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APPROXIMATE BI-HOMOMORPHISMS AND BI-DERIVATIONS IN C^*-TERNARY ALGEBRAS

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of bi-homomorphisms in C^*-ternary algebras and of bi-derivations on C^*-ternary algebras for the following bi-additive functional equation

$$f(x+y,z-w)+f(x-y,z+w)=2f(x,z)-2f(y,w).$$

This is applied to investigate bi-isomorphisms between C^*-ternary algebras.

1. Introduction and preliminaries

Ternary algebraic operations were considered in the 19th century by several mathematicians, such as Cayley [9] who introduced the notion of cubic matrix, which, in turn, was generalized by Kapranov [15] et al. The simplest example of such nontrivial ternary operation is given by the following composition rule:

$$\{a,b,c\}_{ijk}=\sum_{1\le l,m,n\le N} a_{nil}b_{ljm}c_{mkn} \quad (i,j,k=1,2,\cdots,N).$$

Ternary structures and their generalization, the so-called n-ary structures, raise certain hopes in view of their applications in physics. Some significant physical applications are as follows (see [16, 17]):

(1) The algebra of *'nonions'* generated by two matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix} \quad \left(\omega=e^{\frac{2\pi i}{3}}\right)$$

was introduced by Sylvester as a ternary analog of Hamilton's quaternions [1].

(2) The quark model inspired a particular brand of ternary algebraic systems. The so-called *'Nambu mechnics'* is based on such structures [10].

There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics, supersymmetric theory, and Yang–Baxter equation [1, 17, 39].

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A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [2, 40]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a C^*-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Let A and B be C^*-ternary algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary algebra isomorphism.* A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [2, 22]).

Definition. Let A and B be C^*-ternary algebras. A $\mathbb{C}$-bilinear mapping $H : A \times A \to B$ is called a *C^*-ternary algebra bi-homomorphism* if it satisfies

$$H([x, y, z], w) = [H(x, w), H(y, w), H(z, w)]$$

and

$$H(x, [y, z, w]) = [H(x, y), H(x, z), H(x, w)]$$

for all $x, y, z, w \in A$. A $\mathbb{C}$-bilinear mapping $\delta : A \times A \to A$ is called a *C^*-ternary bi-derivation* if it satisfies

$$\delta([x, y, z], w) = [\delta(x, w), y, z] + [x, \delta(y, w), z] + [x, y, \delta(z, w)]$$

and

$$\delta(x, [y, z, w]) = [\delta(x, y), z, w] + [y, \delta(x, z), w] + [y, z, \delta(x, w)]$$

for all $x, y, z, w \in A$.

In 1940, S. M. Ulam [38] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with

$$\rho(f(x), h(x)) < \epsilon$$

for all $x \in G$?

In 1941, Hyers [13] gave the first partial solution to Ulam's question for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Then, Aoki [3] and Bourgin [8] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [35] generalized the theorem of Hyers [13] by considering the stability problem with unbounded Cauchy differences. In 1991, Gajda [11] following the same approach as by Rassias [35] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11] as well as by Rassias and Šemrl [36], that one cannot prove a Rassias-type theorem when $p = 1$. Găvruta [12] obtained the generalized result of Rassias's theorem which allows the Cauchy difference to be controlled by a general unbounded function. During the past two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [4]-[7], [11, 12, 14], [18]-[21], [23], [25]-[31], [36, 37]). The instability of characteristic flows of solutions of partial differential equations is related to the Ulam stability of functional equations [24, 33]. On the other hand, Park [24] and Kim [34] have contributed works to the stability problem of ternary homomorphisms and ternary derivations.

Let X and Y be real or complex linear spaces. For a mapping $f : X \times X \to Y$, consider the functional equation:

$$f(x+y, z-w) + f(x-y, z+w) = 2f(x,z) - 2f(y,w). \tag{1}$$

Recently, the authors [32] showed that a mapping $f : X \times X \to Y$ satisfies the equation (1) if and only if the mapping f is bi-additive. We investigate the generalized Hyers-Ulam stability in C^*-ternary algebras for the bi-additive mappings satisfying (1).

2. Stability of bi-homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

Lemma 2.1. *Let V and W be $\mathbb{C}$-linear spaces and let $f : V \times V \to W$ be a bi-additive mapping such that $f(\lambda x, \mu y) = \lambda\mu f(x,y)$ for all $\lambda, \mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y \in V$, then f is $\mathbb{C}$-bilinear.*

Proof. Since f is bi-additive, we get $f\left(\frac{1}{2}x, \frac{1}{2}y\right) = \frac{1}{4}f(x,y)$ for all $x, y \in V$. Now let $\sigma, \tau \in \mathbb{C}$ and M an integer greater than $2(|\sigma| + |\tau|)$. Since $\left|\frac{\sigma}{M}\right| < \frac{1}{2}$ and $\left|\frac{\tau}{M}\right| < \frac{1}{2}$, there is $s, t \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ such that $\left|\frac{\sigma}{M}\right| = \cos s = \frac{e^{is}+e^{-is}}{2}$ and $\left|\frac{\tau}{M}\right| = \cos t = \frac{e^{it}+e^{-it}}{2}$. Now $\frac{\sigma}{M} = \left|\frac{\sigma}{M}\right|\lambda$ and $\frac{\tau}{M} = \left|\frac{\tau}{M}\right|\mu$ for some $\lambda, \mu \in \mathbb{T}^1$.

Thus we have

$$\begin{aligned}
f(\sigma x, \tau y) &= f\Big(M\frac{\sigma}{M}x, M\frac{\tau}{M}y\Big) = M^2 f\Big(\frac{\sigma}{M}x, \frac{\tau}{M}y\Big)\\
&= M^2 f\Big(\Big|\frac{\sigma}{M}\Big|\lambda x, \Big|\frac{\tau}{M}\Big|\mu y\Big) = M^2 f\bigg(\frac{e^{is}+e^{-is}}{2}\lambda x, \frac{e^{it}+e^{-it}}{2}\mu y\bigg)\\
&= \frac{1}{4}M^2 f(e^{is}\lambda x + e^{-is}\lambda x, e^{it}\mu y + e^{-it}\mu y)\\
&= \frac{1}{4}M^2\,[\,e^{is}e^{it}\lambda\mu f(x,y) + e^{is}e^{-it}\lambda\mu f(x,y) + e^{-is}e^{it}\lambda\mu f(x,y)\\
&\qquad + e^{-is}e^{-it}\lambda\mu f(x,y)\,]\\
&= \sigma\tau f(x,y)
\end{aligned}$$

for all $x, y \in V$. So the mapping $f : V \times V \to W$ is $\mathbb{C}$-bilinear. □

Lemma 2.2. *Let V and W be $\mathbb{C}$-linear spaces and let $f : V \times V \to W$ be a mapping such that*

$$f(\lambda x + \lambda y, \mu z - \mu w) + f(\lambda x - \lambda y, \mu z + \mu w) = 2\lambda\mu f(x, z) - 2\lambda\mu f(y, w)$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in V$. Then f is $\mathbb{C}$ -bilinear.

Proof. Letting $\lambda = \mu = 1$, by Theorem 2.1 in [32], f is bi-additive. Putting $y = w = 0$ in the given functional equation, we get $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, z \in V$. So by Lemma 2.1, the mapping f is $\mathbb{C}$-bilinear. □

For a given mapping $f : A \times A \to B$, we define

$$\begin{aligned}
&D_{\lambda,\mu} f(x, y, z, w)\\
&:= f(\lambda x + \lambda y, \mu z - \mu w) + f(\lambda x - \lambda y, \mu z + \mu w) - 2\lambda\mu f(x, z) + 2\lambda\mu f(y, w)
\end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in C^*-ternary algebras for the functional equation $D_{\lambda,\mu} f(x, y, z, w) = 0$.

Theorem 2.3. *Let $p < 2$ and θ be positive real numbers, and let $f : A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_B \le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p), \tag{2}$$

$$\begin{aligned}
&\|f([x, y, z], w) - [f(x, w), f(y, w), f(z, w)]\|_B\\
&\quad + \|f(x, [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\|_B\\
&\le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p)
\end{aligned}\tag{3}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary algebra bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x, y) - H(x, y)\|_B \le \frac{6\theta}{4 - 2^p}(\|x\|_A^p + \|y\|_A^p) + \frac{4}{3}\|f(0, 0)\|_B \tag{4}$$

for all $x, y \in A$.

Proof. Letting $\lambda = \mu = 1$, $y = x$ and $w = -z$ in (2), we gain

$$\|f(2x,2z) - 2f(x,z) + 2f(x,-z)\|_B \le 2\theta(\|x\|_A^p + \|z\|_A^p) + \|f(0,0)\|_B \tag{5}$$

for all $x, z \in A$. Putting $\lambda = \mu = 1$ and $x = z = 0$ in (2), we get

$$\|f(y,-w) + f(-y,w) + 2f(y,w)\|_B \le \theta(\|y\|_A^p + \|w\|_A^p) + 2\|f(0,0)\|_B$$

for all $y, w \in A$. Replacing y by x and w by z in the above inequality, we have

$$\|f(x,-z) + f(-x,z) + 2f(x,z)\|_B \le 2\theta(\|x\|_A^p + \|z\|_A^p) + 2\|f(0,0)\|_B \tag{6}$$

for all $x, z \in A$. Setting $\lambda = \mu = 1$, $y = -x$ and $w = z$ in (2), we obtain

$$\|f(2x,2z) - 2f(x,z) + 2f(-x,z)\|_B \le 2\theta(\|x\|_A^p + \|z\|_A^p) + \|f(0,0)\|_B \tag{7}$$

for all $x, z \in A$. By (5) and (6), we gain

$$\|f(2x,2z) - 4f(x,z) + f(x,-z) - f(-x,z)\|_B \le 4\theta(\|x\|_A^p + \|z\|_A^p) + 3\|f(0,0)\|_B$$

for all $x, z \in A$. By (5) and (7), we get

$$\|f(x,-z) - f(-x,z)\|_B \le 2\theta(\|x\|_A^p + \|z\|_A^p) + \|f(0,0)\|_B$$

for all $x, z \in A$. By the above two inequalities, we have

$$\|f(2x,2z) - 4f(x,z)\|_B \le 6\theta(\|x\|_A^p + \|z\|_A^p) + 4\|f(0,0)\|_B \tag{8}$$

for all $x, z \in A$. Replacing x by $2^j x$ and z by $2^j z$ and dividing 4^{j+1} in the above inequality, we obtain that

$$\begin{aligned} &\left\| \frac{1}{4^j} f(2^j x, 2^j z) - \frac{1}{4^{j+1}} f(2^{j+1}x, 2^{j+1}z) \right\|_B \\ &\le \frac{6}{4^{j+1}} 2^{jp} \theta(\|x\|_A^p + \|z\|_A^p) + \frac{1}{4^j} \|f(0,0)\|_B \end{aligned}$$

for all $x, z \in A$ and all $j = 0, 1, 2, \ldots$. For given integers l, m $(0 \le l < m)$, we obtain that

$$\begin{aligned} &\left\| \frac{1}{4^l} f(2^l x, 2^l z) - \frac{1}{4^m} f(2^m x, 2^m z) \right\|_B \\ &\le \sum_{j=l}^{m-1} \left[\frac{6}{4^{j+1}} 2^{jp} \theta(\|x\|_A^p + \|z\|_A^p) + \frac{1}{4^j} \|f(0,0)\|_B \right] \end{aligned} \tag{9}$$

for all $x, z \in A$. By the above inequality, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ is a Cauchy sequence for all $x, y \in A$. Since B is complete, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ converges for all $x, y \in A$. Define $H : A \times A \to B$ by

$$H(x,y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$. By (2), we have

$$
\begin{aligned}
&\|H(x+y, z-w) + H(x-y, z+w) - 2H(x,z) + 2H(y,w)\|_B \\
&= \lim_{n\to\infty} \left\| \frac{1}{4^n} f(2^n(x+y), 2^n(z-w)) + \frac{1}{4^n} f(2^n(x-y), 2^n(z+w)) \right. \\
&\qquad \left. - \frac{2}{4^n} f(2^n x, 2^n z) + \frac{2}{4^n} f(2^n y, 2^n w) \right\|_B \\
&\le \lim_{n\to\infty} \frac{2^{np}}{4^n} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p)
\end{aligned}
$$

for all $x, y, z, w \in A$ and all $n = 0, 1, 2, \ldots$. Letting $n \to \infty$, we see that H satisfies (1). By Lemma 2.2, the mapping $H : A \times A \to B$ is $\mathbb{C}$-bilinear. Setting $l = 0$ and taking $m \to \infty$ in (9), one can obtain the inequality (4).

It follows from (3) that

$$
\begin{aligned}
&\|H([x,y,z], w) - [H(x,w), H(y,w), H(z,w)]\|_B \\
&\quad + \|H(x, [y,z,w]) - [H(x,y), H(x,z), H(x,w)]\|_B \\
&= \lim_{n\to\infty} \frac{1}{4^n} \big(\|f([2^n x, 2^n y, 2^n z], 2^n w) \\
&\quad - [f(2^n x, 2^n w), f(2^n y, 2^n w), f(2^n z, 2^n w)]\|_B \\
&\quad + \|f(2^n x, [2^n y, 2^n z, 2^n w]) \\
&\quad - [f(2^n x, 2^n y), f(2^n x, 2^n z), f(2^n x, 2^n w)]\|_B \big) \\
&\le \lim_{n\to\infty} \frac{2^{np}}{4^n} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) = 0
\end{aligned}
$$

for all $x, y, z, w \in A$. So

$$H([x,y,z], w) = [H(x,w), H(y,w), H(z,w)]$$

and

$$H(x, [y,z,w]) = [H(x,y), H(x,z), H(x,w)]$$

for all $x, y, z, w \in A$.

Now, let $T : A \times A \to B$ be another bi-additive mapping satisfying (4). Then we have

$$
\begin{aligned}
&\|H(x,y) - T(x,y)\|_B = \frac{1}{4^n} \|H(2^n x, 2^n y) - T(2^n x, 2^n y)\|_B \\
&\le \frac{1}{4^n} \|H(2^n x, 2^n y) - f(2^n x, 2^n y)\|_B + \frac{1}{4^n} \|f(2^n x, 2^n y) - T(2^n x, 2^n y)\|_B \\
&\le \frac{2}{4^n} \left[\frac{6\theta}{4 - 2^p} 2^{np} (\|x\|_A^p + \|y\|_A^p) + \frac{4}{3} \|f(0,0)\|_B \right],
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x, y \in A$. So we can conclude that $H(x,y) = T(x,y)$ for all $x, y \in A$. This proves the uniqueness of H.

Thus the mapping H is a unique C^*-ternary algebra bi-homomorphism satisfying (4). $\square$

We obtain the Hyers-Ulam stability for the functional equation

$$D_{\lambda,\mu} f(x, y, z, w) = 0$$

as follows.

Theorem 2.4. *Let $\varepsilon > 0$ be a real number, and let $f : A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu} f(x, y, z, w)\|_B \le \varepsilon$$

and

$$\|f([x, y, z], w) - [f(x, w), f(y, w), f(z, w)]\|_B + \|f(x, [y, z, w]) - [f(x, y), f(x, z), f(x, w)]\|_B \le \varepsilon$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary algebra bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x, y) - H(x, y)\|_B \le \varepsilon + \frac{4}{3}\|f(0, 0)\|_B$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Theorem 2.5. *Let $p > 2$ and θ be positive real numbers, and let $f : A \times A \to B$ be a mapping satisfying (2), (3) and $f(0, 0) = 0$. Then there exists a unique C^*-ternary algebra bi-homomorphism $H : A \times A \to B$ such that*

$$\|f(x, y) - H(x, y)\|_B \le \frac{6\theta}{2^p - 4} \cdot (\|x\|_A^p + \|y\|_A^p)$$

for all $x, y \in A$.

Proof. It follows from (8) that

$$\left\|f(x, y) - 4f\left(\frac{x}{2}, \frac{y}{2}\right)\right\|_B \le \frac{6\theta}{2^p}(\|x\|_A^p + \|y\|_A^p)$$

for all $x, y \in A$. So

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - 4^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\right\|_B \\ &\le \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}\right)\right\|_B \\ &\le \frac{6\theta}{2^p} \sum_{j=l}^{m-1} \frac{4^j}{2^{pj}}(\|x\|_A^p + \|y\|_A^p) \end{aligned} \tag{10}$$

for all nonnegative integers m and l with $m > l$ and all $x, y \in A$. It follows from (10) that the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ is a Cauchy sequence for all $x, y \in A$. Since B is complete, the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ converges for all $x, y \in A$. So one can define the mapping $H : A \times A \to B$ by

$$H(x, y) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (10), we get the desired inequality for f and H.

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.6. *Let $p < \frac{1}{2}$ and θ be positive real numbers, and let $f : A \times A \to B$ be a mapping such that*

$$\left\|D_{\lambda,\mu} f(x,y,z,w)\right\|_B \le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p, \tag{11}$$

$$\begin{aligned} &\left\|f([x,y,z],w) - [f(x,w), f(y,w), f(z,w)]\right\|_B \\ &+ \left\|f(x,[y,z,w]) - [f(x,y), f(x,z), f(x,w)]\right\|_B \\ &\le \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p \end{aligned} \tag{12}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique C^*-ternary algebra bi-homomorphism $H : A \times A \to B$ such that

$$\|f(x,y) - H(x,y)\|_B \le \frac{\theta}{2 - 2^{4p-1}} \|x\|_A^{2p} \cdot \|y\|_A^{2p} + \frac{4}{3}\|f(0,0)\|_B$$

for all $x, y \in A$.

Proof. Letting $\lambda = \mu = 1$, $y = x$ and $w = -z$ in (11), we gain

$$\|f(2x,2z) - 2f(x,z) + 2f(x,-z)\|_B \le \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + \|f(0,0)\|_B \tag{13}$$

for all $x, z \in A$. Putting $\lambda = \mu = 1$ and $x = z = 0$ in (11), we get

$$\|f(y,-w) + f(-y,w) + 2f(y,w)\|_B \le 2\|f(0,0)\|_B$$

for all $y, w \in A$. Replacing y by x and w by z in the above inequality, we have

$$\|f(x,-z) + f(-x,z) + 2f(x,z)\|_B \le 2\|f(0,0)\|_B \tag{14}$$

for all $x, z \in A$. Setting $\lambda = \mu = 1$, $y = -x$ and $w = z$ in (11), we obtain

$$\|f(2x,2z) - 2f(x,z) + 2f(-x,z)\|_B \le \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + \|f(0,0)\|_B \tag{15}$$

for all $x, z \in A$. By (13) and (14), we gain

$$\|f(2x,2z) - 4f(x,z) + f(x,-z) - f(-x,z)\|_B \le \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + 3\|f(0,0)\|_B$$

for all $x, z \in A$. By (13) and (15), we get

$$\|f(x,-z) - f(-x,z)\|_B \le \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + \|f(0,0)\|_B$$

for all $x, z \in A$. By the above two inequalities, we have

$$\|f(2x,2z) - 4f(x,z)\|_B \le 2\theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + 4\|f(0,0)\|_B \tag{16}$$

for all $x, z \in A$. Replacing x by $2^j x$ and z by $2^j z$ and dividing 4^{j+1} in the above inequality, we obtain that

$$\begin{aligned} &\left\|\frac{1}{4^j} f(2^j x, 2^j z) - \frac{1}{4^{j+1}} f(2^{j+1}x, 2^{j+1}z)\right\|_B \\ &\le \frac{2^{4jp+1}}{4^{j+1}} \cdot \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + \frac{1}{4^j}\|f(0,0)\|_B \end{aligned}$$

for all $x, z \in A$ and all $j = 0, 1, 2, \ldots$. For given integers $l, m\,(0 \le l < m)$, we obtain that

$$\left\| \frac{1}{4^l} f(2^l x, 2^l z) - \frac{1}{4^m} f(2^m x, 2^m z) \right\|_B$$

$$\le \sum_{j=l}^{m-1} \left(\frac{2^{4jp+1}}{4^{j+1}} \cdot \theta \cdot \|x\|_A^{2p} \cdot \|z\|_A^{2p} + \frac{1}{4^j} \|f(0,0)\|_B \right) \tag{17}$$

for all $x, z \in A$. It follows from (17) that the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ is a Cauchy sequence for all $x, y \in A$. Since B is complete, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ converges for all $x, y \in A$. So one can define the mapping $H : A \times A \to B$ by

$$H(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (17), we get the desired inequality for f and H.

The rest of the proof is similar to the proof of Theorem 2.3. □

Theorem 2.7. *Let $p > \frac{1}{2}$ and θ be positive real numbers, and let $f : A \times A \to B$ be a mapping satisfying (11), (12) and $f(0,0) = 0$. Then there exists a unique C^*-ternary algebra bi-homomorphism $H : A \times A \to B$ such that*

$$\|f(x, y) - H(x, y)\|_B \le \frac{\theta}{2^{4p-1} - 2} \|x\|_A^{2p} \|y\|_A^{2p}$$

for all $x, y \in A$.

Proof. It follows from (16) that

$$\left\| f(x, y) - 4f\left(\frac{x}{2}, \frac{y}{2}\right) \right\|_B \le \frac{\theta}{2^{4p-1}} \cdot \|x\|_A^{2p} \cdot \|y\|_A^{2p}$$

for all $x, y \in A$. So

$$\|4^l f\left(\frac{x}{2^l}, \frac{y}{2^l}\right) - 4^m f\left(\frac{x}{2^m}, \frac{y}{2^m}\right)\Big\|_B$$

$$\le \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}, \frac{y}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}, \frac{y}{2^{j+1}}\right) \right\|_B$$

$$\le \frac{\theta}{2^{4p-1}} \sum_{j=l}^{m-1} \frac{4^j}{2^{4pj}} \|x\|_A^{2p} \|y\|_A^{2p} \tag{18}$$

for all nonnegative integers m and l with $m > l$ and all $x, y \in A$. It follows from (18) that the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ is a Cauchy sequence for all $x, y \in A$. Since B is complete, the sequence $\{4^n f(\frac{x}{2^n}, \frac{y}{2^n})\}$ converges for all $x, y \in A$. So one can define the mapping $H : A \times A \to B$ by

$$H(x, y) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)$$

for all $x, y \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (18), we get the desired inequality for f and H.

The rest of the proof is similar to the proof of Theorem 2.3. □

3. Stability of bi-derivations on C^*-ternary algebras and bi-isomorphisms between C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

We prove the generalized Hyers-Ulam stability of bi-derivations on C^*-ternary algebras for the functional equation $D_{\lambda,\mu} f(x, y, z, w) = 0$.

Theorem 3.1. *Let $p < 2$ and θ be positive real numbers, and let $f : A \times A \to A$ be a mapping satisfying* (2) *such that*

$$\begin{aligned} &\|f([x, y, z], w) - [f(x, w), y, z] - [x, f(y, w), z] - [x, y, f(z, w)]\|_A \\ &+ \|f(x, [y, z, w]) - [f(x, y), z, w] - [y, f(x, z), w] - [y, z, f(x, w)]\|_A \\ &\le \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) \end{aligned} \tag{19}$$

for all $x, y, z, w \in A$. If f satisfies

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^{3n} y) = \lim_{n\to\infty} \frac{1}{4^n} f(2^{3n} x, 2^n y)$$

for all $x, y \in A$, then there is a unique C^*-ternary bi-derivation $\delta : A \times A \to A$ such that

$$\|f(x, y) - \delta(x, y)\|_A \le \frac{6\theta}{4 - 2^p} (\|x\|_A^p + \|y\|_A^p) + \frac{4}{3}\|f(0, 0)\|_B$$

for all $x, y \in A$.

Proof. By the proof of Theorem 2.3, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ is a Cauchy sequence for all $x, y \in A$. Since A is complete, the sequence $\{\frac{1}{4^n} f(2^n x, 2^n y)\}$ converges. So one can define the mapping $\delta : A \times A \to A$ by

$$\delta(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$. By the same reasoning as in the proof of Theorem 2.3, we get the desired inequality for f and δ and the fact that δ is $\mathbb{C}$-bilinear.

It follows from (19) that

$$\begin{aligned} &\|\delta([x, y, z], w) - [\delta(x, w), y, z] - [x, \delta(y, w), z] - [x, y, \delta(z, w)]\|_A \\ &+ \|\delta(x, [y, z, w]) - [\delta(x, y), z, w] - [y, \delta(x, z), w] - [y, z, \delta(x, w)]\|_A \\ &= \lim_{n\to\infty} \left(\left\| \frac{1}{4^{3n}} f\big(2^{3n}[x, y, z], 2^{3n} w\big) - \left[\frac{1}{4^n} f(2^n x, 2^n w), y, z\right] \right. \right. \end{aligned}$$

$$- \left[x, \frac{1}{4^n} f(2^n y, 2^n w), z\right] - \left[x, y, \frac{1}{4^n} f(2^n z, 2^n w)\right]\Big\|_A$$
$$+ \Big\| \frac{1}{4^{3n}} f\big(2^{3n} x, 2^{3n}[y, z, w]\big) - \left[\frac{1}{4^n} f(2^n x, 2^n y), z, w\right]$$
$$- \left[y, \frac{1}{4^n} f(2^n x, 2^n z), w\right] - \left[y, z, \frac{1}{4^n} f(2^n x, 2^n w)\right]\Big\|_A \Big)$$
$$= \lim_{n\to\infty} \Big(\Big\| \frac{1}{4^{3n}} f\big([2^n x, 2^n y, 2^n z], 2^{3n} w\big) - \frac{1}{4^{3n}} \left[f\big(2^n x, 2^{3n} w\big), 2^n y, 2^n z\right]$$
$$- \frac{1}{4^{3n}} \left[2^n x, f\big(2^n y, 2^{3n} w\big), 2^n z\right] - \frac{1}{4^{3n}} \left[2^n x, 2^n y, f\big(2^n z, 2^{3n} w\big)\right]\Big\|_A$$
$$+ \Big\| \frac{1}{4^{3n}} f\big(2^{3n} x, [2^n y, 2^n z, 2^n w]\big) - \frac{1}{4^{3n}} \left[f\big(2^{3n} x, 2^n y\big), 2^n z, 2^n w\right]$$
$$- \frac{1}{4^{3n}} \left[2^n y, f\big(2^{3n} x, 2^n z\big), 2^n w\right] - \frac{1}{4^{3n}} \left[2^n y, 2^n z, f\big(2^{3n} x, 2^n w\big)\right]\Big\|_A \Big)$$
$$\leq \lim_{n\to\infty} \Big[\frac{\theta}{4^{3n}} (2^{np}\|x\|_A^p + 2^{np}\|y\|_A^p + 2^{np}\|z\|_A^p + 2^{3np}\|w\|_A^p)$$
$$+ \frac{\theta}{4^{3n}} (2^{3np}\|x\|_A^p + 2^{np}\|y\|_A^p + 2^{np}\|z\|_A^p + 2^{np}\|w\|_A^p) \Big] = 0$$

for all $x, y, z, w \in A$. So

$$\delta([x, y, z], w) = [\delta(x, w), y, z] + [x, \delta(y, w), z] + [x, y, \delta(z, w)]$$

and

$$\delta(x, [y, z, w]) = [\delta(x, y), z, w] + [y, \delta(x, z), w] + [y, z, \delta(x, w)]$$

for all $x, y, z, w \in A$.

By the same argument as in the proof of Theorem 2.3, the mapping $\delta : A \times A \to A$ is a unique C^*-ternary bi-derivation satisfying (19). □

In Theorem 3.1, for the case $p > 2$, one can obtain a similar result.

Theorem 3.2. *Let $p \neq \frac{1}{2}$ and θ be positive real numbers, and let $f : A \times A \to A$ be a mapping satisfying* (11) *such that*

$$\|f([x, y, z], w) - [f(x, w), y, z] - [x, f(y, w), z] - [x, y, f(z, w)]\|_A$$
$$+ \|f(x, [y, z, w]) - [f(x, y), z, w] - [y, f(x, z), w] - [y, z, f(x, w)]\|_A$$
$$\leq \theta \cdot \|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p \tag{20}$$

for all $x, y, z, w \in A$. Then there exists a unique C^*-ternary bi-derivation $\delta : A \times A \to A$ such that

$$\|f(x, y) - \delta(x, y)\|_B \leq \begin{cases} \frac{2\theta}{2^{4p}-4}\|x\|_A^{2p}\|y\|_A^{2p} & \left(p > \frac{1}{2}\right) \\ \frac{2\theta}{4-2^p}\|x\|_A^{2p} \cdot \|y\|_A^{2p} + \frac{4}{3}\|f(0,0)\|_B & \left(p < \frac{1}{2}\right) \end{cases}$$

for all $x, y \in A$.

Proof. Let $p < \frac{1}{2}$. By the same argument of the proof of Theorem 2.6, one can define the mapping $\delta : A \times A \to A$ by

$$\delta(x, y) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x, 2^n y)$$

for all $x, y \in A$ satisfying the desired inequality for f and δ.

The rest of the proof is similar to the proof of Theorem 3.1.

If $p > \frac{1}{2}$, the proof is similar to the proofs of Theorems 2.7, 3.1 and the case $p < \frac{1}{2}$. □

From now on, assume that A is a unital C^*-ternary algebra with norm $\|\cdot\|_A$ and unit e, and that B is a unital C^*-ternary algebra with norm $\|\cdot\|_B$ and unit e'.

We investigate isomorphisms between C^*-ternary algebras associated with the functional equation $D_{\lambda,\mu} f(x, y, z, w) = 0$.

Theorem 3.3. *Let $p \neq 2$ and θ be positive real numbers, and let $f : A \times A \to B$ be a bijective mapping satisfying* (2) *and* (3)*. If*

$$e' = \begin{cases} \lim_{n\to\infty} f(e, 2^n x) & (p < 2) \\ \lim_{n\to\infty} f\left(e, \frac{x}{2^n}\right) & (p > 2) \end{cases}$$

for all $x \in A$, then the mapping $f : A \times A \to B$ is a C^*-ternary algebra bi-isomorphism.

Proof. Define $H : A \times A \to B$ by

$$H(x, y) := \begin{cases} \lim_{j\to\infty} \frac{1}{4^j} f(2^j x, 2^j y) & (p < 2) \\ \lim_{j\to\infty} 4^j f\left(\frac{1}{2^j} x, \frac{1}{2^j} y\right) & (p > 2) \end{cases}$$

for all $x, y \in A$. By the same reasoning as in the proof of Theorems 2.3 and 2.5, the mapping H is a C^*-ternary algebra bi-homomorphism.

For the case $p < 2$, it follows from (3) that

$$\begin{aligned} H(x, y) = H([e, e, x], y) &= \lim_{n\to\infty} \frac{1}{4^n} f(2^n [e, e, x], 2^n y) \\ &= \lim_{n\to\infty} \frac{1}{4^n} f([e, e, 2^n x], 2^n y) \\ &= \lim_{n\to\infty} \frac{1}{4^n} [f(e, 2^n y), f(e, 2^n y), f(2^n x, 2^n y)] \\ &= \lim_{n\to\infty} \left[f(e, 2^n y), f(e, 2^n y), \frac{1}{4^n} f(2^n x, 2^n y)\right] \\ &= [e', e', f(x, y)] = f(x, y) \end{aligned}$$

for all $x, y \in A$. Hence the bijective mapping $f : A \times A \to B$ is a C^*-ternary algebra bi-isomorphism.

Similarly, the bijective mapping $f : A \times A \to B$ is also a C^*-ternary algebra bi-isomorphism for the case $p > 2$. □

Theorem 3.4. *Let $p \neq \frac{1}{2}$ and θ be positive real numbers, and let $f : A \times A \to B$ be a bijective mapping satisfying* (11) *and* (12)*. If*

$$e' = \begin{cases} \lim_{n\to\infty} f(e, 2^n x) & \left(p < \frac{1}{2}\right) \\ \lim_{n\to\infty} f\left(e, \frac{x}{2^n}\right) & \left(p > \frac{1}{2}\right) \end{cases}$$

for all $x \in A$, then the mapping $f : A \times A \to B$ is a C^*-ternary algebra bi-isomorphism.

Proof. The proof is similar to the proof of Theorem 3.3. □

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