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THE PROPAGATION PHENOMENON OF WEIGHTED SHIFTS

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Abstract. This paper concerns the propagation phenomenon of weighted shifts. We here establish the existence of positive real numbers b and c $(1 < b < c)$ such that the recursive weighted shift $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically but not cubically hyponormal.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional complex Hilbert spaces, let $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \ldots$ (called *weights*), the (*unilateral*) *weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if (cf. [4, II.1.9])

$$\text{(0.1)} \qquad \begin{pmatrix} I & T^* & \ldots & T^{*k} \\ T & T^*T & \ldots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \ldots & T^{*k}T^k \end{pmatrix} \geq 0 \qquad (\text{all } k \geq 1).$$

Let $[A,B] := AB - BA$ denote the commutator of two operators A and B, and define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$\text{(0.2)} \qquad M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k$$

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is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (0.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (0.1); the Bram-Halmos criterion can be then rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([11]).

Recall ([1], [5], [11]) that $T \in \mathcal{B}(\mathcal{H})$ is said to be *weakly k-hyponormal* if $\alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_k T^k$ is hyponormal for each $(\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k$, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([11])

$$(0.3) \quad \left\langle M_k(T) \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_1, \ldots, \lambda_k \in \mathbb{C}.$$

If $k = 2$, then T is said to be *quadratically hyponormal*, and if $k = 3$, then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{B}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general.

In this paper we consider a propagation phenomenon of the cubic hyponormality. Although examples abound of nontrivial quadratically hyponormal recursive shifts with two equal weights (cf. [9]), we conjecture that the same is not true for cubic hyponormality. To this end, we show in Theorem 3 the existence of positive numbers b and c such that the recursive shift $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal but not cubically hyponormal. This gives an explicit description of the gap between quadratic hyponormality and cubic hyponormality for recursive shifts.

2. The main result

J. Stampfli [15] showed that for subnormal weighted shifts W_α, a *propagation* phenomenon occurs which forces the flatness of W_α whenever two equal weights are present. Later, A. Joshi proved in [13] that the shift with weights $\alpha_0 = \alpha_1 = a$, $\alpha_2 = \alpha_3 = \cdots = b$, $0 < a < b$, is *not* quadratically hyponormal, and P. Fan [12] established that for $a = 1$, $b = 2$, and $0 < s < \sqrt{5}/5$, $W_\alpha + s\,W_\alpha^2$ is *not* hyponormal. On the other hand, it was shown in [6, Theorem 2] that a hyponormal weighted shift with *three* equal weights cannot be quadratically hyponormal without being flat: *If W_α is quadratically hyponormal and $\alpha_n = \alpha_{n+1} = \alpha_{n+2}$ for some $n \geq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$, i.e., W_α is subnormal.* Furthermore, in [6, Proposition 11] it was shown that, in the presence of quadratic hyponormality, two consecutive pairs of equal weights again force flatness, thereby subnormality.

Theorem 1 (Propagation). *Let W_α be a weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$.*

(i) ([15, Theorem 6]) *Let W_α be subnormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat, i.e., $\alpha_1 = \alpha_2 = \alpha_3 = \cdots$.*
(ii) ([6, Corollary 6]) *Let W_α be 2-hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 0$, then α is flat.*
(iii) ([2, Theorem 1]) *Let W_α be quadratically hyponormal. If $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$, then α is flat.*

Before we proceed, we consider the selfcommutator $[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$. Let W_α be a hyponormal weighted shift. For $s \in \mathbb{C}$, we write

$$D(s) := [(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]$$

and we let
(1.1)

$$D_n(s) := P_n[(W_\alpha + s\,W_\alpha^2)^*, W_\alpha + s\,W_\alpha^2]P_n = \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \dots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \dots & 0 & 0 \\ 0 & r_1 & q_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \dots & r_{n-1} & q_n \end{pmatrix},$$

where P_n is the orthogonal projection onto the subspace generated by $\{e_0, \dots, e_n\}$,

$$\begin{cases} q_n := u_n + |s|^2 v_n \\ r_n := s\sqrt{w_n} \\ u_n := \alpha_n^2 - \alpha_{n-1}^2 \\ v_n := \alpha_n^2\alpha_{n+1}^2 - \alpha_{n-1}^2\alpha_{n-2}^2 \\ w_n := \alpha_n^2(\alpha_{n+1}^2 - \alpha_{n-1}^2)^2, \end{cases} \tag{1.2}$$

and, for notational convenience, $\alpha_{-2} = \alpha_{-1} = 0$. Clearly, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for all $s \in \mathbb{C}$ and all $n \geq 0$. Let $d_n(\cdot) := \det(D_n(\cdot))$. Then d_n satisfies the following 2–step recursive formula:

$$d_0 = q_0, \quad d_1 = q_0q_1 - |r_0|^2, \quad d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n. \tag{1.3}$$

If we let $t := |s|^2$, we observe that d_n is a polynomial in t of degree $n+1$, and if we write $d_n \equiv \sum_{i=0}^{n+1} c(n,i)t^i$, then the coefficients $c(n,i)$ satisfy a double-indexed recursive formula, namely
(1.4)

$$c(n+2,i) = u_{n+2}\,c(n+1,i) + v_{n+2}\,c(n+1,i-1) - w_{n+1}\,c(n,i-1),$$
$$c(n,0) = u_0 \cdots u_n, \quad c(n,n+1) = v_0 \cdots v_n, \quad c(1,1) = u_1v_0 + v_1u_0 - w_0$$

$(n \geq 0,\ i \geq 1)$. We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for every $n \geq 0$, $0 \leq i \leq n+1$ (cf. [7]). Evidently, positively quadratically hyponormal $\Rightarrow$ quadratically hyponormal. The converse, however, is not true in general (cf. [3]).

The idea of the proof of Theorem 1 (iii) is based on the following observation: if W_α is quadratically hyponormal with $\alpha_1 = \alpha_2 = 1$, then a straightforward calculation shows that

$$d_4(t) = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 t^2 + c(4,3)t^3 + c(4,4)t^4 + c(4,5)t^5,$$

so

$$\lim_{t\to 0+} \frac{d_4(t)}{t^2} = \alpha_0^2\alpha_4^2(\alpha_0^2 - 1)(\alpha_3^2 - 1)^3 \ \geq 0,$$

which forces $\alpha_0 = 1$ or $\alpha_3 = 1$, so that three equal weights are present and hence by [6, Theorem 2], flatness occurs.

Note that in Theorem 1 (iii) the condition "$n \geq 1$" cannot be relaxed to "$n \geq 0$". For example, if (cf. [6, Proposition 7])

$$\alpha_0 = \alpha_1 = \sqrt{\frac{2}{3}}, \quad \alpha_n = \sqrt{\frac{n+1}{n+2}} \ (n \geq 2), \tag{1.5}$$

then W_α is quadratically hyponormal but not cubically hyponormal (and hence not subnormal); indeed if we let

$$C_5(t) := \det\left(P_5\left[(W_\alpha + t\,W_\alpha^2 + t^2\,W_\alpha^3)^*,\ (W_\alpha + t\,W_\alpha^2 + t^2\,W_\alpha^3)\right]P_5\right)$$

then

$$\lim_{t\to 0+} \frac{C_5(t)}{t^8} = -\frac{1}{2041200} < 0.$$

We briefly pause to recall that when $\alpha_0 = \alpha_1 = 1$, quadratic hyponormality implies

$$\alpha_2 < \sqrt{2} \quad \text{and} \quad \alpha_3 \geq (2 - \alpha_2^2)^{-2} \tag{1.6}$$

(cf. [7, p. 78]).

At this point one might guess that *every cubically hyponormal weighted shift with two equal weights is subnormal.* To affirm this, it would suffice to show, in view of Theorem 1 (iii), that if W_α is cubically hyponormal and if $\alpha_0 = \alpha_1$, then W_α is flat.

In the sequel, given $\alpha_0 < \alpha_1 < \alpha_2$ we denote by $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ the recursive weighted shift whose weights are calculated according to the recursive relation

$$\alpha_{n+1}^2 = \varphi_1 + \varphi_0 \frac{1}{\alpha_n^2}, \tag{1.7}$$

where

$$\varphi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \varphi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}. \tag{1.8}$$

It is well-known that $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ is subnormal with 2–atomic Berger measure. Let $W_{x\,(\alpha_0,\alpha_1,\alpha_2)^\wedge}$ denote the weighted shift whose weight sequence consists of the initial weight x followed by the weight sequence of $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$. In [8], it was shown that there exists $1 < b < c$ such that $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal.

In this stage we would pose:

Question 2. Does there exist $1 < b < c$ such that $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is cubically hyponormal?

In [10, Example 3.1] it was shown that $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ is 2-hyponormal if and only if it is subnormal. Thus one might guess that $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal if and only if it is polynomially hyponormal. However, the 2-hyponormality of $W_{\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$ never admits a as a value for x. By contrast, quadratic hyponormality does admit the value a. Thus, the situation for weak k-hyponormality becomes more delicate. In view of the preceding considerations, we conjecture that the answer to Question 2 is negative; the following theorem provides a strong evidence.

Theorem 3. *There exists $1 < b < c$ such that $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal but not cubically hyponormal.*

Proof. Let W_α be a hyponormal weighted shift with weight sequence $\{\alpha_n\}_{n=0}^\infty$. For $s, t \in \mathbb{C}$ we let

$$C_n(s,t) := P_n \left[(W_\alpha + s\,W_\alpha^2 + t\,W_\alpha^3)^*,\ W_\alpha + s\,W_\alpha^2 + t\,W_\alpha^3\right] P_n;$$

$C_n(s,t)$ is a pentadiagonal matrix:

$$C_n(s,t) = \begin{pmatrix} q_0 & r_0 & u_0 & 0 & & & & \\ \bar{r}_0 & q_1 & r_1 & u_1 & 0 & & & \\ \bar{u}_0 & \bar{r}_1 & q_2 & r_2 & u_2 & 0 & & \\ 0 & \bar{u}_1 & \bar{r}_2 & q_3 & r_3 & u_3 & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & u_{n-2} \\ & & & & \ddots & \ddots & \ddots & \ddots & r_{n-1} \\ & & & & & 0 & \bar{u}_{n-2} & \bar{r}_{n-1} & q_n \end{pmatrix}, \tag{3.1}$$

where

$$\begin{aligned} q_n &:= (\alpha_n^2 - \alpha_{n-1}^2) + (\alpha_n^2\alpha_{n+1}^2 - \alpha_{n-2}^2\alpha_{n-1}^2)\,|s|^2 \\ &\quad + (\alpha_n^2\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-3}^2\alpha_{n-2}^2\alpha_{n-1}^2)\,|t|^2, \\ r_n &:= \alpha_n\,(\alpha_{n+1}^2 - \alpha_{n-1}^2)\,\bar{s} + \alpha_n\,(\alpha_{n+1}^2\alpha_{n+2}^2 - \alpha_{n-1}^2\alpha_{n-2}^2)\,s\,\bar{t}, \\ u_n &:= \alpha_n\alpha_{n+1}\,(\alpha_{n+2}^2 - \alpha_{n-1}^2)\,\bar{t}, \end{aligned}$$

and, for notational convenience, $\alpha_{-3} = \alpha_{-2} = \alpha_{-1} = 0$. Then W_α is cubically hyponormal if and only if $\det C_n(s,t) \geq 0$ for every $s, t \in \mathbb{C}$ and every $n \geq 0$. Put $\alpha_0 = \alpha_1 = 1$. Then a straightforward calculation shows that

$$\det C_4(s,0) = \alpha_2^2(\alpha_2^2 - 1)(\alpha_4^2 - \alpha_3^2)\bigg(\alpha_3^2(2 - \alpha_2^2) - 1\bigg)\,s^4\,(1 + p_1(s))$$

and

$$\det C_4(s,s^2) = \alpha_2^2\alpha_3^2(\alpha_2^2-1)(\alpha_3^2-\alpha_2^2)\Big(\alpha_4^2(2-\alpha_3^2)-1\Big)\, s^6\,(1+p_2(s)),$$

where $p_i(s)$ is a polynomial in s with $p_i(0)=0$ for each $i=1,2$. Therefore, if W_α is cubically hyponormal, then

$$\alpha_2^2(\alpha_2^2-1)(\alpha_4^2-\alpha_3^2)\Big(\alpha_3^2(2-\alpha_2^2)-1\Big)\geq 0$$

and

$$\alpha_2^2\alpha_3^2(\alpha_2^2-1)(\alpha_3^2-\alpha_2^2)\Big(\alpha_4^2(2-\alpha_3^2)-1\Big)\geq 0.$$

Thus, if $\{\alpha_n\}_{n=1}^\infty$ is strictly increasing, then

$$\alpha_{k+1}^2 \geq \frac{1}{2-\alpha_k^2} \quad \text{and} \quad 1<\alpha_k^2<2 \quad \text{for } k=2,3. \tag{3.2}$$

Write $\{\alpha_n\}_{n=0}^\infty : 1,(1,\sqrt{b},\sqrt{c})^\wedge$ $(1<b<c)$. In view of Theorem 1 (iii), $\{\alpha_n\}_{n=1}^\infty$ is strictly increasing. Put

$$\varphi_1 := \frac{b(c-1)}{b-1} \quad \text{and} \quad \varphi_0 := -\frac{b(c-b)}{b-1}.$$

Then $\alpha_4^2=\varphi_1+\frac{\varphi_0}{c}=\frac{b(c^2-2c+b)}{c(b-1)}$. Thus by (3.2), we have that $c\geq\frac{1}{2-b}$ and $\alpha_4^2\geq\frac{1}{2-c}$, i.e.,

$$\frac{b(c^2-2c+b)}{c(b-1)}\geq\frac{1}{2-c}.$$

Therefore, if $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is cubically hyponormal, then

$$g(b,c) := c^3-4c^2+(b+5-\frac{1}{b})c-2b\leq 0. \tag{3.3}$$

It is known ([8, Proposition 4.6]) that if $b=\frac{11}{10}$, then there exists a value of c between $\frac{1142}{1000}$ and $\frac{1143}{1000}$ for which $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal. But a direct calculation shows that $g(\frac{11}{10},c)>0$ for $\frac{1142}{1000}<c<\frac{1143}{1000}$, so that the corresponding shift is not cubically hyponormal. □

As a strategy to answer Question 2, recall an argument in [8, Theorem 4.3]. Let $0<a<b<c$, let $\alpha:\sqrt{x},(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge$ and let

$$h_2^+ := \Big(\sup\{x : W_\alpha \text{ is positively quadratically hyponormal}\}\Big)^{\frac{1}{2}}.$$

As before, write $\varphi_1:=\frac{b(c-a)}{b-a}$ and $\varphi_0:=-\frac{ab(c-b)}{b-a}$. Put

$$L^2 := \frac{\varphi_1+\sqrt{\varphi_1^2+4\varphi_0}}{2} \quad \text{and} \quad K := -\frac{\varphi_1^2L^2}{\varphi_0}. \tag{3.4}$$

In [8], it was shown that

$$h_2^+ = \min\left\{\sqrt{a}\,,\quad f(a,b,c)\right\}, \tag{3.5}$$

where

$$f(a,b,c) := \left(\frac{a^2b^2c + ab^2(c-a)K + ab(c-b)K^2}{a^3b + ab(c-a)K + (a^2+bc-2ab)K^2}\right)^{\frac{1}{2}}.$$

Note that the second term under min in (3.5) may be greater than the first: for example if $a = \frac{17}{5}$, $b = \frac{58}{17}$ and $c = \frac{99}{29}$, then

$$f(a,b,c) \approx \sqrt{3.4218} > \sqrt{\frac{17}{5}} = \sqrt{a}.$$

Thus, in this case, $\alpha : \sqrt{\frac{17}{5}}, (\sqrt{\frac{17}{5}}, \sqrt{\frac{58}{17}}, \sqrt{\frac{99}{29}})^\wedge$ induces a positively quadratically hyponormal shift. For $\alpha : \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ write

$$h_2 := \left(\sup\left\{x : W_\alpha \text{ is quadratically hyponormal}\right\}\right)^{\frac{1}{2}}.$$

Clearly, $h_2^+ \le h_2$. In [14, Theorem 4.6], it was shown that $h_2 \le f(a,b,c)$; consequently, we have

$$h_2 = \min\left\{\sqrt{a}\,,\quad f(a,b,c)\right\}. \tag{3.6}$$

If $1 < b < c$ write

$$\mathfrak{H}_2 := \left\{(b,c) : W_{1,(1,\sqrt{b},\sqrt{c})^\wedge} \text{ is quadratically hyponormal}\right\};$$

$$\mathfrak{H}_3 := \left\{(b,c) : W_{1,(1,\sqrt{b},\sqrt{c})^\wedge} \text{ is cubically hyponormal}\right\}.$$

Corollary 4. *If $1 < b < c$ we let*

$$f(b,c) := b(bc-1) + b(b-1)(c-1)K - (b-1)^2K^2; \tag{4.1}$$

$$g(b,c) := c^3 - 4c^2 + (b + 5 - \frac{1}{b})c - 2b, \tag{4.2}$$

where K is given by (3.4) *with $a = 1$. Then we have*:

(i) $\mathfrak{H}_2 = \{(b,c) :\ f(b,c) \ge 0\}$;
(ii) $\mathfrak{H}_3 \subseteq \{(b,c) :\ f(b,c) \ge 0 \quad \textit{and} \quad g(b,c) \le 0\}$.

Proof. By (3.6), $W_{1,(1,\sqrt{b},\sqrt{c})^\wedge}$ is quadratically hyponormal if and only if $1 \le f(a,b,c)$, or equivalently,

$$b(bc-1) + b(b-1)(c-1)K - (b-1)^2K^2 \ge 0,$$

where

$$K = \frac{b(c-1)^2\left(b(c-1) + \sqrt{b^2(c-1)^2 - 4b(b-1)(c-b)}\right)}{2(b-1)^2(c-b)}.$$

This proves assertion (i). Assertion (ii) follows from assertion (i) and (3.3). $\square$

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