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LEFT JORDAN DERIVATIONS ON BANACH ALGEBRAS AND RELATED MAPPINGS

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Abstract. In this note, we obtain range inclusion results for left Jordan derivations on Banach algebras: (i) Let δ be a spectrally bounded left Jordan derivation on a Banach algebra A. Then δ maps A into its Jacobson radical. (ii) Let δ be a left Jordan derivation on a unital Banach algebra A with the condition $\sup\{r(c^{-1}\delta(c)) : c \in A \text{ invertible}\} < \infty$. Then δ maps A into its Jacobson radical.

Moreover, we give an exact answer to the conjecture raised by Ashraf and Ali in [2, p. 260]: every generalized left Jordan derivation on 2-torsion free semiprime rings is a generalized left derivation.

1. Introduction

Throughout this note, R will represent an associative ring with center $Z(R)$ and we will write $[a, b]$ for the commutator $ab - ba$. The Jacobson radical of R which is the intersection of all primitive ideals of R will be denoted by $rad(R)$.

Recall that R is *semiprime* (resp. *prime*) if $aRa = 0$ implies $a = 0$ (resp. $aRb = 0$ implies $a = 0$ or $b = 0$) and that R is semisimple if $rad(R) = \{0\}$. An additive mapping $\delta : R \to R$ is called a *derivation* (resp. *Jordan derivation*) if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in R$ (resp. $\delta(a^2) = a\delta(a) + \delta(a)a$ for all $a \in A$). Obviously, every derivation is a Jordan derivation. The converse, in general, is not true. Brešar [4] proved that every Jordan derivation on a 2-torsion free semiprime ring is a derivation.

An additive mapping $d : R \to R$ is said to be a *left Jordan derivation* or *Jordan left derivation* (resp. *left derivation*) if $d(a^2) = 2ad(a)$ for all $a \in R$ (resp. $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$). Brešar, Vukman ([7], [17]), Deng [8] and Ashraf *et al.* [1] studied left Jordan derivations and left derivations on prime rings and semiprime rings, which are in a close connection with so-called commuting mappings.

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Now let us introduce some principal results concerning derivations and related mappings in Banach algebra theory. The non-commutative Singer-Wermer conjecture states that if δ is a linear derivation on a Banach algebra A such that $[\delta(a), a] \in rad(A)$ for all $a \in A$, then $\delta(A) \subseteq rad(A)$. This is equivalent to the fact that all primitive ideals of A are invariant under δ [9]. There is some evidence for the validity of the conjecture [16]. It is known to be true if δ is continuous ([7], [10]) or if A is commutative [15], while the classical Singer-Wermer theorem [14] is affirmative if both hypotheses are satisfied. Also, as one of non-commutative versions of the Singer-Wermer theorem (for example, [9]), Brešar and Vukman [7] proved that every continuous linear left derivation on a Banach algebra A maps A into $rad(A)$. And they raised the problem whether the above conclusion holds for any continuous linear left Jordan derivation [7]. In case A is commutative, the problem is equivalent to the classical Singer-Wermer theorem.

In Section 2, we improve the results in [13] as non-commutative versions of the Singer-Wermer theorem. Moreover, in Section 3, we give an exact answer to a conjecture raised by Ashraf and Ali [2] and investigate the spectral boundedness of generalized left derivations.

2. Range inclusion results for left Jordan derivations

Definition 2.1. Let A and B be Banach algebras. A linear mapping $T : A \to B$ is called *spectrally bounded* if there is $M \geq 0$ such that $r(T(a)) \leq Mr(a)$ for all $a \in A$. If $r(T(a)) = r(a)$ for all $a \in A$, we say that T is a *spectral isometry*. If $r(a) = 0$, then a is called *quasinilpotent*. (Herein, $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ denotes the *spectral radius* of the element a).

Remark 2.2. Brešar and Mathieu [6] showed that if δ is a linear derivation on a unital Banach algebra A, then the three conditions 'δ is spectrally bounded', '$\sup\{r(c^{-1}\delta(c)) | \, c \in A$ invertible$\} < \infty$' and '$\delta(A) \subseteq rad(A)$' are equivalent each other.

We proved the following results concerning left Jordan derivations in [13] motivated by the Brešar and Mathieu's results in Remark 2.2:

(i) Every spectrally bounded left Jordan derivation δ on a Banach algebra A such that $[\delta(a), a] \in rad(A)$ for all $a \in A$, maps A into $rad(A)$.

(ii) Every linear left Jordan derivation δ on a unital Banach algebra A with the condition $\sup\{r(c^{-1}\delta(c)) : \, c \in A$ invertible$\} < \infty$ such that $[\delta(a), a] \in rad(A)$ for all $a \in A$, maps A into $rad(A)$.

Here we obtain our main results without the condition $[\delta(a), a] \in rad(A)$ for all $a \in A$.

Theorem 2.3. *Let A be a Banach algebra. If δ is a spectrally bounded left Jordan derivation on A, then we have $\delta(A) \subseteq rad(A)$.*

Proof. Let $r(\delta(a)) \leq Mr(a)$ for some $M \geq 0$ and all $a \in A$. Since the canonical epimorphism $\pi : A \to A/rad(A)$ is a spectral isometry, it follows from [7, Proposition 1.1(1°)] that

$$\begin{aligned} r(2a\delta(b)) &= r(\delta(ab+ba) - 2b\delta(a)) \\ &= r(\pi(\delta(ab+ba) - 2b\delta(a))) \\ &= r(\pi(\delta(ab+ba) - \pi(2b\delta(a))) \\ &= r(\pi(\delta(ab+ba))) \\ &= r(\delta(ab+ba)) \\ &\leq Mr(ab+ba) = 0, \ a \in A, \ b \in rad(A). \end{aligned}$$

Therefore we obtain that $r(a\delta(b)) = 0$ for all $a \in A$ and $b \in rad(A)$ and so Proposition 1 in [3, p. 126] tells us that $\delta(rad(A)) \subseteq rad(A)$. Then we can define a linear left Jordan derivation $\bar{\delta}$ on the semisimple Banach algebra $A/rad(A)$ by $\bar{\delta}(a + rad(A)) = \delta(a) + rad(A)$ for all $a \in A$. Hence, from [19, Theorem 4], we conclude that $\bar{\delta} = 0$, i.e., $\delta(A) \subseteq rad(A)$. This completes the proof of the theorem. □

Theorem 2.4. *Let A be a unital Banach algebra. If δ is a linear left Jordan derivation on A with the condition* $\sup\{r(c^{-1}\delta(c)) : c \in A \text{ invertible}\} < \infty$, *then we have $\delta(A) \subseteq rad(A)$.*

Proof. Let $\pi : A \to A/rad(A)$ be the canonical epimorphism. Let

$$s = \sup\{r(c^{-1}d(c)) : c \in A \text{ invertible}\} < \infty.$$

We claim that $\delta(rad(A)) \subseteq rad(A)$. Given $c \in rad(A)$, we have $(1+c)^{-1} = 1 - c(1+c)^{-1} \in 1 + rad(A)$ and hence

$$\begin{aligned} r((1+c)^{-1}d(1+c)) &= r((1 - c(1+c)^{-1})\delta(c)) \\ &= r(\delta(c) - c(1+c)^{-1}\delta(c)) \\ &= r(\pi(\delta(c) - c(1+c)^{-1}\delta(c))) \\ &= r(\pi(\delta(c)) - \pi(c(1+c)^{-1}\delta(c))) \\ &= r(\pi(\delta(c))) \\ &= r(\delta(c)). \end{aligned}$$

By the assumption, it follows that $r(d(c)) \leq s < \infty$ for all $c \in rad(A)$, hence $r(d(c)) = 0$ for all $c \in rad(A)$. It follows from [7, Proposition 1.1(1°)] that

$$\begin{aligned} r(2a\delta(b)) &= r(\delta(ab+ba) - 2b\delta(a)) \\ &= r(\pi(\delta(ab+ba) - 2b\delta(a))) \\ &= r(\pi(\delta(ab+ba) - \pi(2b\delta(a))) \\ &= r(\pi(\delta(ab+ba))) \\ &= r(\delta(ab+ba)) = 0, \ a \in A, \ b \in rad(A). \end{aligned}$$

Therefore we see that $r(a\delta(b)) = 0$ for all $a \in A$ and $b \in rad(A)$, and so $\delta(rad(A)) \subseteq rad(A)$, as claimed. The remainder follows the same fashion as in the proof of Theorem 2.3. Hence we obtain $\delta(A) \subseteq rad(A)$. This completes the proof. □

3. Spectrally boundedness of generalized left derivations and generalized left Jordan derivations on semiprime rings

An additive mapping $f : R \to R$ is called a *generalized derivation* (resp. *generalized Jordan derivation*) if there exists a derivation $\delta : R \to R$ (resp. a Jordan derivation $\delta : R \to R$) such that $f(ab) = af(b) + \delta(a)b$ holds for all $a, b \in R$ (resp. $f(a^2) = af(a) + \delta(a)a$ holds for all $a \in R$). The concept of generalized derivation has been introduced by Brešar [5]. Jing and Lu [11] proved that every generalized Jordan derivation on 2-torsion free prime ring is a generalized derivation. In case when R is semiprime, they conjectured that the result above may be still true, and Vukman [18] proved the conjecture.

An additive mapping $g : R \to R$ is called a *generalized left derivation* (resp. *generalized left Jordan derivation*) if there exists a left derivation $d : R \to R$ (resp. a left Jordan derivation $d : R \to R$) such that $g(ab) = ag(b) + bd(a)$ holds for all $a, b \in R$ (resp. $g(a^2) = ag(a) + ad(a)$ holds for all $a \in R$).

Brešar and Mathieu [6, Theorem 2.8] obtained a necessary and sufficient condition for a generalized derivation to be spectrally bounded on a unital Banach algebra:

Let $f = \tau_t + \delta$ be a generalized derivation with $t = f(1)$ associated with a derivation δ on a unital Banach algebra A, where τ_t is a left multiplication by t. The following conditions are equivalent.

(i) *f is spectrally bounded.*
(ii) *Both τ_t and δ are spectrally bounded.*

Let R be a ring. It is easy to prove that $g : R \to R$ is a generalized left derivation if and only if g is of the form $g = \lambda + d$, where $\lambda : R \to R$ is a right centralizer and $d : R \to R$ is a left derivation. If R contains a unit element, then it is easy to see that g is a of the form $g = \mu_k + d$, where μ_k is a right multiplication by $k = \lambda(1)$. We here apply the Brešar and Mathieu's result above to arbitrary spectrally bounded generalized left derivations.

Theorem 3.1. *Let $g = \mu_k + d$ be a generalized left derivation with $k = \lambda(1)$ on a unital Banach algebra A. The following conditions are equivalent.*

(i) *g is spectrally bounded.*
(ii) *Both μ_k and d are spectrally bounded.*

Proof. Let $\pi : A \to A/rad(A)$ be the canonical epimorphism. Suppose that both μ_k and d are spectrally bounded. By [12, Theorem 3.12], $d(A) \subseteq rad(A)$

whence

$$\begin{aligned} r(g(a)) &= r(\mu_k(a) + d(a)) \\ &= r(\pi(\mu_k(a) + d(a))) \\ &= r(\pi(ak + \pi(d(a))) \\ &= r(\pi(ak)) \\ &= r(ak) \leq Mr(a) \end{aligned}$$

for some $M \geq 0$ and all $a \in A$. Hence g is spectrally bounded.

Conversely, suppose that g is spectrally bounded. It suffices to show that d is spectrally bounded. For then, since we know that $d(A) \subseteq rad(A)$ by [12, Theorem 3.12] and that $r(ak) = r(g(a))$ for all $a \in A$ as the above relation, it follows that μ_k is spectrally bounded with the same constant as g.

From the relation

$$\begin{aligned} r(ad(b)) &= r(d(ab) - bd(a)) \\ &= r(g(ab) - abk - bd(a)) \\ &= r(\pi(g(ab) - abk - bd(a))) \\ &= r(\pi(g(ab)) - \pi(abk) - \pi(bd(a))) \\ &= r(\pi(g(ab))) \\ &= r(g(ab)) \leq Mr(ab) = 0 \end{aligned}$$

for some $M \geq 0$ and for all $a \in A$ and $b \in rad(A)$, we arrive at

$$r(ad(b)) = 0$$

for all $a \in A$ and $b \in rad(A)$ which implies that

$$d(rad(A)) \subseteq rad(A).$$

Since every left derivation on semisimple Banach algebras is zero by [12, Corollary 3.7], the induced derivation on the semisimple Banach algebra $A/rad(A)$ yields $d(A) \subseteq rad(A)$. Therefore, d is spectrally bounded by [12, Theorem 3.12]. □

An additive mapping $\lambda : R \to R$ is called a left (resp. right) centralizer if $\lambda(ab) = \lambda(a)b$ (resp. $\lambda(ab) = a\lambda(b)$) holds for all $a, b \in R$. An additive mapping $\lambda : R \to R$ is called left (resp. right) Jordan centralizer if $\lambda(a^2) = \lambda(a)a$ (resp. $\lambda(a^2) = a\lambda(a)$) holds for all $a \in R$.

Obviously, every left (resp. right) centralizer is a left (resp. right) Jordan centralizer. Zalar has proved the following fact.

Lemma 3.2 ([20, Proposition 1.4])**.** *Let R be a 2-torsion free semiprime ring. If $\lambda : R \to R$ is a left (resp. right) Jordan centralizer, then λ is a left (resp. right) centralizer.*

Recently, Ashraf and Ali [2] proved that every generalized left Jordan derivation on prime rings is a generalized left derivation. In [2], they also conjectured that every generalized left Jordan derivation on semiprime rings may be a generalized left derivation. Finally, we give an exact answer to this conjecture:

Theorem 3.3. *Let R be a 2-torsion free semiprime ring. If $g : R \to R$ is a generalized left Jordan derivation, then g is a generalized derivation.*

Proof. Suppose that there exists $d : R \to R$ is a left Jordan derivation such that

$$g(a^2) = ag(a) + ad(a)$$

is fulfilled for all $a \in R$. Let us denote $g - d$ by λ. Using the relation above, we get

$$\begin{aligned} \lambda(a^2) &= g(a^2) - d(a^2) \\ &= ag(a) + ad(a) - 2ad(a) \\ &= a(g(a) - d(a)) \\ &= a\lambda(a) \end{aligned}$$

for all $a \in R$. We have therefore $\lambda(a^2) = a\lambda(a)$ for all $a \in R$. In other words, λ is a right Jordan centralizer of R. Since R is a 2-torsion free semiprime ring, it follows from [19, Theorem 2] and Lemma 3.2 that $d : R \to R$ is a derivation such that $d(R) \subseteq Z(R)$ and λ is a right centralizer of R, respectively. Since we know that $g = \lambda + d$, we see that the equality $g(ab) = ag(b) + d(a)b$ holds for all $a, b \in R$. That is, we conclude that g is a generalized derivation. The proof of the theorem is complete. □

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