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# NOTES ON CRITICAL ALMOST HERMITIAN STRUCTURES

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Abstract. We discuss the critical points of the functional $\mathcal{F}_{\lambda,\mu}(J,g) = \int_M (\lambda\tau+\mu\tau^*)dv_g$ on the spaces of all almost Hermitian structures $\mathcal{AH}(M)$ with $(\lambda,\mu) \in \mathbb{R}^2 - (0,0)$, where $\tau$ and $\tau^*$ being the scalar curvature and the $*$-scalar curvature of $(J,g)$, respectively. We shall give several characterizations of Kähler structure for some special classes of almost Hermitian manifolds, in terms of the critical points of the functionals $\mathcal{F}_{\lambda,\mu}(J,g)$ on $\mathcal{AH}(M)$. Further, we provide the almost Hermitian analogy of the Hilbert's result.

## 1. Introduction

Let $M$ be a compact orientable smooth manifold of dimension $m$. We denote by $\mathcal{M}(M)$ the set of all Riemannian metrics on $M$ and $\mathcal{M}_c(M) = \{g \in \mathcal{M}(M) \mid Vol(M,g) = c\}$, where $c$ is a positive constant. It is well-known that a Riemannian metric $g \in \mathcal{M}_c(M)$ is a critical point of the (so-called) Einstein-Hilbert functional $\mathcal{F}$ on $\mathcal{M}_c(M)$ denoted by

$$\mathcal{F}(g) = \int_M \tau dv_g \tag{1.1}$$

if and only if $g$ is an Einstein metric, where $\tau$ is the scalar curvature of $g$ and $dv_g$ is the volume element of $g$ [5]. Now, let $M$ be a compact smooth manifold of dimension $m = 2n$ admitting an almost complex structure. We denote by $\mathcal{AH}(M)$ the set of all almost Hermitian structures on $M$. It is also known that the space $\mathcal{AH}(M)$ is a contractible Frechet space. Let $\tau$ and $\tau^*$ be the scalar curvature and the $*$-scalar curvature of $(J,g)$, respectively. Let $(\lambda,\mu) \in \mathbb{R}^2 - (0,0)$. In [6], Koda introduced and studied the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$, which is a generalization of the Einstein-Hilbert functional $\mathcal{F}$ in almost Hermitian case, that is defined by

$$\mathcal{F}_{\lambda,\mu}(J,g) = \int_M (\lambda\tau + \mu\tau^*)dv_g, \tag{1.2}$$

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and discussed the critical points on $\mathcal{AH}(M)$ and

$$\mathcal{AH}_c(M) = \{\, (J,g) \in \mathcal{AH}(M) \mid g \in \mathcal{M}_c(M)\}.$$

We here note that, if $(J,g)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$, then $(J,g)$ is also a critical point of the functional $\mathcal{F}_{s\lambda,s\mu}$ on $\mathcal{AH}(M)$ for any non-zero real number $s$, and vice versa. In the present paper, we shall reconsider the critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ and $\mathcal{AH}_c(M)$ by using a slightly different method from the ones used by Koda [6], and give conditions for an almost Hermitian manifold to be a Kähler manifold in terms of critical almost Hermitian structures for certain $(\lambda,\mu)$. In the last section, we shall remark on the critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$.

## 2. Preliminaries

In this section, we prepare some fundamental tools which we need in our arguments. Let $M = (M,J,g)$ be a $2n$-dimensional almost Hermitian manifold with almost Hermitian structure $(J,g)$ and $\Omega$ be the Kähler form of $M$ defined by $\Omega(X,Y) = g(X,JY)$ for $X$, $Y \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ denoting the Lie algebra of all smooth vector fields $X$, $Y$ on $M$. We assume that $M$ is oriented by the volume form $dv_g = \frac{(-1)^n}{n!}\Omega^n$. We denote by $\nabla$, $R$, $\rho$ and $\tau$ the Riemannian connection, the curvature tensor, the Ricci tensor and scalar curvature of $M$, respectively. The curvature tensor is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

for $X$, $Y$, $Z \in \mathfrak{X}(M)$. A tensor field $\rho^*$ on $M$ of type (0,2) defined by

$$\begin{aligned} \rho^*(X,Y) &= \mathrm{tr}\ (Z \mapsto R(X,JZ)JY) \\ &= \frac{1}{2}\mathrm{tr}\ (Z \mapsto R(X,JY)JZ) \end{aligned} \tag{2.1}$$

is called a Ricci $*$-tensor, for $X$, $Y$, $Z \in \mathfrak{X}(M)$, respectively. We denote by $\tau^*$ the $*$-scalar curvature of $M$, which is the trace of the linear endomorphism $Q^*$ defined by $g(Q^*X,Y) = \rho^*(X,Y)$. We remark that $\rho^*$ satisfies

$$\rho^*(X,Y) = \rho^*(JY,JX) \tag{2.2}$$

for any $X$, $Y \in \mathfrak{X}(M)$. Thus $\rho^*$ is symmetric if and only if $\rho^*$ is J-invariant. In this paper, for any orthonormal basis, (resp. local orthonormal frame field) $\{e_i\}_{i=1,\dots,2n}$ at any point $p \in M$ (resp. on a neighborhood of $p$), we shall adopt the following notational conventions:

$$\begin{aligned} R_{ijkl} &= g(R(e_i,e_j)e_k,e_l), \quad & R_{\bar{i}\bar{j}\bar{k}\bar{l}} &= g(R(Je_i,Je_j)Je_k,Je_l), \\ \rho_{ij} &= \rho(e_i,e_j), & \rho_{\bar{i}\bar{j}} &= \rho(Je_i,Je_j), \\ \rho^*_{ij} &= \rho^*(e_i,e_j), & \rho^*_{\bar{i}\bar{j}} &= \rho^*(Je_i,Je_j), \\ J_{ij} &= g(Je_i,e_j), & \nabla_i J_{jk} &= g((\nabla_{e_i}J)e_j,e_k), \end{aligned} \tag{2.3}$$

and so on, where the Latin indices run over the range $1,2,\dots,2n$.

Then we have

$$J_{ij} = -J_{ji}, \quad \Omega_{ij} = -J_{ij}, \quad \nabla_i J_{jk} = -\nabla_i J_{kj}, \quad \nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}. \tag{2.4}$$

We here recall some special classes of almost Hermitian manifolds [4]. We denote by $\mathcal{K}$, $\mathcal{AK}$, $\mathcal{NK}$, $\mathcal{QK}$, $\mathcal{SK}$ and $\mathcal{H}$ the sets of all Kähler manifolds, almost Kähler manifolds, nearly Kähler manifolds, quasi Kähler manifolds, semi Kähler manifolds and Hermitian manifolds, respectively.

(I) $\mathcal{K}$ (Class of Kähler manifolds): An almost Hermitian manifold $M = (M, J, g)$ is called a Kähler manifold if $\nabla J = 0$ on $M$.

(II) $\mathcal{AK}$ (Class of almost Kähler manifolds) : An almost Hermitian manifold is called an almost Kähler manifold if $d\Omega = 0$ on $M$. The condition $d\Omega = 0$ is equivalent to the condition $\nabla_i J_{jk} + \nabla_j J_{ki} + \nabla_k J_{ij} = 0$.

(III) $\mathcal{NK}$ (Class of nearly Kähler manifolds): An almost Hermitian manifold is called a nearly Kähler manifold if $(\nabla_X J)Y + (\nabla_Y J)X = 0$ ($\nabla_i J_{jk} + \nabla_j J_{ik} = 0$) on $M$.

(IV) $\mathcal{QK}$ (Class of quasi Kähler manifolds): An almost Hermitian manifold is called a quasi Kähler manifold if $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$ ($\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0$) on $M$.

(V) $\mathcal{SK}$ (Class of semi Kähler manifolds): An almost Hermitian manifold is called a semi Kähler manifold if $\delta\Omega = 0$ on $M$. The condition $\delta\Omega = 0$ is equivalent to the condition $\sum_a \nabla_a J_{aj} = 0$.

(VI) $\mathcal{H}$ (Class of Hermitian manifolds): An almost Hermitian manifold with the integrable almost complex structure $J$ is called a Hermitian manifold. It is well-known that $M$ is a Hermitian manifold if and only if $(\nabla_X J)Y - (\nabla_{JX} J)(JY) = 0$ ($\nabla_i J_{jk} - \nabla_{\bar{i}} J_{\bar{j}k} = 0$) on $M$.

Now we shall recall some fundamental identities on the above classes (II)∼(VI).

(II) Let $M = (M, J, g)$ be an almost Kähler manifold. Then, addition to (2.4), we have [3]

$$\nabla_{\bar{i}} J_{\bar{j}k} + \nabla_i J_{jk} = 0, \tag{2.5}$$

$$\begin{aligned} &2\sum_a (\nabla_a J_{ij}) \nabla_a J_{kl} \\ &= R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}}. \end{aligned} \tag{2.6}$$

From (2.6), we have

$$\rho^*_{ij} + \rho^*_{ji} - \rho_{ij} - \rho_{\bar{i}\bar{j}} = \sum_{a,b} (\nabla_a J_{ib}) \nabla_a J_{jb}, \tag{2.7}$$

and further

$$|\nabla J|^2 = 2(\tau^* - \tau). \tag{2.8}$$

(III) Let $M = (M, J, g)$ be a nearly Kähler manifold. Then, it is well-known that the following identities exist on $M$ [2], we have

$$\nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0, \tag{2.9}$$

$$\rho_{\bar{i}\bar{j}} = \rho_{ij}, \quad \rho^*_{\bar{i}\bar{j}} = \rho^*_{ij}, \tag{2.10}$$

$$\rho_{ij} - \rho^*_{ij} = \sum_{a,b} (\nabla_i J_{ab}) \nabla_j J_{ab}. \tag{2.11}$$

From (2.11), we have

$$|\nabla J|^2 = \tau - \tau^* (= \text{constant}). \tag{2.12}$$

(V) Let $M = (M, J, g)$ be a semi Kähler manifold. Then, it is well-known that the following identities on $M$ [2]:

$$\tau - \tau^* = \sum (\nabla_k J_{ji}) \nabla_j J_{ik}. \tag{2.13}$$

Now, let $M = (M, J, g)$ be a 4-dimensional almost Hermitian manifold. Then we have

$$\tau - \tau^* = 2\delta\omega + |\omega|^2 - \frac{1}{8}|N|^2, \tag{2.14}$$

where $\omega$ is the Lee form of $M$ and $N$ is the Nijenhuis tensor of $M$ [10].

(VI) Let $M = (M, J, g)$ be a 4-dimensional Hermitian manifold. Then from (2.14), in particular, we have

$$\tau - \tau^* = 2\delta\omega + |\omega|^2. \tag{2.15}$$

Here we note the inclusion relations between the classes (I)$\sim$(VI).

$$\mathcal{K} \begin{matrix} \subset \mathcal{AK} \subset \\ \subset \mathcal{NK} \subset \end{matrix} \mathcal{QK} \subset \mathcal{SK}, \qquad \mathcal{QK} \cap \mathcal{H} = \mathcal{K}.$$

We denote by $\mathcal{K}(M)$, $\mathcal{AK}(M)$, $\mathcal{NK}(M)$, $\mathcal{QK}(M)$, $\mathcal{SK}(M)$ and $\mathcal{H}(M)$, the subsets of $\mathcal{AH}(M)$ of all Kähler structures, almost Kähler structures, nearly Kähler structures, quasi Kähler structures, semi Kähler structures and Hermitian structures on $M$. Then, we may obtain the inclusion relations between these sets corresponding to the above inclusion ones between the classes (I)$\sim$(IV). In the sequel, we assume always that $M$ is a $2n (n \geqq 2)$-dimensional compact orientable smooth manifold and $\mathcal{AH}(M) \neq \emptyset$ unless otherwise specified.

## 3. Critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$

Let $M$ be a $2n$-dimensional compact orientable smooth manifold admitting almost complex structure. Let $(J, g) \in \mathcal{AH}(M)$ and consider a smooth curve $(J(t), g(t)) \in \mathcal{AH}(M)$ through $(J, g)$. We shall also call it a (1-parameter) deformation of $(J, g)$. We denote by $\Omega(t)$ the Kähler form of $(J(t), g(t))$ and set $\alpha(t) = \Omega(t) - \Omega$. We denote further by $\nabla^{(t)}, R(t), \rho(t), \rho^*(t), \tau(t)$ and $\tau^*(t)$ the Riemannian connection, the curvature tensor, the Ricci tensor, the Ricci $*$-tensor, the scalar curvature and the $*$-scalar curvature of

$(J(t), g(t))$, respectively. Let $(U; x_1, \ldots, x_{2n})$ be a local coordinate system on coordinate neighborhood $U$ of $M$. With respect to the natural frame $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1,\ldots,2n}$, we set $g(t)(\partial_i, \partial_j) = g(t)_{ij}$, $J(t)\partial_i = J(t)_i{}^j \partial_j$, $(\nabla^{(t)}_{\partial_i} J(t))\partial_j = (\nabla^{(t)}_i J(t)_j{}^k)\partial_k$, $R(t)(\partial_i, \partial_j)\partial_k = R(t)_{ijk}{}^l \partial_l$, $\rho(t)(\partial_i, \partial_j) = \rho(t)_{ij}$, $\rho^*(t)(\partial_i, \partial_j) = \rho^*(t)_{ij}$, $\alpha(t)(\partial_i, \partial_j) = \alpha(t)_{ij}$ and $g(t)^{ij} = (g(t)_{ij})^{-1}$. In particular, we have $g(0)_{ij} = g_{ij}$, $J(0)_i{}^j = J_i{}^j$, $\nabla^{(0)}_i J(0)_j{}^k = \nabla_i J_j{}^k$, $R(0)_{ijk}{}^l = R_{ijk}{}^l$, $\rho(0)_{ij} = \rho_{ij}$, $\rho^*(0)_{ij} = \rho^*_{ij}$, $\tau(0) = \tau$, $\tau^*(0) = \tau^*$ and $\alpha(0)_{ij} = 0$.
Further, we set

$$\left.\frac{d}{dt}\right|_{t=0} g(t)_{ij} = h_{ij},\ \left.\frac{d}{dt}\right|_{t=0} J(t)_j{}^i = K_j{}^i,\ \left.\frac{d}{dt}\right|_{t=0} \alpha(t)_{ij} = A_{ij}. \tag{3.1}$$

Then we see that $A = (A_{ij})$ is a 2-form, $h = (h_{ij})$ is a symmetric $(0,2)$-tensor field on $M$ and we also have

$$\left.\frac{d}{dt}\right|_{t=0} g(t)^{ij} = -h^{ij}, \tag{3.2}$$

where we adopt the standard notational convention of tensor analysis: for example $h^{ij}$ means $h^{ij} = g^{ia}g^{jb}h_{ab}$. We denote by $dv_{g(t)}$ the volume of $(M, g(t))$. Then, we have

$$\left.\frac{d}{dt}\right|_{t=0} dv_{g(t)} = \frac{1}{2}(g^{ij}h_{ij})dv_g. \tag{3.3}$$

From (3.2), we see that the coefficients $\Gamma(t)_{ij}{}^k$ of $\nabla^{(t)}$ satisfy

$$\left.\frac{d}{dt}\right|_{t=0} \Gamma(t)_{ij}{}^k = \frac{1}{2}g^{ka}(\nabla_i h_{aj} + \nabla_j h_{ia} - \nabla_a h_{ij}). \tag{3.4}$$

Thus, from (3.4), the derivations of $R(t)_{ijk}{}^l$, $\rho(t)_{ij}$ and $\tau(t)$ at $t = 0$ are given respectively by

(3.5)
$$\left.\frac{d}{dt}\right|_{t=0} R(t)_{ijk}{}^l = \frac{1}{2}(-R_{ijk}{}^a h_a{}^l + R_{ija}{}^l h_k{}^a + \nabla_i\nabla_k h_j{}^l - \nabla_j\nabla_k h_i{}^l - \nabla_i\nabla^l h_{jk} + \nabla_j\nabla^l h_{ik}),$$

(3.6)
$$\left.\frac{d}{dt}\right|_{t=0} \rho(t)_{ij} = \frac{1}{2}(-R_{aij}{}^b h_b{}^a + \rho_{ia}h_j{}^a + \nabla_a\nabla_j h_i{}^a - \nabla_i\nabla_j h_a{}^a - \nabla^a\nabla_a h_{ij} + \nabla_i\nabla_a h_j{}^a),$$

$$\left.\frac{d}{dt}\right|_{t=0} \tau(t) = -\rho_{ij}h^{ij} + \nabla^i\nabla^j h_{ij} - \nabla^i\nabla_i h_a{}^a. \tag{3.7}$$

Further, since $(J(t), g(t)) \in \mathcal{AH}(M)$, we have

$$K_a{}^i J_j{}^a + J_a{}^i K_j{}^a = 0, \tag{3.8}$$

$$h_{ij} = h_{ab}J_i{}^a J_j{}^b + K_{ia}J_j{}^a + J_{ia}K_j{}^a, \tag{3.9}$$

$$K_j{}^i = -h_a{}^i J_j{}^a - A_j{}^i, \tag{3.10}$$

$$h_{ij} = -h_{ab}J_i{}^aJ_j{}^b + J_i{}^aA_{aj} + J_j{}^aA_{ai}. \tag{3.11}$$

From (3.10) and (3.11), we have also

$$K_j{}^i = h_j{}^aJ_a{}^i - A_b{}^aJ_a{}^iJ_j{}^b. \tag{3.12}$$

Conversely, let $(h, A)$ be a pair of a symmetric $(0, 2)$-tensor $h = (h_{ij})$ and a 2-form $A = (A_{ij})$ satisfying (3.11) and define a $(1, 1)$-tensor $K$ by (3.12). Then, we may easily check that the equalities (3.8) and (3.9) hold. This means that for a given $(J, g) \in \mathcal{AH}(M)$ and a pair $(h, A)$ satisfying (3.11), there exists a curve $(J(t), g(t)) \in \mathcal{AH}(M)$ through $(J, g)$ for sufficiently small $t$, where tangent vector at $t = 0$ is $(K, h)$. We here introduce several explicit examples of such curves.

(i) Blair-Ianus deformations [1]: The curve $(J(t), g(t))$ through $(J, g)$ which corresponds to $\Omega(t) = \Omega$, where $\Omega$ is the Kähler form of $(J, g)$. The curve $(J(t), g(t))$ can be regarded as a curve in $\mathcal{AH}(M)$ through $(J, g)$ with an initial condition $(h, A)$ such that $A = 0$ and $J$-skew invariant $h$.

(ii) The curve $(J(t), g(t))$ through $(J, g)$ with initial condition $(h, A)$ given by $h_{ij} = \frac{1}{2}(J_i{}^aA_{aj} + J_j{}^aA_{ai})$ for any 2-form $A = (A_{ij})$ on $M$.

From (3.3) and (3.4), taking account of (3.10), (3.11), (3.12), we have further

$$\left.\frac{d}{dt}\right|_{t=0} J(t)^{ij} = -\, h^{ia}J_a{}^j + g^{ia}K_a{}^j = A_{ab}J^{ia}J^{jb}, \tag{3.13}$$

(3.14)
$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \rho^*(t)_{ij} = \rho^*_{ia}h_j{}^a - \frac{1}{2}R_{iua}{}^bJ_j{}^uJ^{ac}h_{bc} - \frac{1}{2}J^{ab}J_j{}^c\nabla_i\nabla_ah_{bc}\\ + \frac{1}{2}J^{ab}J_j{}^c\nabla_c\nabla_ah_{bi} + \frac{1}{2}(2J_j{}^q\rho^*{}_i{}^p - J_j{}^uJ^{pa}J^{qb}R_{iuab})A_{pq},\end{aligned}$$

$$\left.\frac{d}{dt}\right|_{t=0} \tau^*(t) = \rho^*_{ab}h^{ab} - J^{ia}J^{jb}\nabla_a\nabla_bh_{ij} - 2J^{ip}\rho^*_{iq}A_p{}^q. \tag{3.15}$$

Now, we are ready to compute the first variation of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$. We shall adapt the notational convention (2.3) with respect to a local orthonormal frame field $\{e_i\}_{i=1,\ldots,2n}$. By (3.4), (3.7) and (3.15), we have

(3.16)
$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_{\lambda,\mu}(J(t), g(t)) &= \left.\frac{d}{dt}\right|_{t=0} \int_M (\lambda\tau(t) + \mu\tau^*(t))dv_{g(t)}\\ &= \int_M \sum(-\lambda\rho_{ij} + \mu\rho^*_{ij} + \frac{1}{2}(\lambda\tau + \mu\tau^*)g_{ij})h_{ij}dv_g\\ &\quad - \mu\int_M \sum J_{ia}J_{jb}\nabla_a\nabla_bh_{ij}dv_g + 2\mu\int_M \sum \rho^*_{ij}A_{ij}dv_g\\ &= \int_M \sum_{i,j} \{\big( - \lambda\rho_{ij} + \mu\rho^*_{ij} + \frac{1}{2}(\lambda\tau + \mu\tau^*)g_{ij}\end{aligned}$$

$$- \mu \sum_{a,b} \nabla_b \nabla_a (J_{ia} J_{jb}) \big) h_{ij} + 2\mu \rho^*_{\bar{i}j} A_{ij} \} dv_g.$$

Here, we get
(3.17)

$$\begin{aligned}
&\sum_{a,b} \nabla_b \nabla_a (J_{ia} J_{jb}) = \sum_{a,b} \nabla_b ((\nabla_a J_{ia}) J_{jb} + J_{ia} \nabla_a J_{jb}) \\
=& \sum_{a,b} (\nabla_b \nabla_a J_{ia}) J_{jb} + \sum_{a,b} (\nabla_a J_{ia}) \nabla_b J_{jb} + \sum_{a,b} (\nabla_b J_{ia}) \nabla_a J_{jb} + \sum_{a,b} J_{ia} \nabla_b \nabla_a J_{jb} \\
=& \sum_{a,b} (\nabla_b \nabla_a J_{ia}) J_{jb} + \sum_{a,b} (\nabla_a J_{ia}) \nabla_b J_{jb} + \sum_{a,b} (\nabla_b J_{ia}) \nabla_a J_{jb} \\
&+ \sum_{a,b} J_{ia} \nabla_a \nabla_b J_{jb} - \sum_{a,b,c} J_{ia} R_{bajc} J_{cb} - \sum_{a,b,c} J_{ia} R_{babc} J_{jc} \\
=& \sum_{a,b} (\nabla_b \nabla_a J_{ia}) J_{jb} + \sum_{a,b} (\nabla_a J_{ia}) \nabla_b J_{jb} + \sum_{a,b} (\nabla_b J_{ia}) \nabla_a J_{jb} \\
&+ \sum_{a,b} (\nabla_a \nabla_b J_{jb}) J_{ia} - \frac{1}{2} \sum_{a,b,c} J_{ia} (R_{abcj} - R_{acbj}) J_{cb} + \sum J_{ia} J_{jc} \rho_{ac} \\
=& \sum_{a,b} J_{jb} \nabla_b \nabla_a J_{ia} + \sum_{a,b} J_{ia} \nabla_a \nabla_b J_{jb} + \sum_{a,b} (\nabla_b J_{ia}) \nabla_a J_{jb} \\
&+ \sum_{a,b} (\nabla_a J_{ia}) \nabla_b J_{jb} - \rho^*_{ji} + \rho_{\bar{i}\bar{j}}.
\end{aligned}$$

We denote by $T = (T_{ij})$ the symmetric $(0,2)$-tensor defined by
(3.18)

$$\begin{aligned}
T_{ij} =& - \lambda \rho_{ij} - \mu \rho_{\bar{i}\bar{j}} + \mu (\rho^*_{ij} + \rho^*_{ji}) + \frac{1}{2} (\lambda \tau + \mu \tau^*) g_{ij} \\
&- \mu \sum_{a,b} (J_{ia} \nabla_a \nabla_b J_{jb} + J_{ja} \nabla_a \nabla_b J_{ib} + (\nabla_a J_{ia}) \nabla_b J_{jb} + (\nabla_a J_{jb}) \nabla_b J_{ia}).
\end{aligned}$$

Thus, from (3.16), (3.17), (3.18), we have the following.

**Lemma 1.** *$(J, g)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ if and only if $(J, g)$ satisfies*

$$\int_M \sum_{i,j} (T_{ij} h_{ij} + 2\mu \rho^*_{\bar{i}j} A_{ij}) dv_g = 0 \tag{3.19}$$

*for any pair $(h, A)$ of a symmetric $(0,2)$-tensor $h = (h_{ij})$ and a 2-form $A = (A_{ij})$ satisfying* (3.11)*, where $T = (T_{ij})$ is the symmetric $(0,2)$-tensor defined by* (3.18)*.*

Now, we recall the following fact due to Blair-Ianus [1]:

**Lemma 2.** *Let $B = (B_{ij})$ be a symmetric $(0,2)$-tensor on $M$. Then*

$$\int_M \sum_{i,j} B_{ij} D_{ij} dv_g = 0$$

*for all symmetric $(0,2)$-tensor $D$ satisfying $D_{ij} + D_{\bar{i}\bar{j}} = 0$ if and only if $B$ is $J$-invariant.*

Now, let $(J, g)$ be a critical point of $\mathcal{F}_{\lambda,\mu}$ and consider the Blair-Ianus deformation $(J(t), g(t))$ of $(J, g)$. Then, by Lemma 2, we see that the tensor $T$ is $J$-invariant. Next, we consider the deformation $(J(t), g(t))$ of $(J, g)$ of type (ii). Then, we have

$$\begin{aligned}
&\int_M \sum_{i,j} (\frac{1}{2} T_{ij} A_{\bar{i}j} + \frac{1}{2} T_{ij} A_{\bar{j}i} + 2\mu\rho^*_{ij} A_{ij}) dv_g \\
&= \int_M \sum_{i,j} (-\frac{1}{2} T_{\bar{i}j} + \frac{1}{2} T_{i\bar{j}} + 2\mu\rho^*_{ij}) A_{ij} dv_g \\
&= \int_M \sum_{i,j} (T_{i\bar{j}} + 2\mu\rho^*_{ij}) A_{ij} dv_g = 0
\end{aligned} \tag{3.20}$$

for all 2-forms $A = (A_{ij})$ on $M$. Thus, from (3.20), we have

$$T_{i\bar{j}} + 2\mu\rho^*_{ij} = 0. \tag{3.21}$$

Thus, from (3.21), we see in particular that $\rho^*$ is symmetric if $\mu \neq 0$.

Conversely, we assume that $T_{ij} - 2\mu\rho^*_{i\bar{j}} = 0$ and $T = (T_{ij})$ is $J$-invariant. Then, for any $(h, A)$ satisfying (3.11), we have

$$\begin{aligned}
&\int_M \sum_{i,j} (T_{ij} h_{ij} + 2\mu\rho^*_{ij} A_{ij}) dv_g \\
&= 2\mu \int_M \sum_{i,j} \rho^*_{ij} (h_{ij} - A_{\bar{i}j}) dv_g \\
&= \mu \int_M \sum_{i,j} \rho^*_{ij} (h_{ij} + h_{\bar{i}\bar{j}} - A_{\bar{i}j} - A_{\bar{j}i}) dv_g \\
&= 0.
\end{aligned} \tag{3.22}$$

By virtue of (3.11), and hence, from Lemma 1, we see that $(J, g)$ is a critical point of $\mathcal{F}_{\lambda,\mu}$. Thus, summing up the above arguments, we have finally the following theorem.

**Theorem 3.** *$(J, g)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$ if and only if $(J, g)$ satisfies $T_{ij} = T_{\bar{i}\bar{j}}$ and $T_{i\bar{j}} + 2\mu\rho^*_{ij} = 0$ (and hence, in particular $\rho^*$ is symmetric for a critical point $(J, g)$ of $\mathcal{F}_{\lambda,\mu} (\mu \neq 0)$).*

*Remark* 1. From (3.18), taking account of the equality $T_{ij} = T_{\bar{i}\bar{j}}$, we may easily check that the equality $T_{i\bar{j}} + 2\mu\rho^*_{\bar{i}j} = 0$ in Theorem 3 can be rewritten as

(3.23)

$$\lambda\rho_{ij} + \mu\rho_{\bar{i}\bar{j}} - \frac{1}{2}(\lambda\tau + \mu\tau^*)g_{ij} + \mu\sum_{a,b}(J_{ia}\nabla_a\nabla_b J_{jb} + J_{ja}\nabla_a\nabla_b J_{ib} + (\nabla_a J_{ia})\nabla_b J_{jb} + (\nabla_a J_{jb})\nabla_b J_{ia}) = 0.$$

Further, by (3.18), we see that the equality $T_{ij} = T_{\bar{i}\bar{j}}$ is equivalent to the following equality.

(3.24)

$$(\lambda-\mu)(\rho_{ij} - \rho_{\bar{i}\bar{j}}) + \mu\Big(\sum_b \nabla_i\nabla_b J_{\bar{j}b} + \sum_b \nabla_j\nabla_b J_{\bar{i}b} + \sum_b \nabla_{\bar{i}}\nabla_b J_{jb} + \sum_b \nabla_{\bar{j}}\nabla_b J_{ib} - \sum_{a,b}(\nabla_a J_{\bar{i}a})\nabla_b J_{\bar{j}b} - \sum_{a,b}(\nabla_a J_{\bar{j}b})\nabla_b J_{\bar{i}a} + \sum_{a,b}(\nabla_a J_{ia})\nabla_b J_{jb} + \sum_{a,b}(\nabla_a J_{jb})\nabla_b J_{ia}\Big) = 0.$$

Let $(J, g)$ be a critical point of $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$. Thus, from (3.23), we have further

(3.25)

$$(\lambda+\mu)\tau - n(\lambda\tau + \mu\tau^*) + \mu\sum\Big(2J_{ia}\nabla_a\nabla_b J_{ib} + (\nabla_a J_{ia})\nabla_b J_{ib} + (\nabla_a J_{ib})\nabla_b J_{ia}\Big) = 0.$$

Here, we have

$$\begin{aligned}
&\sum J_{ia}\nabla_a\nabla_b J_{ib} \\
&= \sum \nabla_a(J_{ia}\nabla_b J_{ib}) - \sum(\nabla_a J_{ia})\nabla_b J_{ib} \\
&= -\sum \nabla_a(J_{ib}\nabla_b J_{ia}) - \sum(\nabla_a J_{ia})\nabla_b J_{ib} \\
&= -\sum(\nabla_a J_{ib})\nabla_b J_{ia} - \sum J_{ib}\nabla_a\nabla_b J_{ia} - \sum(\nabla_a J_{ia})\nabla_b J_{ib} \\
&= -\sum J_{ib}\nabla_b\nabla_a J_{ia} + \sum J_{ib}(R_{abij}J_{ja} + R_{abaj}J_{ij}) \\
&\quad - \sum(\nabla_a J_{ia})\nabla_b J_{ib} - \sum(\nabla_a J_{ib})\nabla_b J_{ia} \\
&= -\sum J_{ia}\nabla_a\nabla_b J_{ib} + \tau^* - \tau \\
&\quad - \sum(\nabla_a J_{ia})\nabla_b J_{ib} - \sum(\nabla_a J_{ib})\nabla_b J_{ia},
\end{aligned} \tag{3.26}$$

and hence,

$$2\sum J_{ia}\nabla_a\nabla_b J_{ib} = \tau^* - \tau - \sum(\nabla_a J_{ia})\nabla_b J_{ib} - \sum(\nabla_a J_{ib})\nabla_b J_{ia}. \tag{3.27}$$

Thus, from Theorem 3, (3.26) and (3.28), we have the following:

**Theorem 4.** *Let $(J, g)$ be a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$. Then, we have*

$$\lambda\tau + \mu\tau^* = 0.$$

From Theorems 3 and 4, taking account of (2.8) and (2.12), we have immediately the following Corollaries 5 and 6.

**Corollary 5.** *$(J, g) \in \mathcal{AK}(M)$ is a critical point of functional $\mathcal{F}_{-1,1}$ on $\mathcal{AH}(M)$ if and only if $(J, g)$ is a Kähler structure on $M$.*

**Corollary 6.** *$(J, g) \in \mathcal{NK}(M)$ is a critical point of functional $\mathcal{F}_{-1,1}$ on $\mathcal{AH}(M)$ if and only if $(J, g)$ is a Kähler structure on $M$.*

*Remark* 2. The result of Corollary 6 itself is weaker than the one in ([8], Corollary 4).

*Remark* 3. Let $M = (M, \Omega)$ be a compact symplectic manifold. We denote by $\mathcal{AK}(M, [\Omega])$ the set of all almost Kähler structures on $M$ with the same Kähler class $[\Omega]$ in the de Rham cohomology group of degree 2. Then in [9], Oguro, Sekigawa, and Yamada showed that $\mathcal{F}_{\frac{1}{2},\frac{1}{2}} = \frac{4\pi}{(n-1)!}(c_1 \cdot [\Omega]^{n-1})[M]$ for any $(J, g) \in \mathcal{AK}(M, [\Omega])$, where $c_1$ is the first Chern class of $(M, J, g)$. Thus, if $c_1 \cdot [\Omega]^{n-1} \neq 0$, then there does not exist a critical point in $\mathcal{AK}(M, [\Omega])$ of the functional $\mathcal{F}_{\frac{1}{2},\frac{1}{2}}$ on $\mathcal{AH}(M)$.

From Theorem 3 and (3.24), we have also the following.

**Theorem 7.** *Let $(J, g) \in \mathcal{SK}(M)$ be a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}(M)$. Then, we have*

$$(\lambda - \mu)(\rho_{ij} - \rho_{\bar{i}\bar{j}}) + \mu \sum_{a,b} \left((\nabla_a J_{ib})\nabla_b J_{ja} - (\nabla_a J_{\bar{i}b})\nabla_b J_{\bar{j}a}\right) = 0.$$

**Corollary 8.** *Let $(J, g) \in \mathcal{QK}(M)$ be a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ $(\lambda \neq \mu)$ on $\mathcal{AH}(M)$. Then, $\rho$ is $J$-invariant, and moreover, if $\mu \neq 0$, $\rho^*$ is also $J$-invariant.*

Now let $M = (M, J, g)$ be a 4-dimensional Hermitian manifold. Then by (2.4) and (VI), we have the following equalities

$$\sum_a (\nabla_a J_{ib})\nabla_b J_{ia} = 0 \quad \text{and} \quad \sum_a (\nabla_a J_{ia})\nabla_b J_{ib} = |\omega|^2. \tag{3.28}$$

Thus, from Theorem 4, (VI) in page 3, and (3.23), we easily have the following.

**Theorem 9.** *Let $M = (M, J, g)$ be a 4-dimensional Hermitian manifold. $(J, g) \in \mathcal{H}(M)$ is a critical point of $\mathcal{F}_{1,-1}$ on $\mathcal{AH}(M)$ if and only if $(J, g)$ is a Kähler structure on $M$.*

## 4. Critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$

In this section, we shall give a condition for an almost Hermitian structure to be a critical points of the functional $\mathcal{F}_{\lambda,\mu}$ on the space $\mathcal{AH}_c(M)$ for certain positive constant $c$. Let $(J,g)$ be a critical point of $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$. First, consider a deformation $(J(t),g(t))$ of $(J,g)$ of type (i) in Section 3, namely a Blair-Ianus deformation. Then, since $\Omega(t)=\Omega$, we see that $(J(t),g(t))\in\mathcal{AH}_c(M)$ for sufficiently small $|t|$. Thus, applying the similar argument in Section 3 to the present case, we see that the tensor field $T$ defined by (3.18) is $J$-invariant. Next, we consider the deformation $(J(t),g(t))$ in $\mathcal{AH}_c(M)$ of $(J,g)$ of type (ii). Since $(J(t),g(t))\in\mathcal{AH}_c(M)$, the 2-form $A=(A_{ij})$ have to satisfy the following equality

$$\int_M \sum_{ij} J_{ij}A_{ij}\,dv_g = 0. \tag{4.1}$$

Further, by taking account of the arguments in Section 3, we see that the 2-form $A$ satisfies the equality (3.20).

Therefore, from (4.1), (3.20) and Lemma 2, by applying the Lagrange's multiplier method, we have finally the following:

**Theorem 10.** *$(J,g)$ is a critical point of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$ if and only if $\lambda\tau+\mu\tau^*$ is constant, and $T_{ij}=T_{\bar{i}\bar{j}}$ , $T_{ij}-2\mu\rho^*_{ij}=Cg_{ij}$ hold on $M$ with respect to $(J,g)$, where $C=\frac{n-1}{2n}(\lambda\tau+\mu\tau^*)$.*

From Theorem 10, we can easily deduce the following.

**Corollary 11.** *$(J,g)$ is a critical point of the functional $\mathcal{F}_{1,0}$ on $\mathcal{AH}_c(M)$ if and only if $(M,J,g)$ is Einstein.*

Corollary 11 is the almost Hermitian analogy of the result by Hilbert [5]. From Theorem 10, taking account of (2.10), (2.11), (2.12) and (3.11) in [2], for the critical points $(J,g)\in\mathcal{NK}(M)$ of the functional $\mathcal{F}_{\lambda,\mu}$ on $\mathcal{AH}_c(M)$, we easily have the following.

**Corollary 12.** *Let $(J,g)\in\mathcal{NK}(M)$ be a critical point of the functional $\mathcal{F}_{\lambda,\mu}(\lambda+\mu=0)$ on $\mathcal{AH}_c(M)$ for some positive constant $c$ if and only if*

*(1) $(J,g)$ is a Kähler structure on $M$, or*

*(2) $(J,g)$ is an Einstein and $*$-Einstein non-Kähler, nearly Kähler structure on $M$ with $\tau=5\tau^*$.*

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