\}.$$

By applying the genus formula 5, we have

$$
\begin{aligned}
2g - 2 &= n' - 2n + \sum (v_k - 1) \\
&= \dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle f, f_y \rangle + (v_\infty - \sum v_k) - 2n + \sum (v_k - 1) \\
&= (n - 1) + n - 2n + \sum_k (-1) = -1 - l,
\end{aligned}
$$

where l is the number of branches for the singular points of C. The curve defined by the f is irreducible, hence the genus of the curve C is always greater than zero. So we show that C has a unique singular branch and genus zero. The unique singular point of C lies on the line at infinity $z = 0$ by the assumption. The polynomial f has at most one point at infinity. $\square$

2.3. Proof of the main theorem

Before proving the main theorem, it might be good place to mention the previous results on the Jacobian conjecture and the invertibility of Keller map. In 1979, the following theorem is proved by M. Razar [12].

Theorem 5 ([12]). *A Keller map $F = (P, Q)$ has an inverse if P is a rational polynomial and all fibers $P = c, c \in \mathbb{C}$, are irreducible.*

Later in 1990 R. Heitmann [9] presented another algebraic proof for Razar's observation. All the proofs for the theorem ends up with showing that the restriction of Q to each fiber of P is a proper map. Theorem 5 has been extended to the generic case when the generic fiber of P or Q is rational polynomial.

Theorem 6 ([4, Lê]). *A Keller map $F = (P, Q)$ has an inverse if P or Q is a rational polynomial.*

Here the statement that P is rational polynomial is meant to be that the generic fiber of P is a rational curve, which is a two dimensional topological sphere with a finite number of punctures. Let me sketch the main idea of the theorem [4] done by N. V. Chau. Given a Keller map $F = (P, Q)$ with rational polynomial P, let $f = (p, q) : X \to \mathbb{P} \times \mathbb{P}$ be a regular extension over a compactification X of $\mathbb{C}^2$. The divisor at infinity $\mathcal{D} := X \backslash \mathbb{C}^2$ so called *horizontal component* of P which is an irreducible component D of $\mathcal{D}$ such that the restriction to D of p is not constant. And D is called a *section* of P if the degree of the restriction $p|_D$ is one. He proved that P must has a unique section, which implies that every fiber $P = c$ has a unique irreducible branch at infinity. Hence by the Theorem 5, $F = (P, Q)$ is invertible.

Theorem 7. *Let P be a Jacobian polynomial of degree n and $\deg_y P = \deg P = n$. Suppose $\dim_{\mathbb{C}} \mathbb{C}[x, y]/\langle P, P_y \rangle = n - 1$, then any Keller map defined by a pair of Jacobian polynomials (P, Q) is invertible.*

Proof. First of all, it is clear that $P-c$ is irreducible for generic $c \in \mathbb{C}$. Moreover we have

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P - c, P_y \rangle = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = n - 1 \text{ for all } c \in \mathbb{C}.$$

By the Proposition 4, we can show that $P-c$ is a rational curve with one point at infinity for generic $c \in \mathbb{C}$. Thus P is a rational polynomial. Hence by the Theorem 6, the Keller map (P, Q) has a polynomial inverse. $\qquad\square$

Remark 8. First of all, one can note that if we prove that *the polar-intersection* condition $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = n - 1$ is satisfied for any Jacobian polynomial we can directly prove the Jacobian conjecture is followed by the theorem of S. Abhyankar [1, 6]. Secondly, we can show that the Keller map (P, Q) is proper easily if one can prove the $P - c$ has a polynomial parameterization for all $c \in \mathbb{C}$.

3. Equi-dimensional conjecture

3.1. Equi-dimension conjecture

Let us discuss what the condition, $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = n - 1$ meant to be. And how many of the plane curves satisfies such condition. So far we have investigated, only examples we have known are the defining equations of the embedding of lines in $\mathbb{C}^2$ which is a smooth rational curve with one point at infinity.

Example 2 (Embedding of line). Let P be a defining equation of the embedding of line in $\mathbb{C}^2$. The affine curve defined by the polynomial P has a parametric representation by $x = x(t), y = y(t)$, where $x(t), y(t) \in \mathbb{C}[t]$. Then we have $f_y(x(t), y(t)) = c_0 x'(t)$ for some $c_0 \in \mathbb{C}^*$ [7], where $f(x(t), y(t)) \equiv 0$. Hence $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = \deg_t x'(t) = n - 1$.

Among all the polynomial curves of parameterization $x = x(t), y = y(t)$ where $\deg_t x(t) = n, \deg_t y(t) = m$, the only possible pair of degrees for the embeddings of line has to be like that m divides n under the assumption $m \leq n$ [2]. Moreover there are more complicated structure for the parameter spaces of smooth polynomial curves, i.e., embeddings of line for given degree m, n [10].

As a result, it naturally gives rise to have the following question whether the condition $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = n - 1$ is satisfied for the Jacobian polynomial P to fix the plane Jacobian conjecture.

Question. *Can the condition for a Jacobian polynomial P,*

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = n - 1, \ \text{where} \ \deg P = \deg_y P = n$$

be derived from the constant Jacobian condition for the pair (P, Q)?

Suppose the polynomial P is a Jacobian polynomial, i.e., there exists a polynomial Q such that the pair $P, Q \in \mathbb{C}[x,y]$ has constant Jacobian. And suppose moreover the polynomial is monic in y-variable and $\deg P = \deg_y P$,

then the ideal generated by $xPx + yP_y$ and P_y is same as the one generated by x, P_y since x is generated by xP_x, P_y.

$$x = x(P_x Q_y - P_y Q_x) = Q_y(xP_x) + (-xQ_x)P_y.$$

Thus we have

$$
\begin{aligned}
\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle xP_x + yP_y, P_y \rangle &= \dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle x, P_y \rangle \\
&= \dim_{\mathbb{C}} \mathbb{C}[y]/\langle P_y(0,y) \rangle \\
&= \deg P - 1 = n - 1.
\end{aligned}
$$

It gives rise to make a following conjecture.

Conjecture 1. *Let P be a polynomial of two variables which is monic in y and $\deg_y P = \deg P$. Can one describe a family of polynomial of two variables which satisfy the following condition?*

(6) $\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle xP_x + yP_y, P_y \rangle.$

We can not expect the *equi-dimension* condition (6) is true for all polynomial $P \in \mathbb{C}[x,y]$. However we can not find any example polynomial P which does not satisfy the equation in the conjecture. We have checked the equation holds for many bi-variate polynomials by running the singular program. We will presents the some of the examples in the end of this paper. The following examples explain Conjecture 1 is true for each invertible Jacobian polynomial, which is a defining equation for the embedding of line [2].

3.2. Gröbner basis approach to Jacobian conjecture

There is a way to approach Conjecture 1 via the Gröbner basis theory. Let $P(x,y) \in \mathbb{C}[x,y]$ and $P_x = \frac{\partial P}{\partial x}, P_y = \frac{\partial P}{\partial y}$. Let's assume that $I = \langle P, P_y \rangle, J = \langle xP_x + yP_y, P_y \rangle$ is the associated zero-dimensional ideals. Suppose we choose a monomial order $>$ on $\mathbb{C}[x,y]$, we can define the associated monomial ideal $\langle \mathrm{LT}(I) \rangle$ called initial ideal, which is generated by the leading terms of all elements in I. The Gröbner basis for I is a finite set of generators of the ideal which also generate the initial ideal $\langle \mathrm{LT}(I) \rangle$ for given monomial order. For the ideals $I = \langle P, P_y \rangle, J = \langle xP_x + yP_y, P_y \rangle$ in Conjecture 1, we have investigated the initial ideals $\tilde{I}, \tilde{J}$ with respect to the standard lexicographic order as $x < y$ via the Singular program. The computation run by the Singular shows that even though the initial ideals $\tilde{I}, \tilde{J}$ are not equal to each other, the equation in Conjecture 1 holds for substantially many polynomials $P \in \mathbb{C}[x,y]$ which are monic in y and $\deg P = \deg_y P$. The following examples are the one of them. This may be the part of evidence why Conjecture 1 should be true, henceforth the Jacobian conjecture in dimension two should be true. For the following examples, we denote that $\tilde{I}, \tilde{J}$ be the initial ideal of $I = \langle P, P_y \rangle, J = \langle xP_x + yP_y, P_y \rangle$ for any given polynomials $P \in \mathbb{C}[x,y]$. The lexicographic order with $x < y$ is fixed for the computation of the following examples. Let $d = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle P, P_y \rangle$, $s = \dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle xP_x + yP_y, P_y \rangle$ be the dimensions respectively.

Example 3 (Computational evidence for equation (6))**.**

$$P_1 = ((y + (x + y^3)^2)^2 + (x + y^3))^2 + y + (x + y^3)^2, \implies d = s = 23$$
$$P_2 = ((y + (x + y^3)^2)^2 + (x + y^3))^2 + x, \implies d = s = 38$$
$$P_3 = x^{12} + x^{11}y^2 + x^2y^{20} + y^{24} + y, \implies d = s = 276$$
$$P_4 = x^{24} + y^{24} + y^{13} + y, \implies d = s = 552.$$

Note that the polynomial P_1 in the example is a coordinate of $\mathbb{C}^2$, in fact a defining equation of an embedding of line in $\mathbb{C}^2$.

3.3. Discussions

The possible counterexample for the Jacobian conjecture in dimension two has to be a curve of higher genus or a curve having two points at infinity. There would be closed conditions of the polynomials of given degree for which equation (6) does not hold. If then, the last challenge one can take would be to prove that Jacobian condition is complementary to it.

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DEPARTMENT OF MATHEMATICS EDUCATION
KONKUK UNIVERSITY
SEOUL 143-701, KOREA
E-mail address: dosjoe@konkuk.ac.kr