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A NOTE ON FUNCTIONS OF MEAN BLOCH TYPES

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Abstract. A characterization of the holomorphic function spaces of mean Bloch type on the unit disc is deduced in terms of the induced distance.

1. Introduction

We introduce basic definitions, previous results, and the goal of this paper that we will involve.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane $\mathbb{C}$ and let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the boundary of D.

For $\alpha > 0$, let $\mathcal{B}_\alpha(D)$ be the space of holomorphic functions on D satisfying

$$\sup_{z \in D} \left(1 - |z|^2\right)^\alpha |f'(z)| < \infty.$$

For $\alpha > 0$ and $1 \le p < \infty$, let $\mathcal{B}^p_\alpha(D)$ be the space holomorphic functions satisfying

$$\sup_{0<r<1} (1 - r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} < \infty.$$

The spaces $\mathcal{B}_\alpha(D)$ and $\mathcal{B}^p_\alpha(D)$ occurred in the literature in connection with the Lipschitz space $Lip_\alpha(D)$ and the mean Lipschitz space $Lip^{\ p}_\alpha(D)$ which, for $0 < \alpha < 1$ and $1 \le p < \infty$, are defined to consist of f holomorphic in D such that

$$|f(z) - f(w)| \le C|z - w|^\alpha, \quad z, w \in D,$$

and of $f \in H^p(D)$ such that

$$\left(\int_{\mathrm{T}} |f(\zeta) - f(\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \le C|1 - \eta|^\alpha, \ \eta \in \mathrm{T}$$

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respectively. Here $H^p(D)$ denotes the classical Hardy space on D. A famous theorem of Hardy and Littlewood verified the connection

$$f \in Lip_\alpha^p(D) \Longleftrightarrow \sup_{0<p<1} (1-r^2)^{1-\alpha} \left(\int_{\mathrm{T}} |f'(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} < \infty.$$

See [1].

We, in this note, are interested in the induced distances on the spaces. K. Zhu defined a distance on D and characterized the space $\mathcal{B}_\alpha(D)$ in terms of the distance as follows.

Theorem A ([2, Proposition 16 and Theorem 17]). *For $\alpha > 0$, and $z, w \in D$, let*

$$d_\alpha(z,w) = \sup\{|f(z) - f(w)| : f \in \mathcal{B}_\alpha(D),\ \sup_{u \in D} \left(1-|u|^2\right)^\alpha |f'(u)| \le 1\}.$$

Then d_α is a distance on D and

$$\lim_{w \to z} \frac{d_\alpha(z,w)}{|z-w|} = \left(1-|z|^2\right)^{-\alpha}.$$

Theorem B ([2, Theorem 18]). *Suppose $\alpha > 0$ and f is holomorphic on D. Then*

$$f \in \mathcal{B}_\alpha(D) \Longleftrightarrow |f(z) - f(w)| \le C\ d_\alpha(z,w),\ z, w \in D$$

for some positive constant C. Moreover, for all $f \in \mathcal{B}_\alpha(D)$,

$$\sup_{z \in D} \left(1-|z|^2\right)^\alpha |f'(z)| = \sup_{\substack{z,w \in D \\ z \ne w}} \frac{|f(z) - f(w)|}{d_\alpha(z,w)}.$$

See [3] for the case $\alpha = 1$.

The goal of this note is to find a variant of Theorem A and Theorem B under the settings of $\mathcal{B}_\alpha^p(D)$. See Section 2 for our results of this paper.

We note that, when $0 < \alpha < \infty$ and $1 \le p < \infty$, $\mathcal{B}_\alpha^p(D)$ is a Banach space equipped with the norm

$$\|f\|_{\mathcal{B}_\alpha^p(D)} := |f(0)| + \sup_{0 \le r < 1} (1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}.$$

2. Results

For simplicity, the class of holomorphic functions on D will be denoted by $H(D)$ and we will make use of the customary notation:

$$M_p(r,f) := \left(\int_T |f(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

We define a distance and obtain Theorem 2.3 and Theorem 2.4 which correspond to Theorem A and Theorem B.

Definition 2.1. For $\alpha > 0,\ 1 \le p < \infty$, $0 < r < 1$, and $\eta \in T$, let $d_{p,\alpha,r}(1,\eta)$ be defined by

$$d_{p,\alpha,r}(1,\eta) = \sup\left\{ \left(\int_T |f(r\zeta) - f(r\bar{\eta}\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} : f \in H(D), (1-r^2)^\alpha M_p(r,f') \le 1 \right\}.$$

Theorem 2.2. *If we extensively define*

$$d_{p,\alpha,r}(\zeta,\eta) = d_{p,\alpha,r}(1,\bar{\zeta}\eta), \quad \zeta,\eta \in T,$$

then $d_{p,\alpha,r}$ is a metric on T.

Theorem 2.3. *For $\alpha > 0,\ 1 \le p < \infty$, and $0 < r < 1$,*

$$\lim_{\eta \to \zeta} \frac{d_{p,\alpha,r}(\zeta,\eta)}{r|\zeta - \eta|} \ = \ \left(1 - r^2\right)^{-\alpha}.$$

Theorem 2.4. *Suppose $\alpha > 0,\ 1 \le p < \infty$, and $f \in H(D)$. Then*

$$f \in \mathcal{B}^p_\alpha(D) \iff \left(\int_T |f(r\zeta) - f(r\bar{\eta}\zeta)|^p \ d\sigma(\zeta) \right)^{\frac{1}{p}} \le C \ d_{p,\alpha,r}(1,\eta),\ \eta \in T, 0 < r < 1,$$

for some positive constant C. Moreover, for $0 < r < 1$,

$$(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} = \sup_{\substack{\eta \in T \\ \eta \ne 1}} \frac{1}{d_{p,\alpha,r}(1,\eta)} \left(\int_T |f(r\zeta) - f(r\bar{\eta}\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}.$$

Corollary 2.5. *Suppose $\alpha > 0,\ 1 \le p < \infty$, and $f \in H(D)$. Then*

$$\|f\|_{\mathcal{B}^p_\alpha(D)} \ = \ |f(0)| + \sup_{0<r<1} \sup_{\substack{\eta \in T \\ \eta \ne 1}} \frac{1}{d_{p,\alpha,r}(1,\eta)} \left(\int_T |f(r\zeta) - f(r\bar{\eta}\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}.$$

3. A lemma

Lemma 3.1. (1) *Let $1 < p < \infty$ and $\alpha = \frac{1}{p}$. Then there is a function $f \in H^p(D)$ for which $\|f\|_{\mathcal{B}^p_\alpha(D)} = 1$. Moreover, we may take f such that*

$$\left(\int_T |f'(\rho\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} \ = \ \frac{1}{(1-\rho^2)^\alpha} \tag{3.1}$$

for every $\rho : 0 < \rho < 1$.

(2) *Let $0 < p < \infty$ and $\alpha = \frac{1}{p}$. Then there is a function $f \in H(D)$ satisfying* (3.1) *for every $\rho : 0 < \rho < 1$.*

(3) *Let $0 < p < \infty$ and $0 < \alpha < 1$. For a fixed $r : 0 < r < 1$, there is $f \in H(D)$ for which*

$$\left(\int_T |f'(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} \ = \ \frac{1}{(1-r^2)^\alpha}.$$

Proof. (3) is obvious. For example, take $f(z) = (1-r^2)^{-\alpha}z$. We prove (2) and (3). Take

$$f(z) = \begin{cases} (1-\frac{2}{p})^{-1}\left[(1-z)^{1-\frac{2}{p}} - 1\right], & \text{if } p \neq 2 \\ \log(1-z), & \text{if } p = 2. \end{cases}$$

Then f is holomorphic on D with $f(0) = 0$, and

$$\begin{aligned}\int_T |f'(\rho\zeta)|^p d\sigma(\zeta) &= \int_T \left| -\frac{(1-\frac{2}{p})(1-\rho\zeta)^{-\frac{2}{p}}}{|1-\frac{2}{p}|} \right|^p d\sigma(\zeta) \\ &= \int_T |1-\rho\zeta|^{-2} d\sigma(\zeta) = \frac{1}{1-\rho^2},\end{aligned}$$

so that

$$\left(\int_T |f'(\rho\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} = \frac{1}{(1-\rho^2)^{\frac{1}{p}}}.$$

This verifies (2). The same function satisfies $\|f\|_{\mathcal{B}^p_\alpha(D)} = 1$. Since

$$\sup_r \int_T |1-r\zeta|^{p-2} d\sigma(\zeta) < \infty \quad \text{if } 1 < p < \infty$$

and

$$\sup_r \int_T |\log(1-r\zeta)|^2 \, d\sigma(\zeta) < \infty,$$

we have $f \in H^p(D)$. This verifies (1). □

4. Proofs of the results

Proof of Theorem 2.2. If $d_{p,\alpha,r}(\zeta,\eta) = 0$, then $\zeta = \eta$ obviously. Triangular inequality follows from Minkowski's inequality. □

Proof of Theorem 2.3. Note first that we may assume $\zeta = 1$. Fix r and take $f \in H(D)$ with $(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}} = 1$. Then by the definition of $d_{p,\alpha,r}$ and Fatou's Lemma, it follows that

$$\begin{aligned}\liminf_{\eta\to 1} \frac{d_{p,\alpha,r}(1,\eta)}{r|1-\eta|} &\geq \liminf_{\eta\to 1} \frac{1}{|r-r\eta|}\left(\int_T |f(r\zeta) - f(r\bar{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \\ &= \left(\liminf_{\eta\to 1} \int_T \left|\frac{f(r\zeta)-f(r\bar{\eta}\zeta)}{r\zeta - r\bar{\eta}\zeta}\right|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \\ &\geq \left(\int_T \liminf_{\eta\to 1} \left|\frac{f(r\zeta)-f(r\bar{\eta}\zeta)}{r\zeta - r\bar{\eta}\zeta}\right|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \\ &= \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \\ &= \left(1-r^2\right)^{-\alpha}.\end{aligned}$$

Hence, for the conclusion of Theorem 2.3, we are sufficient to show

$$\limsup_{\eta\to 1} \frac{d_{p,\alpha,r}(1,\eta)}{r|1-\eta|} \le (1-r^2)^{-\alpha}.$$

Let $f \in H(D)$ with $(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}} \le 1$. It simply follows that

$$|f(r\zeta) - f(re^{ih}\zeta)| = \left|\int_0^h \frac{d}{dt}\left[f(re^{it}\zeta)\right] dt\right| \le r\int_0^h |f'(re^{it}\zeta)| dt,$$

so that by Minkowski's inequality

$$\left(\int_T |f(r\zeta) - f(re^{ih}\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}} \le r|h| \left(\int_T |f'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}}.$$

Hence, for $1 \ne \eta \in T$,

$$\left(\int_T |f(r\zeta) - f(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \le r|\text{Arg } \eta| \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}.$$

Taking the supremum over all $f \in H(D)$ with $(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \le 1$, we obtain

$$d_{p,\alpha,r}(1,\eta) \le (1-r^2)^{-\alpha} \, r|\text{Arg } \eta|.$$

Thus,

$$\frac{d_{p,\alpha,r}(1,\eta)}{|r-r\eta|} \le (1-r^2)^{-\alpha} \frac{r|\text{Arg } \eta|}{|r-r\eta|}.$$

Since

$$\lim_{h\to 0} \frac{|h|}{|1-e^{ih}|} = 1,$$

we finally obtain

$$\limsup_{\eta\to 1} \frac{d_{p,\alpha,r}(1,\eta)}{|r-r\eta|} \le (1-r^2)^{-\alpha}.$$

The proof is complete. □

Proof of Theorem 2.4. Fix r. If $\eta \ne 1$, then by the definition of $d_{p,\alpha,r}$,

$$\frac{1}{d_{p,\alpha,r}(1,\eta)} \left(\int_T |g(r\zeta) - g(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \le 1$$

if $g \in H(D)$ with

$$(1-r^2)^\alpha \left(\int_T |g'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}} \le 1.$$

By considering

$$g = \frac{f}{(1-r^2)^\alpha (\int_T |f'(r\zeta)|^p d\sigma(\zeta))^{\frac{1}{p}}}$$

for a nonconstant holomorphic f, it follows that

$$\begin{aligned}&\frac{1}{d_{p,\alpha,r}(1,\eta)}\left(\int_T |f(r\zeta)-f(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &\leq (1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{\substack{\eta\in T\\ \eta\neq 1}} \frac{1}{d_{p,\alpha,r}(1,\eta)}\left(\int_T |f(r\zeta)-f(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &\leq (1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p \, d\sigma(\zeta)\right)^{\frac{1}{p}}.\end{aligned} \tag{4.1}$$

Conversely, by Fatou's Lemma and Theorem 2.3,

$$\begin{aligned}&\sup_{\substack{\eta\in T\\ \eta\neq 1}} \frac{1}{d_{p,\alpha,r}(1,\eta)}\left(\int_T |f(r\zeta)-f(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &\geq \liminf_{\eta\to 1} \frac{1}{d_{p,\alpha,r}(1,\eta)}\left(\int_T |f(r\zeta)-f(r\overline{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &= \liminf_{\eta\to 1}\left[\left(\int_T \left|\frac{f(r\zeta)-f(r\overline{\eta}\zeta)}{r-r\eta}\right|^p d\sigma(\zeta)\right)^{\frac{1}{p}} \left|\frac{r-r\eta}{d_{p,\alpha,r}(1,\eta)}\right|\right]\\ &\geq (1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}.\end{aligned} \tag{4.2}$$

By (4.1) and (4.2) we have

$$\begin{aligned}&(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &= \sup_{\substack{\eta\in T\\ \eta\neq 1}} \frac{1}{d_\alpha(r,r\eta)}\left(\int_T |f(r\zeta)-f(r\bar{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}.\end{aligned} \tag{4.3}$$

□

Proof of Corollary 2.5. Taking $\sup_{0<r<1}$ on both sides of (4.3), we obtain

$$\begin{aligned}&\sup_{0<r<1}(1-r^2)^\alpha \left(\int_T |f'(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}\\ &= \sup_{0<r<1}\sup_{\substack{\eta\in T\\ \eta\neq 1}} \frac{1}{d_\alpha(r,r\eta)}\left(\int_T |f(r\zeta)-f(r\bar{\eta}\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}}.\end{aligned}$$

Hence follows the conclusion. □

We remark that our distance is actually restricted on T. We do not know whether we can extend the distance to D (for example, by using more powerful version of Lemma 3.1).

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