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INTEGRATION WITH RESPECT TO ANALOGUE OF WIENER MEASURE OVER PATHS IN ABSTRACT WIENER SPACE AND ITS APPLICATIONS

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Abstract. In 1992, the author introduced the definition and the properties of Wiener measure over paths in abstract Wiener space and this measure was investigated extensively by some mathematicians. In 2002, the author and Dr. Im presented an article for analogue of Wiener measure and its applications which is the generalized theory of Wiener measure theory. In this note, we will derive the analogue of Wiener measure over paths in abstract Wiener space and establish two integration formulae, one is similar to the Wiener integration formula and another is similar to simple formula for conditional Wiener integral. Furthermore, we will give some examples for our formulae.

1. Introduction

In 1827, Robert Brown, the British botanist, observed the motions of small particles while viewing pollen suspended in water through a microscope [18]. This motion is called the Brownian motion. He hypothesized that this movement was either due to a particular phenomenon of living matter or the power source of life. As a result, the novelty was investigated extensively by many scientists of the time, among whom were Cantoni, Oehl, Delsaux, Exner and Guoy. The results of these studies showed that Brown's original hypotheses were wrong and Brownian motion is very irregular and rapid.

In 1905, Albert Einstein suggested a probabilistic approach to Brownian motion in his paper that is associated with the special theory of relativity. By 1923, Wiener had established a theory for the reasonable probability measure m_w, the one-dimensional Wiener measure, on the space $C_0[a,b]$ of all real-valued continuous functions on a closed bounded interval $[a,b]$ that vanish at the initial point a [19]. One dimensional Wiener measure space satisfies Einstein's

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suggestion, as well as, the results of related scientists. Since then, the theory of this measure has been studied by many mathematicians and many mathematical physicists, such as Carmeron, Martin, Kac, Skoug, Johnson, Lapidus, Chang, etc, and applied to various objects.

In 1965, Gross presented the theory for the abstract Wiener measure ω on $\mathbb{B}$, the infinite dimensional real separable Banach space [4]. The concrete Wiener space $C_0[a,b]$ is an example of abstract Wiener space and the measure space $(\mathbb{B},\omega)$ is a typical measure space on infinite dimensional Banach space. In 1973, Kuelb and LePage claimed the existence of non-zero, stationary-increment Gaussian measure $m^{\mathbb{B}}$ over paths in abstract Wiener space $C_0(\mathbb{B})$, the space of all $\mathbb{B}$-valued continuous functions on $[a,b]$ that vanish at a [13]. In 1992, the author established the existence of such a measure $m^{\mathbb{B}}$, found the integration formula for it, similar to Wiener integration formula and applied to the theory of an operator-valued function space integral [13, 17].

In 2002, the author and Professor Im defined the analogue of Wiener measure ω_φ on the space $C[a,b]$, the space of all real-valued continuous functions on $[a,b]$, associated with a complex Borel measure φ on $\mathbb{R}$ [5, 14, 15, 16]. Indeed, if we take $\varphi=\delta_0$, the Dirac measure at the origin 0 in $\mathbb{R}$, ω_φ is the concrete Wiener measure m_w. This theory is the theory of many particles such that each particle moves under the Brownian process. For simplicity, we ignore the collision between particles in this article.

In this article, we introduce an analogue of Wiener measure $m_\varphi^{\mathbb{B}}$ on $C(\mathbb{B})$, the space of all $\mathbb{B}$-valued continuous functions on $[a,b]$, associated with a Borel measure φ on $\mathbb{B}$. Indeed, if φ is the Dirac measure δ_0 at the origin, $m_\varphi^{\mathbb{B}}$ is the concrete Wiener measure $m^{\mathbb{B}}$ on $C_0(\mathbb{B})$. We describe the various properties of the analogue of Wiener measure $m_\varphi^{\mathbb{B}}$ in Section 2. In Section 3, we derive the measure-valued measure V_φ^X on $C(\mathbb{B})$, associated with the random variable X on $C(\mathbb{B})$ and a Borel measure φ on $\mathbb{B}$; establish two integration formulae, one is similar to the Wiener integration formula and another is similar to the simple formula for conditional expectation. We give some examples for our formulae in Section 4.

2. Analogue of Wiener measure $m_\varphi^{\mathbb{B}}$ on the space of paths in $\mathbb{B}$

In this section, we establish the existence of analogue of Wiener measure $m_\varphi^{\mathbb{B}}$ on the space of paths in an arbitrary abstract Wiener space $\mathbb{B}$ and investigate the various properties of it.

Let $\mathbb{B}$ be an infinite dimensional real separable Banach space and let $(\mathbb{B},\mathcal{B}(\mathbb{B}),\omega)$ be an abstract Wiener measure space, associated with the measurable norm $\|\cdot\|_{\mathbb{B}}$ [7]. For positive real number λ, let ω_λ be a Borel measure on $\mathbb{B}$ given by

$$\omega_\lambda(B)=\omega(\lambda^{-1}B) \tag{1}$$

for Borel subsets B of $\mathbb{B}$. For two Borel measures μ and ν, the convolution measure $\mu * \nu$ of μ and ν is given by

$$\mu * \nu(B) = \mu \times \nu(\{(x, y) \text{ in } \mathbb{B} \times \mathbb{B} \,|\, x + y \text{ is in } B\}) \tag{2}$$

for Borel subsets B of $\mathbb{B}$. Then for two positive real numbers s and t,

$$\omega_s * \omega_t = \omega_{\sqrt{s^2+t^2}} \tag{3}$$

and $\mu * \delta_0 = \mu$ where δ_0 is the Dirac measure at the origin in $\mathbb{B}$. Let $\mathbb{B}^*$ be the dual space of $\mathbb{B}$.

Let a and b be two real numbers with $a < b$. Let $C(\mathbb{B})$ denote the space of all $\mathbb{B}$-valued continuous functions on a closed bounded interval $[a, b]$. Then $C(\mathbb{B})$ is a real separable Banach space in the norm $||y||_{C(\mathbb{B})} \equiv \sup_{a \le t \le b} ||y(t)||_{\mathbb{B}}$. Let $\overrightarrow{t} = (t_0, t_1, \ldots, t_n)$ be given with $a = t_0 < t_1 < t_2 < \cdots < t_n \le b$ and let $T_{\overrightarrow{t}} : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ be a function given by

$$T_{\overrightarrow{t}}(x_0, x_1, \ldots, x_n) = (x_0, x_0 + \sqrt{t_1 - t_0}\; x_1, \ldots, x_0 + \sum_{j=1}^{n}(\sqrt{t_j - t_{j-1}}\; x_j)). \tag{4}$$

Let φ be a non-negative finite measure on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$. We define a set function $\nu^{\varphi}_{\overrightarrow{t}}$ on $\mathcal{B}(\mathbb{B}^{n+1})$ given by

$$\nu^{\varphi}_{\overrightarrow{t}}(B) = \int_{\mathbb{B}} \Big[\int_{\mathbb{B}^n} \chi_B \circ T_{\overrightarrow{t}}(x_0, x_1, \ldots, x_n)\; d\Big(\prod_{j=1}^{n} \omega\Big)(x_1, \ldots, x_n)\Big] d\varphi(x_0), \tag{5}$$

where χ_B is a characteristic function associated with B. Then $\nu^{\varphi}_{\overrightarrow{t}}$ is a Borel measure on $(\mathbb{B}^{n+1}, \mathcal{B}(\mathbb{B}^{n+1}))$. Let $J_{\overrightarrow{t}} : C(\mathbb{B}) \to \mathbb{B}^{n+1}$ be a function with

$$J_{\overrightarrow{t}}(y) = (y(t_0), y(t_1), \ldots, y(t_n)). \tag{6}$$

For Borel subsets $B_0, B_1, \ldots, B_n$ in $\mathcal{B}(\mathbb{B})$, the subset $J^{-1}_{\overrightarrow{t}}(\prod_{j=0}^{n} B_j)$ of $C(\mathbb{B})$ is called an interval. Let $\mathcal{I}$ be the set of all such intervals. Then $\mathcal{I}$ is an semi-algebra. We define a set function M_φ on $\mathcal{I}$ by

$$M_\varphi(J^{-1}_{\overrightarrow{t}}(\prod_{j=0}^{n} B_j)) = \nu^{\varphi}_{\overrightarrow{t}}(\prod_{j=0}^{n} B_j). \tag{7}$$

Then $(x_0, x_1, \ldots, x_n)$ is in $T^{-1}_{\overrightarrow{t}}(\prod_{j=0}^{n} B_j)$ if and only if for $k = 1, 2, \ldots, n$, x_0 is in B_0 and $x_0 + \sum_{j=1}^{k}(\sqrt{t_j - t_{j-1}}\; x_j)$ is in B_k, so, we have

$$M_\varphi(J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)) \tag{8}$$
$$= \int_{\mathbb{B}} [\int_{\mathbb{B}^n} \prod_{j=1}^{n} \chi_{B_j}(x_0 + \sum_{i=1}^{j}(\sqrt{t_i - t_{i-1}}\ x_i)) d(\prod_{i=1}^{n}\omega)(x_1, \ldots, x_n)]$$
$$\times \chi_{B_0}(x_0) d\varphi(x_0).$$

Theorem 2.1. *M_φ is well-defined on $\mathcal{I}$.*

Proof. There is a minimal representation for an interval $I \equiv J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ as in (7). Any alternate representation for I must involve additional points. We will show that, in the case of one additional point, the corresponding formula for $M_\varphi(I)$ agrees with the formula associated with the minimal representation for $M_\varphi(I)$. The case of N additional points can be done by applying the procedure below N times. Suppose that the interval $I \equiv J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ is the minimal representation and that the extra point, say s, satisfies $t_k < s < t_{k+1}$ for some $k = 0, 1, \ldots, n-1$. Let $I^* = \{y$ in $C(\mathbb{B})|$ for $j = 0, 1, \ldots, n,\ y(t_j)$ is in B_j and $y(s)$ is in $\mathbb{B}\}$ and let $X_k = x_0 + \sum_{i=1}^{k}(\sqrt{t_i - t_{i-1}}\ x_i)$. Then

$$\int_{\mathbb{B}} [\int_{\mathbb{B}^{n-k}} \chi_{\mathbb{B}}(X_k + \sqrt{s - t_k}\ x_s) \prod_{u=1}^{n-k} \chi_{B_{k+u}}(X_k + \sqrt{s - t_k}\ x_s \tag{9}$$
$$+ \sqrt{t_{k+1} - s}\ x_{k+1} + \sum_{j=k+2}^{k+u}(\sqrt{t_j - t_{j-1}}\ x_j)$$
$$d(\prod_{j=1}^{n-k}\omega)(x_{k+1}, x_{k+2}, \ldots, x_n)\big] d\omega(x_s)$$
$$\overset{\text{(i)}}{=} \int_{\mathbb{B}} [\int_{\mathbb{B}^{n-k}} \chi_{\mathbb{B}}(X_k + x_s) \prod_{u=1}^{n-k} \chi_{B_{k+u}}(X_k + x_s + x_{k+1} + \sum_{j=k+2}^{k+u} x_j)$$
$$d(\omega_{\sqrt{t_{k+1}-s}} \times \prod_{u=1}^{n-k-1} \omega_{\sqrt{t_{k+u+1}-t_{k+u}}})(x_{k+1}, (x_{k+2}, \ldots, x_n))\big] d\omega_{\sqrt{s-t_k}}(x_s)$$
$$\overset{\text{(ii)}}{=} \int_{\mathbb{B}^{n-k}} \prod_{u=1}^{n-k} \chi_{B_{k+u}}(X_k + x^* + \sum_{j=k+2}^{k+u} x_j) d\big((\omega_{\sqrt{s-t_k}} * \omega_{\sqrt{t_{k+1}-s}})$$
$$\times (\prod_{u=1}^{n-k-1} \omega_{\sqrt{t_{k+u+1}-t_{k+u}}})\big)(x^*, (x_{k+2}, \ldots, x_n))$$
$$\overset{\text{(iii)}}{=} \int_{\mathbb{B}^{n-k}} \prod_{u=1}^{n-k-1} \chi_{B_{k+u}}(X_k + x^* + \sum_{j=k+2}^{k+u} x_j)$$

$$d\Big(\prod_{u=0}^{n-k}\omega_{\sqrt{t_{k+u+1}-t_{k+u}}}\Big)(x^*,x_{k+2},\dots,x_n)$$

$$\stackrel{\text{(iv)}}{=}\int_{\mathbb{B}^{n-k}}\prod_{u=1}^{n-k}\chi_{B_{k+u}}\Big(X_k+\sum_{j=k}^{k+u-1}(\sqrt{t_{j+1}-t_j}\;x_{j+1})\Big)d\Big(\prod_{j=1}^{n-k}\omega\Big)(x_{k+1},\dots,x_n).$$

Step (i) and (iv) follow from the change of variables theorem in [9] and (1). Step (ii) results from (2). By (3), we obtain Step (iii). Using (8) and (9), we can easily show that $M_\varphi(I)=M_\varphi(I^*)$, as desired. □

By the essentially same method as in the proofs of Theorem 2.1 and Theorem 5.1 in [11], $\mathcal{B}(C(\mathbb{B}))$, the set of all Borel subsets of $C(\mathbb{B})$, coincides with smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique measure $m_\varphi^{\mathbb{B}}$ on $(C(\mathbb{B}),\mathcal{B}(C(\mathbb{B})))$ such that $m_\varphi^{\mathbb{B}}(I)=M_\varphi(I)$ for all I in $\mathcal{I}$.

From the change of variables theorem, we have the following theorems which is one of main theorems.

Theorem 2.2 (The Wiener integration formula). *If $f:\mathbb{B}^{n+1}\to\mathbb{R}$ is a Borel measurable function, then following equality holds*;

$$\text{(10)}\qquad \int_{C(\mathbb{B})}f(y(t_0),y(t_1),\dots,y(t_n))dm_\varphi^{\mathbb{B}}(y)$$

$$\stackrel{*}{=}\int_{\mathbb{B}}\Big[\int_{\mathbb{B}^n}(f\circ T_{\vec{t}})(x_0,x_1,\dots,x_n)d\Big(\prod_{j=1}^{n}\omega\Big)(x_1,x_2,\dots,x_n)\Big]d\varphi(x_0)$$

where $\stackrel{*}{=}$ means that if one side exists, then both sides exist and the two values are equal.

Remark 2.3. (1) Let φ_1 and φ_2 be two finite non-negative Borel measures on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$ and let α and β be two non-negative real numbers. Then, letting $\varphi=\alpha\varphi_1+\beta\varphi_2$, $m_\varphi^{\mathbb{B}}=\alpha m_{\varphi_1}^{\mathbb{B}}+\beta m_{\varphi_2}^{\mathbb{B}}$.

(2) For a finite non-negative Borel measure φ on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$, $m_\varphi^{\mathbb{B}}(C(\mathbb{B}))=\varphi(\mathbb{B})$, if φ is a probability measure on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$, then $m_\varphi^{\mathbb{B}}$ is also a probability measure.

(3) Let $\mathcal{M}(\mathbb{B})$ and $\mathcal{M}(C(\mathbb{B}))$ be the space of all finite complex-valued countably additive measures on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$ and $(C(\mathbb{B}),\mathcal{B}(C(\mathbb{B})))$, respectively. Then $\mathcal{M}(\mathbb{B})$ and $\mathcal{M}(C(\mathbb{B}))$ are two Banach space with respect to its total variation norm. For φ in $\mathcal{M}(\mathbb{B})$ with the Jordan decomposition $\varphi=\varphi_1-\varphi_2+i\varphi_3-i\varphi_4$, we let $m_\varphi^{\mathbb{B}}=m_{\varphi_1}^{\mathbb{B}}-m_{\varphi_2}^{\mathbb{B}}+im_{\varphi_3}^{\mathbb{B}}-im_{\varphi_4}^{\mathbb{B}}$. Then $m_\varphi^{\mathbb{B}}$ is well-defined. Furthermore, let a function $m^{\mathbb{B}}:\mathcal{M}(\mathbb{B})\to\mathcal{M}(C(\mathbb{B}))$ be given by $m^{\mathbb{B}}(\varphi)=m_\varphi^{\mathbb{B}}$, $m^{\mathbb{B}}$ is a bounded linear operator by bound 4, in the operator norm sense.

Theorem 2.4. *For $a<s<b$ and for a non-zero element b^* of $\mathbb{B}^*$,*

$$\text{(11)}\quad m_\varphi^{\mathbb{B}}(\{y\text{ in }C(\mathbb{B})|b^*(y(s))<\alpha\})$$

$$= \int_{-\infty}^{\alpha} \big[\int_{\mathbb{B}} \frac{1}{\sqrt{2\pi(s-a)} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\Big\{ - \frac{(v - b^*(x_0))^2}{2(s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}} \Big\} d\varphi(x_0) \big] dm_L(v)$$

for any real number α, where m_L is the Lebesgue measure on $\mathbb{R}$.

Proof. For $a < s < b$, for a non-zero b^* in $\mathbb{B}^*$ and for real number α,

$$(12) \quad m_{\varphi}^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | b^*(y(s)) < \alpha\})$$

$$\overset{(1)}{=} \int_{C(\mathbb{B})} \chi_{b^{*-1}((-\infty,\alpha))}(y(s)) dm_{\varphi}^{\mathbb{B}}(y)$$

$$\overset{(2)}{=} \int_{\mathbb{B}} \big[\int_{\mathbb{B}} \chi_{b^{*-1}((-\infty,\alpha))}(x_0 + \sqrt{s-a}x_1) d\omega(x_1) \big] d\varphi(x_0)$$

$$\overset{(3)}{=} \int_{\mathbb{B}} \big[\int_{\mathbb{B}} \chi_{(-\infty, \frac{1}{\sqrt{s-a}}(\alpha - b^*(x_0)))} b^*(x_1) d\omega(x_1) \big] d\varphi(x_0)$$

$$\overset{(4)}{=} \int_{\mathbb{B}} \Big[\int_{-\infty}^{\frac{1}{\sqrt{s-a}}(\alpha - b^*(x_0))} \frac{1}{\sqrt{2\pi} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\big\{ - \frac{u^2}{2 \parallel b^* \parallel^2_{\mathbb{B}^*}} \big\} dm_L(u) \Big] d\varphi(x_0)$$

$$\overset{(5)}{=} \int_{-\infty}^{\alpha} \Big[\int_{\mathbb{B}} \frac{1}{\sqrt{2\pi(s-a)} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\{ - \frac{(v - b^*(x_0))^2}{2(s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}} \} d\varphi(x_0) \Big] dm_L(v).$$

Step (1) and (3) obtain from the elementary calculus. By the Wiener integration formula for $m_{\varphi}^{\mathbb{B}}$, we have Step (2). Step (4) results from the formula in [7]. By substituting $v = \sqrt{s-a}u + b^*(x_0)$ and the Fubini theorem, we have Step (5).
Hence, the theorem is proved. □

From Theorem 2.4 in the above, we have the following corollaries.

Corollary 2.5. *If $a < s < b$, b^* is a non-zero element of $\mathbb{B}^*$, $\varphi = \delta_0$ and $W(y) = b^*(y(s))$, then W has a normal distribution with mean 0 and variation $(s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}$.*

Corollary 2.6. *If $a < s < b$, b^* is a non-zero element of $\mathbb{B}^*$, $\varphi = \omega$ and $b^*(y(s)) = W(y)$, then W has a normal distribution with mean 0 and variation $(1 + s - a) \parallel b^* \parallel^2_{\mathbb{B}^*}$.*

Proof. For any real number α,

$$(13) \quad m_{\omega}^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | W(y) < \alpha\})$$

$$= \int_{-\infty}^{\alpha} \Big[\int_{\mathbb{B}} \frac{1}{\sqrt{2\pi(s-a)} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\{ - \frac{(v - b^*(x_0))^2}{2(s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}} \} d\omega(x_0) \Big] dm_L(v)$$

$$= \int_{-\infty}^{\alpha} \Big[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(s-a)} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\{ - \frac{(v-u)^2}{2(s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}} \}$$

$$\times \frac{1}{\sqrt{2\pi} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\{ - \frac{u^2}{2 \parallel b^* \parallel^2_{\mathbb{B}^*}} \} d\omega(u_0) \Big] dm_L(v)$$

$$= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi(1+s-a)} \parallel b^* \parallel_{\mathbb{B}^*}} \exp\{-\frac{v^2}{2(1+s-a) \parallel b^* \parallel^2_{\mathbb{B}^*}}\} dm_L(v).$$

Hence, the corollary is proved. □

Theorem 2.7. *Let $W_t(y) = y(t)$ for $a \le t \le b$. Then W_t has stationary increment.*

Proof. Suppose $s > 0$ and $t_1, t_2, t_1 + s$ and $t_2 + s$ are in $[a, b]$ with $t_1 < t_2$. Then for B in $\mathcal{B}(\mathbb{B})$,

$$\begin{aligned} (14) \qquad & m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | W_{t_2+s}(y) - W_{t_1+s}(y) \text{ is in } B\}) \\ = \ & \int_{\mathbb{B}} \big[\int_{\mathbb{B}} \chi_B(\sqrt{t_2 - t_1} x_2) d\omega(x_2) \big] d\varphi(x_0) \\ = \ & m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B\}). \end{aligned}$$

Hence, the theorem is proved. □

Theorem 2.8. *If $a \le t_1 < t_2 < t_3 \le b$ and B_1 and B_2 are both in $\mathcal{B}(\mathbb{B})$, then*

$$\begin{aligned} (15) \quad & \varphi(\mathbb{B}) m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1 \text{ and } y(t_3) - y(t_2) \text{ is in } B_2\}) \\ = \ & m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1\}) \\ & \times m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_3) - y(t_2) \text{ is in } B_2\}). \end{aligned}$$

Proof. For $a \le t_1 < t_2 < t_3 \le b$ and B_1 and B_2 in $\mathcal{B}(\mathbb{B})$

$$\begin{aligned} (16) \quad & m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1 \text{ and } y(t_3) - y(t_2) \text{ is in } B_2\}) \\ = \ & \int_{C(\mathbb{B})} \chi_{B_1}(y(t_2) - y(t_1)) \chi_{B_2}(y(t_3) - y(t_2)) dm_\varphi^{\mathbb{B}}(y) \\ = \ & \int_{\mathbb{B}} \big[\int_{\mathbb{B}\times\mathbb{B}\times\mathbb{B}} \chi_{B_1}(\sqrt{t_2 - t_1} x_2) \chi_{B_2}(\sqrt{t_3 - t_2} x_3) \\ & d\omega \times \omega \times \omega(x_1, x_2, x_3) \big] d\varphi(x_0) \\ = \ & \varphi(\mathbb{B}) \omega_{\sqrt{t_2 - t_1}}(B_1) \omega_{\sqrt{t_3 - t_2}}(B_2) \end{aligned}$$

and

$$\begin{aligned} & m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1\}) \\ & \times m_\varphi^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_3) - y(t_2) \text{ is in } B_2\}) \\ = \ & \int_{C(\mathbb{B})} \chi_{B_1}(y(t_2) - y(t_1)) dm_\varphi^{\mathbb{B}}(y) \int_{C(\mathbb{B})} \chi_{B_2}(y(t_3) - y(t_2)) dm_\varphi^{\mathbb{B}}(y) \\ = \ & \int_{\mathbb{B}} \big[\int_{\mathbb{B}} \chi_{B_1}(\sqrt{t_2 - t_1} x_2) d\omega(x_2) \big] d\varphi(x_0) \\ & \int_{\mathbb{B}} \big[\int_{\mathbb{B}} \chi_{B_2}(\sqrt{t_3 - t_2} x_3) d\omega(x_3) \big] d\varphi(x_0) \\ = \ & \varphi(\mathbb{B})^2 \omega_{\sqrt{t_2 - t_1}}(B_1) \omega_{\sqrt{t_3 - t_2}}(B_2). \end{aligned}$$

Hence, the theorem is proved. □

By the essentially similar method as in the above, we can prove the following theorem.

Theorem 2.9. *If $a \le t_1 < t_2 < t_3 < t_4 \le b$ and B_1 and B_2 are both in $\mathcal{B}(\mathbb{B})$,*

$$\begin{aligned}
(17)\quad & \varphi(\mathbb{B}) m_{\varphi}^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1 \text{ and } y(t_4) - y(t_3) \text{ is in } B_2\}) \\
= \quad & m_{\varphi}^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_2) - y(t_1) \text{ is in } B_1\}) \\
& m_{\varphi}^{\mathbb{B}}(\{y \text{ in } C(\mathbb{B}) | y(t_4) - y(t_3) \text{ is in } B_2\}).
\end{aligned}$$

Remark 2.10. If φ is a probability measure on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$ and $a \le t_1 < t_2 \le t_3 < t_4 \le b$, $y(t_2) - y(t_1)$ and $y(t_4) - y(t_3)$ are stochastically independent.

Theorem 2.11. *If $\int_{\mathbb{B}} \| x \| d\varphi(x)$ is finite, then $F(y) = y$ is Bochner integrable on $C(\mathbb{B})$ and*

$$(B_0) - \int_{C(\mathbb{B})} y dm_{\varphi}^{\mathbb{B}}(y) = (B_0) - \int_{\mathbb{B}} x d\varphi(x). \tag{18}$$

Proof. Let D be the set of all rational numbers in $[a, b]$. Then we can write $D = \{u_n \mid n \text{ is a natural numbers}\}$. For a natural number m, let $D_m = \{u_1, u_2, \ldots, u_m\}$. Then by the monotone convergence theorem,

$$\begin{aligned}
(19)\quad & \int_{C(\mathbb{B})} \| y \|_{C(\mathbb{B})} dm_{\varphi}^{\mathbb{B}}(y) \\
= \quad & \int_{C(\mathbb{B})} \sup_{u \in D} \| y(u) \|_{\mathbb{B}} dm_{\varphi}^{\mathbb{B}}(y) \\
= \quad & \int_{C(\mathbb{B})} \lim_{m \to \infty} \sup_{u \in D_m} \| y(u) \|_{\mathbb{B}} dm_{\varphi}^{\mathbb{B}}(y) \\
= \quad & \lim_{m \to \infty} \int_{C(\mathbb{B})} \sup_{u \in D_m} \| y(u) \|_{\mathbb{B}} dm_{\varphi}^{\mathbb{B}}(y) \\
= \quad & \lim_{m \to \infty} \int_{\mathbb{B}} \Big[\int_{\mathbb{B}} \sup_{u \in D_m} \| \sqrt{u-a} x_1 + x_0 \| d\omega(x_1) \Big] d\varphi(x_0) \\
\le \quad & \sqrt{b-a} \varphi(\mathbb{B}) \int_{\mathbb{B}} \| x_1 \| d\omega(x_1) + \int_{\mathbb{B}} \| x_0 \| d\varphi(x_0).
\end{aligned}$$

Since $\int_{\mathbb{B}} \| x_1 \| d\omega(x_1)$ is finite by Fernique's theorem [3], the right side of (19) is finite, so $\int_{C(\mathbb{B})} \| y \|_{C(\mathbb{B})} dm_{\varphi}^{\mathbb{B}}(y)$ is finite. Since F is weakly measurable and $C(\mathbb{B})$ is separable, by Pettis measurability theorem in [2], F is strongly measurable. Hence, from Theorem 2 in [2], F is Bochner integrable.

Now, if b^* is a non-zero element of $\mathbb{B}^*$ and $a \le s \le b$ then putting $T(y) = b^*(y(s))$, T is a bounded linear functional, so

$$\begin{aligned}
(20)\quad & T\big((B_0) - \int_{C(\mathbb{B})} y dm_{\varphi}^{\mathbb{B}}(y)\big) \\
\overset{(1)}{=} \quad & \int_{C(\mathbb{B})} T(y) dm_{\varphi}^{\mathbb{B}}(y)
\end{aligned}$$

$$\begin{aligned}
&\overset{(2)}{=} \int_{C(\mathbb{B})} b^*(y(s))dm_\varphi^{\mathbb{B}}(y)\\
&\overset{(3)}{=} \int_{\mathbb{B}} \Big[\int_{-\infty}^{\infty} \frac{u}{\sqrt{2\pi(s-a)}\parallel b^*\parallel_{\mathbb{B}^*}} \exp\{-\frac{(u-b^*(x_0))^2}{2(s-a)\parallel b^*\parallel_{\mathbb{B}^*}^2}\}dm_L(u)\Big]d\varphi(x_0)\\
&\overset{(4)}{=} \int_{\mathbb{B}} b^*(x_0)d\varphi(x_0)\\
&\overset{(5)}{=} b^*\big((Bo)-\int_{\mathbb{B}} x_0 d\varphi(x_0)\big).
\end{aligned}$$

Step(1) follows from Theorem 6 in [2]. From the definition of T, we have Step(2). By Theorem 2.4, we obtain Step(3). Using the elementary calculus, we have Step(4). Step(5) results from the assumption and Theorem 2 in [2]. Hence $(B_0)-\int_{C(\mathbb{B})} ydm_\varphi^{\mathbb{B}}(y) = (B_0)-\int_{\mathbb{B}} xd\varphi(x)$. □

Notation 2.12. For y in $C(\mathbb{B})$, let

$$[y](s) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)\big[y(t_{j-1})+\frac{s-t_{j-1}}{t_j-t_{j-1}}(y(t_j)-y(t_{j-1}))\big]+y(b)\chi_{\{b\}}(s) \tag{21}$$

for s in $[a,b]$.

For $(u_0,u_1,\dots,u_n)\in\mathbb{B}^{n+1}$, let

$$[u](s) = \sum_{j=1}^{n} \chi_{[t_{j-1},t_j)}(s)\big[u_{j-1}+\frac{s-t_{j-1}}{t_j-t_{j-1}}(u_j-u_{j-1})\big]+u_n\chi_{\{b\}}(s) \tag{22}$$

for s in $[a,b]$.

Theorem 2.13. *Let φ be a probability measure on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$. Let $a=t_0<t_1<\cdots<s_1<t_{j-1}<s<t_j<s_2<\cdots<t_n=b$ and let b^* be a non-zero element of $\mathbb{B}^*$. Let X, Y and Z be three functions from $C(\mathbb{B})$ into $\mathbb{R}$ with $X(y)=b^*(y(s)-[y](s))$, $Y(y)=b^*(y(s_1))$ and $Z(y)=b^*(y(s_2))$. Then X and Y are stochastically independent and X and Z are stochastically independent.*

Proof. By Theorem 2.4, letting $A=\sqrt{\frac{(t_j-s)(s-t_{j-1})}{t_j-t_{j-1}}}$,

$$\begin{aligned}
&E(\exp\{i\lambda X\}) \\
&= \int_{C(\mathbb{B})} \exp\{i\lambda b^*(y(s)-[y](s))\}dm_\varphi^{\mathbb{B}}(y)\\
&= \int_{\mathbb{B}}\Big(\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}} \exp\{i\lambda(\sqrt{s-t_{j-1}}\frac{t_j-s}{t_j-t_{j-1}}b^*(v_2)\\
&\qquad -\sqrt{t_j-s}\frac{s-t_{j-1}}{t_j-t_{j-1}}b^*(v_3))\}d\omega(v_3)d\omega(v_2)d\omega(v_1)\Big)d\varphi(v_0)\\
&= \int_{\mathbb{B}} \exp\{i\lambda b^*(Av)\}d\omega(v)
\end{aligned} \tag{23}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \exp\{i\lambda x\} \frac{1}{\sqrt{2\pi}A \parallel b^* \parallel_{\mathbb{B}^*}^2} \exp\{-\frac{x}{2A^2 \parallel b^* \parallel_{\mathbb{B}^*}^2}\} dm_L(x) \\
&= \exp\{-\frac{1}{2}A^2 \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda^2\}.
\end{aligned}
$$

Hence X is normal distributed with mean 0 and variance $\frac{(t_j-s)(s-t_{j-1})}{t_j-t_{j-1}} \parallel b^* \parallel_{\mathbb{B}^*}^2$. And

$$
\begin{aligned}
&E(\exp\{i\lambda Y\}) \\
&= \int_{C(\mathbb{B})} \exp\{i\lambda b^*(y(s_1))\} dm_\varphi^{\mathbb{B}}(y) \\
&= \int_{\mathbb{B}} \big(\int_{\mathbb{B}} \exp\{i\lambda b^*(v_0) + i\lambda b^*(\sqrt{s_1 - a} v_1)\} d\omega(v_1)\big) d\varphi(v_0) \\
&= \exp\{-\frac{1}{2}(s_1 - a) \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda^2\} \int_{\mathbb{B}} \exp\{i\lambda b^*(v_0)\} d\varphi(v_0)
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
&E(\exp\{i\lambda Z\}) \\
&= \int_{C(\mathbb{B})} \exp\{i\lambda b^*(y(s_2))\} dm_\varphi^{\mathbb{B}}(y) \\
&= \int_{\mathbb{B}} \big(\int_{\mathbb{B}} \exp\{i\lambda b^*(v_0) + b^*(\sqrt{s_2 - a} v_1)\} d\omega(v_1)\big) d\varphi(v_0) \\
&= \exp\{-\frac{1}{2}(s_2 - a) \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda^2\} \int_{\mathbb{B}} \exp\{i\lambda b^*(v_0)\} d\varphi(v_0).
\end{aligned} \tag{25}
$$

By the basic calculation,

$$
\begin{aligned}
&E(\exp\{i\lambda_1 X + i\lambda_2 Y\}) \\
&= \int_{C(\mathbb{B})} \exp\{i\lambda_1(b^*(y(s)) - b^*([y](s))) + i\lambda_2 b^*(y(s_1))\} dm_\varphi^{\mathbb{B}}(y) \\
&= \int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}} \exp\{i\lambda_1(b^*(\frac{\sqrt{s - t_{j-1}}(t_j - s)}{t_j - t_{j-1}} v_3) \\
&\qquad -b^*(\frac{\sqrt{t_j - s}(s - t_{j-1})}{t_j - t_{j-1}} v_4)) + i\lambda_2 b^*(v_0 + \sqrt{s_1 - a} v_1)\} \\
&\qquad d\omega(v_4) d\omega(v_3) d\omega(v_2) d\omega(v_1) d\varphi(v_0) \\
&= \int_{\mathbb{B}} \exp\{i\lambda_1 b^*(Av)\} d\omega(v) \int_{\mathbb{B}} \exp\{i\lambda_2 b^*(\sqrt{(s_1 - a)} v_1)\} d\omega(v_1) \\
&\qquad \times \int_{\mathbb{B}} \exp\{i\lambda_2 b^*(v_0)\} d\varphi(v_0) \\
&= \exp\{-\frac{1}{2}A^2 \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda_1^2\} \exp\{-\frac{1}{2}(s_1 - a) \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda_2^2\} \\
&\qquad \times \int_{\mathbb{B}} \exp\{i\lambda_2 b^*(v_0)\} d\varphi(v_0)
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
(27)\qquad & E(\exp\{i\lambda_1 X + i\lambda_2 Z\}) \\
= & \int_{C(\mathbb{B})} \exp\{i\lambda_1(b^*(y(s)) - b^*([y](s))) + i\lambda_2 b^*(y(s_2))\} dm_\varphi^{\mathbb{B}}(y) \\
= & \int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}} \exp\{i\lambda_1(b^*(\frac{\sqrt{s-t_{j-1}}(t_j - s)}{t_j - t_{j-1}} v_2) \\
& -b^*(\frac{\sqrt{t_j - s}(s - t_{j-1})}{t_j - t_{j-1}} v_3)) + i\lambda_2 b^*(v_0 + \sqrt{t_{j-1} - a} v_1 \\
& + \sqrt{s - t_{j-1}} v_2 + \sqrt{t_j - s} v_3 + \sqrt{s - t_j} v_4)\} \\
& d\omega(v_4) d\omega(v_3) d\omega(v_2) d\omega(v_1) d\varphi(v_0) \\
= & \exp\{-\frac{1}{2} A^2 \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda_1^2\} \exp\{-\frac{1}{2}(s_2 - a) \parallel b^* \parallel_{\mathbb{B}^*}^2 \lambda_2^2\} \\
& \times \int_{\mathbb{B}} \exp\{i\lambda_2 b^*(v_0)\} d\varphi(v_0).
\end{aligned}
$$

Hence,

$$E(\exp\{i\lambda_1 X + i\lambda_2 Y\}) = E(\exp\{i\lambda_1 X\}) E(\exp\{i\lambda_2 Y\})$$

and

$$E(\exp\{i\lambda_1 X + i\lambda_2 Z\}) = E(\exp\{i\lambda_1 X\}) E(\exp\{i\lambda_2 Z\}).$$

From [8], X and Y are stochastically independent and X and Z are stochastically independent. □

3. The measure-valued measure V_φ^X on $(C(\mathbb{B}), \mathcal{B}(C(\mathbb{B})))$

Using the concept of conditional expectation in [15], Ryu and Im derived a measure-valued measure V_φ^X and they found the integration formula for V_φ^X on the analogue of Wiener measure space $(C[a,b], \omega_\varphi)$. In this section, we define a measure-valued measure V_φ^X on $(C(\mathbb{B}), \mathcal{B}(C(\mathbb{B})))$, associated with the measurable function X and find two integration formulae with respect to V_φ^X.

Let $X : C(\mathbb{B}) \to \mathbb{B}^{n+1}$ be a Borel measurable function. For B in $\mathcal{B}(C(\mathbb{B}))$ and for E in $\mathcal{B}(\mathbb{B}^{n+1})$, we let

$$[V_\varphi^X(B)](E) = m_\varphi^{\mathbb{B}}(B \cap X^{-1}(E)). \tag{28}$$

Then for B in $\mathcal{B}(C(\mathbb{B}))$, $V_\varphi^X(B)$ is a measure on $(\mathbb{B}^{n+1}, \mathcal{B}(\mathbb{B}^{n+1}))$ and $V_\varphi^X : \mathcal{B}(C(\mathbb{B})) \to \mathcal{M}(\mathbb{B}^{n+1})$, where $\mathcal{M}(\mathbb{B}^{n+1})$ is the space of Borel measures on $\mathbb{B}^{n+1}$, is a measure-valued measure in the total variation norm sense, clearly.
Define a measure P_X on $(\mathbb{B}^{n+1}, \mathcal{B}(\mathbb{B}^{n+1}))$ determined by X as follows;

$$P_X(E) = m_\varphi^{\mathbb{B}}(X^{-1}(E)) \tag{29}$$

for E in $\mathcal{B}(\mathbb{B}^{n+1})$.

Let $F : C(\mathbb{B}) \to \mathbb{R}$ be a $m_\varphi^{\mathbb{B}}$-integrable function. For E in $\mathcal{B}(\mathbb{B}^{n+1})$, we let

$$\mu(E) = \int_{X^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y). \tag{30}$$

Then μ is a Borel measure on $\mathbb{B}^{n+1}$ and μ is absolutely continuous with respect to P_X. So, by the Radon-Nikodym theorem, there is a $\mathcal{B}(\mathbb{B}^{n+1})$-measurable and P_X-integrable function f on $\mathbb{B}^{n+1}$ such that

$$\begin{aligned} \int_{X^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y) &= \mu(E) \\ &= \int_E f(x_0, x_1, \ldots, x_n)dP_X(x_0, x_1, \ldots, x_n) \end{aligned} \tag{31}$$

for E in $\mathcal{B}(\mathbb{B}^{n+1})$.

When φ is a probability Borel measure on $\mathcal{B}(\mathbb{B})$, f is called the conditional expectation of F given X and is denoted by $E^\varphi(F|X)$ in [15].

Theorem 3.1. *For all E in $\mathcal{B}(\mathbb{B}^{n+1})$ and for $m_\varphi^{\mathbb{B}}$-integrable function F, the following equality holds*

$$\big[(Ba) - \int_{C(\mathbb{B})} F(y)dV_\varphi^X(y)\big](E) = \int_{X^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y), \tag{32}$$

where $(Ba) - \int_{C(\mathbb{B})} F(y)dV_\varphi^X(y)$ is the Bartle integral.

Proof. Let $F = \chi_B$ where B is in $\mathcal{B}(C(\mathbb{B}))$. Then for E in $\mathcal{B}(\mathbb{B}^{n+1})$,

$$\begin{aligned} \int_{X^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y) &= m_\varphi^{\mathbb{B}}(B \cap X^{-1}(E)) \\ &= [V_\varphi^X(B)](E). \end{aligned} \tag{33}$$

If F is a simple function, then by the basic properties of the Lebesgue integral and the Bartle integral, the equality (32) holds. Suppose F is $m_\varphi^{\mathbb{B}}$-integrable. Then there is an increasing sequence $\langle F_n \rangle$ of simple functions such that $\langle F_n \rangle$ converges to F $m_\varphi^{\mathbb{B}}$-*a.e.*, and $\lim_{n\to\infty} \int_{C(\mathbb{B})} F_n(y)dm_\varphi^{\mathbb{B}}(y) = \int_{C(\mathbb{B})} F(y)dm_\varphi^{\mathbb{B}}(y)$. Then for $n \geq m$,

$$\begin{aligned} &\| (Ba) - \int_{C(\mathbb{B})} F_n(y)dV_\varphi^X(y) - (Ba) - \int_{C(\mathbb{B})} F_m(y)dV_\varphi^X(y) \| \\ &= \| (Ba) - \int_{C(\mathbb{B})} (F_n(y) - F_m(y))dV_\varphi^X(y) \| \\ &= \int_{C(\mathbb{B})} (F_n(y) - F_m(y))dm_\varphi^{\mathbb{B}}(y) \end{aligned} \tag{34}$$

so, $\langle (Ba) - \int_{C(\mathbb{B})} F_n(y) dV_\varphi^X(y) \rangle$ is Cauchy in the total variation norm sense. By the definition of Bartle integral,

$$(Ba) - \int_{C(\mathbb{B})} F(y) dV_\varphi^X(y) = \lim_{n\to\infty} (Ba) - \int_{C(\mathbb{B})} F_n(y) dV_\varphi^X(y)$$

where the convergence means the convergence in the total variation norm sense. Hence, for E in $\mathcal{B}(\mathbb{B}^{n+1})$,

$$\begin{aligned} & \big[(Ba) - \int_{C(\mathbb{B})} F(y) dV_\varphi^X(y)\big](E) \\ = \ & \big[\lim_{n\to\infty} (Ba) - \int_{C(\mathbb{B})} F_n(y) dV_\varphi^X(y)\big](E) \\ = \ & \lim_{n\to\infty} \big[(Ba) - \int_{C(\mathbb{B})} F_n(y) dV_\varphi^X(y)\big](E) \\ = \ & \lim_{n\to\infty} \int_{X^{-1}(E)} F_n(y) dm_\varphi^{\mathbb{B}}(y) \\ = \ & \int_{X^{-1}(E)} F(y) dm_\varphi^{\mathbb{B}}(y) \end{aligned} \tag{35}$$

as desired. □

From Theorem 2.2 and Theorem 3.1, we have directly the following theorem.

Theorem 3.2 (The Wiener integration formula)**.** *Suppose for $k = 1, 2, \dots, n$, i_k is a nonnegative integer such that $m = n + \sum_{j=1}^n i_j + 1$ and $a \equiv t_0 \equiv t_{0,0} < t_{0,1} < t_{0,2} < \cdots < t_{0,i_1} < t_1 \equiv t_{0,i_1+1} \equiv t_{1,0} < t_{1,1} < t_{1,2} < \cdots < t_{n-1,i_n} < t_n \equiv t_{n-1,i_n+1} \equiv b$ and for $j = 1, 2, \dots, n$, $\bar{u}_{0,0} = u_{0,0}$ and $\bar{u}_{j-1,v} = u_{0,0} + \sum_{e=0}^{j-2} \sum_{f=1}^{i_{e+1}+1} \sqrt{t_{e,f} - t_{e,f-1}} u_{e,f} + \sum_{f=0}^{v} \sqrt{t_{j-1,f} - t_{j-1,f-1}} u_{j-1,f}$. If $f : \mathbb{B}^m \to \mathbb{R}$ is a Borel measurable function then the following equality holds*;

$$\begin{aligned} & \big[(Ba) - \int_{C(\mathbb{B})} f(y(t_{0,0}), y(t_{0,1}), \dots, y(t_{n-1,i_n+1})) dV_\varphi^{J_{\vec{t}}}(y)\big](E) \\ \overset{*}{=} \ & \int_{\mathbb{B}} \big[\int_{\mathbb{B}^{m-1}} f(\bar{u}_{0,0}, \bar{u}_{0,1}, \dots, \bar{u}_{n-1,i_n+1}) \prod_{g=0}^{n} \chi_{E^{[g]}}(\bar{u}_{g,0}) \\ & d(\prod_{i=1}^{m-1} \omega)(u_{0,0}, u_{0,1}, \dots, u_{n-1,i_n+1})\big] d\varphi(u_{0,0}), \end{aligned} \tag{36}$$

where $E^{[g]}$ is the g^{th}-section of E.

When using (36) in the above, we calculate an integral of various functions, with respect to $V_\varphi^{J_{\vec{t}}}$, we meet too the difficult problems frequently, so we can need more a simple formula for integral with respect to $V_\varphi^{J_{\vec{t}}}$ which is one of main theorems. Indeed, at 1988, Park and Skoug proved the simple formula for conditional expectation in the concrete Wiener case in [10] and at 2008, D. H.

Cho established the simple formula for conditional expectation in $C_0(\mathbb{B})$ case [1]. Our result is very similar to previous results but the process of its proof is not quite same.

Theorem 3.3 (The simple formula for conditional expectation)**.** *Let φ be a Borel probability measure on $\mathbb{B}$ and let F be $m_\varphi^{\mathbb{B}}$-integrable on $C(\mathbb{B})$. Then for E in $\mathcal{B}(\mathbb{B}^{n+1})$,*

$$
\begin{aligned}
(37)\qquad & \big[(Ba)-\int_{C(\mathbb{B})} F(y)dV_\varphi^{J_{\vec{t}}}(y)\big](E) \\
&= \int_{J_{\vec{t}}^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y) \\
&= \int_E \big(\int_{C(\mathbb{B})} F(y-[y]+[\vec{u}])dm_\varphi^{\mathbb{B}}(y)\big)dP_{J_{\vec{t}}}(\vec{u})
\end{aligned}
$$

Proof. Let A be in $\mathcal{B}(C(\mathbb{B}^n))$ and let $F=\chi_A$. Then for E in $\mathcal{B}(\mathbb{B}^{n+1})$,

$$
\begin{aligned}
(38)\qquad \int_{J_{\vec{t}}^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y) &= m_\varphi^{\mathbb{B}}(A\cap J_{\vec{t}}^{-1}(E)) \\
&= \int_E E(F|J_{\vec{t}})(\vec{u})dP_{J_{\vec{t}}}(\vec{u}) \\
&= \int_E E(F(y-[y]+[\vec{u}])|J_{\vec{t}})(\vec{u})dP_{J_{\vec{t}}}(\vec{u}).
\end{aligned}
$$

From Theorem 2.13, $y-[y]$ and $J_{\vec{t}}(y)$ are weakly stochastically independent. Since $\mathbb{B}$ is separable, by [12], $y-[y]$ and $J_{\vec{t}}(y)$ are stochastically independent. Hence

$$
(39)\qquad \int_{J_{\vec{t}}^{-1}(E)} F(y)dm_\varphi^{\mathbb{B}}(y)=\int_E E(F(y-[y]+[\vec{u}]))dP_{J_{\vec{t}}}(\vec{u}).
$$

Thus, the result holds for the characteristic function of E in $\mathcal{B}(C(\mathbb{B}))$. The general case follows by the usual arguments in Bartle integration theory. □

4. Examples and applications

In this section, we give the various examples for our theorems.

Example 4.1 (The absolutely continuity)**.** Let φ and ψ be two finite positive Borel measures on $\mathbb{B}$. Then φ is absolutely continuous with respect to ψ if and only if $m_\varphi^{\mathbb{B}}$ is absolutely continuous with respect to $m_\psi^{\mathbb{B}}$ and $\frac{dm_\varphi^{\mathbb{B}}}{dm_\psi^{\mathbb{B}}}=\frac{d\varphi}{d\psi}(y(0))$.

Suppose that φ is absolutely continuous with respect to ψ. Then there is a measurable function $g:\mathbb{B}\to\mathbb{R}$ such that for E in $\mathcal{B}(\mathbb{B})$, $\varphi(E)=\int_E g(x)d\psi(x)$. Let $\mu_\psi:\mathcal{B}(C(\mathbb{B}))\to\mathbb{R}$ be a function with $\mu_\psi(B)=\int_B g(y(0))dm_\psi^{\mathbb{B}}(y)$ for B in $\mathcal{B}(C(\mathbb{B}))$. Then $\int_{C(\mathbb{B})}|g(y(0))|dm_\psi^{\mathbb{B}}(y)=\int_{\mathbb{B}}|g(x)|d\psi(x)$, so g is $m_\psi^{\mathbb{B}}$-integrable. Hence μ_ψ is a Borel measure on $C(\mathbb{B})$. Consider a set $\mathcal{J}=\{B$

in $\mathcal{B}(C(\mathbb{B}))|m_{\varphi}^{\mathbb{B}}(B) = \mu_{\psi}(B)\}$. Then by the Radon-Nykodym theorem, $\mathcal{I} \subset \mathcal{J}$. By the routine method as in the measure theory, $\mathcal{J}$ is a σ-algebra, so $\mathcal{B}(C(\mathbb{B})) = \mathcal{J}$. Thus, we have $m_{\varphi}^{\mathbb{B}}(B) = \int_{C(\mathbb{B})} g(y(0))dm_{\psi}^{\mathbb{B}}(y)$.
Now, we assume that $m_{\varphi}^{\mathbb{B}}$ is absolutely continuous with respect to $m_{\psi}^{\mathbb{B}}$ and N is a Borel subset with $\psi(N) = 0$. Let $J_0 : C(\mathbb{B}) \to \mathbb{B}$ be a function with $J_0(y) = y(0)$. Then $m_{\psi}^{\mathbb{B}}(J_0^{-1}(N)) = \int_{\mathbb{B}} \chi_N(x)d\psi(x) = 0$, so $m_{\varphi}^{\mathbb{B}}(J_0^{-1}(N)) = \varphi(N) = 0$, that is, φ is absolutely continuous with respect to ψ.

Example 4.2 (The scale-invariant measurability)**.** We can establish the existence of scale-invariant measurable subsets in $C(\mathbb{B})$.

Given partition $\prod_n$ of $[a, b]$; $a = t_0^n < t_1^n < \cdots < t_{k(n)}^n = b$ with $\mu(\prod_n) = \max_{1\le p\le k(n)} |t_p^n - t_{p-1}^n| \to 0$ as $n \to +\infty$ and y in $C(\mathbb{B})$, let $S_{\prod_n}(y) = \sum_{j=1}^{k(n)} \| y(t_j^n) - y(t_{j-1}^n) \|_{\mathbb{B}}^2$. By [11], $\int_{\mathbb{B}} \| x \|_{\mathbb{B}}^2 \, d\omega(x)$ and $\int_{\mathbb{B}} \| x \|_{\mathbb{B}}^4 \, d\omega(x)$ are finite. Let $\alpha = (b-a)\int_{\mathbb{B}} \| x \|_{\mathbb{B}}^2 \, d\omega(x)$ and $\beta = \int_{\mathbb{B}} \| x \|_{\mathbb{B}}^4 \, d\omega(x) - \big(\int_{\mathbb{B}} \| x \|_{\mathbb{B}}^2 \, d\omega(x)\big)^2$. Then Theorem 2.2,

$$
\begin{aligned}
&\int_{C(\mathbb{B})} S_{\prod_n}(y)dm_{\varphi}^{\mathbb{B}}(y)\\
= &\int_{\mathbb{B}} \Big[\int_{\mathbb{B}^{k(n)}} \sum_{j=1}^{k(n)} \| (x_0 + \sum_{i=1}^{j} \sqrt{t_i^n - t_{i-1}^n}x_i) - (x_0\\
&+ \sum_{i=1}^{j-1} \sqrt{t_i^n - t_{i-1}^n}x_i) \|_{\mathbb{B}}^2 \, d\big(\prod_{j=1}^{k(n)} \omega\big)(x_1, x_2, \ldots, x_{k(n)})\Big] d\varphi(x_0)\\
= &\varphi(\mathbb{B}) \sum_{j=1}^{k(n)} (t_j^n - t_{j-1}^n) \int_{\mathbb{B}^{k(n)}} \| x_j \|_{\mathbb{B}}^2 \, d\big(\prod_{j=1}^{k(n)} \omega\big)(x_1, x_2, \ldots, x_{k(n)})\\
= &\alpha\varphi(\mathbb{B})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{C(\mathbb{B})} (S_{\prod_n}(y) - \alpha)^2 dm_{\varphi}^{\mathbb{B}}(y)\\
= &\int_{\mathbb{B}} \Big[\int_{\mathbb{B}^{k(n)}} \big(\sum_{j=1}^{k(n)} (t_j^n - t_{j-1}^n) \| x_j \|_{\mathbb{B}}^2\big)^2 d\big(\prod_{j=1}^{k(n)} \omega\big)(x_1, x_2, \ldots, x_{k(n)})\Big]\\
&d\varphi(x_0) - \alpha^2\varphi(\mathbb{B})\\
= &\varphi(\mathbb{B})\big(\sum_{p=1}^{k(n)} \sum_{q=1}^{k(n)} (t_p^n - t_{p-1}^n)(t_q^n - t_{q-1}^n)\big)\big(\int_{\mathbb{B}} \| x \|_{\mathbb{B}}^2 \, d\omega(x)\big)^2\\
&+ \varphi(\mathbb{B})\beta \sum_{p=1}^{k(n)} (t_p^n - t_{p-1}^n)^2 - \alpha^2\varphi(\mathbb{B})
\end{aligned}
$$

$$= \quad \varphi(\mathbb{B})\beta\sum_{p=1}^{k(n)}(t_p^n - t_{p-1}^n)^2.$$

By the definition of Riemann integral $\int_a^b xdx$, we have $\lim_{n\to\infty,\mu(\prod_n)\to 0}\sum_{p=1}^{k(n)}$ $(t_p^n - t_{p-1}^n)^2 = 0$, so we can choose a subsequence $\langle\prod_{\sigma(n)}\rangle$ of $\langle\prod_n\rangle$ such that $\sum_{n=1}^{\infty}\sum_{p=1}^{k(n)}(t_p^{\sigma(n)} - t_{p-1}^{\sigma(n)})^2$ is finite, that is, $\sum_{n=1}^{\infty}\int_{C(\mathbb{B})}(S_{\prod_{\sigma(n)}}(y)-\alpha)^2\, dm_\varphi^{\mathbb{B}}(y)$ $= \varphi(\mathbb{B})\beta\sum_{n=1}^{\infty}\sum_{p=1}^{k(\sigma(n))}(t_p^{\sigma(n)} - t_{p-1}^{\sigma(n)})^2$ is finite. From [9], there is a subsequence $\langle\sigma^*(n)\rangle$ of $\langle\sigma(n)\rangle$ such that $\lim_{n\to\infty} S_{\prod_{\sigma^*(n)}}(y) = \alpha$ for $m_\varphi^{\mathbb{B}}$-*a.e.* y.

For $\lambda \geq 0$, let

$$\Omega_\lambda = \{y \text{ in } C(\mathbb{B}) | \lim_{n\to\infty} S_{\prod_{\sigma^*(n)}}(y) = \lambda^2\alpha\}$$

and let

$$D^* = \{y \text{ in } C(\mathbb{B}) | \text{ the limit } \lim_{n\to\infty} S_{\prod_{\sigma^*(n)}}(y) \text{ doesn't exist}\}.$$

Then for two positive real numbers λ_1 and λ_2, $\lambda_1\Omega_{\lambda_2} = \Omega_{\lambda_1\lambda_2}$, Ω_λ $(\lambda \geq 0)$ and D^* are Borel subsets, $C(\mathbb{B})$ is the disjoint union of the sets Ω_λ $(\lambda \geq 0)$ and D^* and $m_\varphi^{\mathbb{B}}(\Omega_\lambda) = 0$ if and only if $\lambda \neq 1$.

For $\lambda > 0$, we define a Borel measure $m_{\varphi,\lambda}^{\mathbb{B}}$ on $\mathcal{B}(C(\mathbb{B}))$ by $m_{\varphi,\lambda}^{\mathbb{B}}(B) = m_\varphi^{\mathbb{B}}(\lambda^{-1}B)$.

For $\lambda > 0$, let $(\mathbb{B}, \mathcal{B}(\bar{C(\mathbb{B})})_\lambda, \bar{m}_{\varphi,\lambda}^{\mathbb{B}})$ be the completion of $(\mathbb{B}, \mathcal{B}(C(\mathbb{B})), m_{\varphi,\lambda}^{\mathbb{B}})$. Let $\mathcal{S} = \cap_{\lambda>0}\mathcal{B}(\bar{C(\mathbb{B})})_\lambda$. The element of $\mathcal{S}$ is called the scale-invariant measurable subset of $C(\mathbb{B})$. Then for $\lambda > 0$, $\mathcal{B}(C(\mathbb{B})) \underset{\neq}{\subset} \mathcal{S} \underset{\neq}{\subset} \mathcal{B}(\bar{C(\mathbb{B})})_\lambda$.

By the elementary calculus and Theorem 2.4, directly we have the following example for the convolution.

Example 4.3 (The convolution). Let φ and ψ be two positive finite Borel measures on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$. Let p and q be two positive real numbers. Then $m_{\varphi,p}^{\mathbb{B}} * m_{\psi,q}^{\mathbb{B}} = m_{\varphi*\psi,\sqrt{p^2+q^2}}^{\mathbb{B}}$ on $\mathcal{B}(C(\mathbb{B}))$.

Let $B_0, B_1, \ldots, B_n$ be in $\mathcal{B}(\mathbb{B})$. Then

$$\begin{aligned}
& m_{\varphi,p}^{\mathbb{B}} * m_{\psi,q}^{\mathbb{B}}(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)) \\
= \quad & m_{\varphi,p}^{\mathbb{B}} \times m_{\psi,q}^{\mathbb{B}}(\{(x,y) \text{ in } C(\mathbb{B}) \times C(\mathbb{B}) | x+y \text{ is in } J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)\ \}) \\
= \quad & \int_{C(\mathbb{B})\times C(\mathbb{B})} \chi_{\prod_{j=0}^n B_j}(J_{\vec{t}}(x+y)) dm_{\varphi,p}^{\mathbb{B}} \times m_{\psi,q}^{\mathbb{B}}(x,y) \\
= \quad & \int_{C(\mathbb{B})\times C(\mathbb{B})} \chi_{\prod_{j=0}^n B_j}(x(t_0)+y(t_0), x(t_1)+y(t_1), \ldots, x(t_n)+y(t_n)) \\
& dm_{\varphi,p}^{\mathbb{B}} \times m_{\psi,q}^{\mathbb{B}}(x,y)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{B}}\int_{\mathbb{B}}\Big[\int_{\mathbb{B}^n}\int_{\mathbb{B}^n}\chi_{\prod_{j=0}^n B_j}(x_0+y_0, x_0+y_0+\sqrt{t_1-t_0}(x_1+y_1),\ldots,\\
&\quad x_0+y_0+\sum_{j=1}^n\sqrt{t_j-t_{j-1}}(x_j+y_j))d(\prod_{j=1}^n\omega_p)(x_1,x_2,\ldots,x_n)\\
&\quad d(\prod_{j=1}^n\omega_q)(y_1,y_2,\ldots,y_n)\Big]d\varphi(x_0)d\psi(y_0)\\
&= \int_{\mathbb{B}}\Big[\int_{\mathbb{B}^n}\chi_{\prod_{j=0}^n B_j}(z_0, z_0+\sqrt{t_1-t_0}z_1,\ldots,z_0+\sum_{j=1}^n\sqrt{t_j-t_{j-1}}z_j)\\
&\quad d(\prod_{j=1}^n\omega_{\sqrt{p^2+q^2}})(z_1,z_2,\ldots,z_n)\Big]d(\varphi*\psi)(z_0)\\
&= m^{\mathbb{B}}_{\varphi*\psi,\sqrt{p^2+q^2}}(J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)).
\end{aligned}
$$

Consider a set $K=\{E \text{ in } \mathcal{B}(C(\mathbb{B}))|m^{\mathbb{B}}_{\varphi,p}*m^{\mathbb{B}}_{\psi,q}(E)=m^{\mathbb{B}}_{\varphi*\psi,\sqrt{p^2+q^2}}(E)\}$. Then K is a monotone class containing all intervals in $C(\mathbb{B})$, so, $K=\mathcal{B}(C(\mathbb{B}))$.

Example 4.4. Let φ be a probability measure on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$ and let b^* be a non-zero element of $\mathbb{B}^*$. Let $t_{j-1}\le s\le t_j$. Then $b^*(y(s))$ is $V_\varphi^{J_{\vec{t}}}$-integrable and

$$
\begin{aligned}
(40)\qquad & E^\varphi(b^*(y(s))|J_{\vec{t}})(u_0,u_1,\ldots,u_n)\\
&= \Big[(Ba)-\int_{C(\mathbb{B})}b^*(y(s))dV_\varphi^{J_{\vec{t}}}(y)\Big](u_0,u_1,\ldots,u_n)\\
&= \frac{s-t_{j-1}}{t_j-t_{j-1}}b^*(u_j)+\frac{t_j-s}{t_j-t_{j-1}}b^*(u_{j-1}).
\end{aligned}
$$

For E in $\mathcal{B}(\mathbb{B}^{n+1})$,

$$
\begin{aligned}
&\Big[(Ba)-\int_{C(\mathbb{B})}b^*(y(s))dV_\varphi^{J_{\vec{t}}}(y)\Big](E)\\
&= \int_E\int_{C(\mathbb{B})}b^*(y(s)-[y](s)+[\vec{u}](s))dm_\varphi^{\mathbb{B}}(y)dP_{J_{\vec{t}}}(\vec{u})\\
&= \int_E\int_{B^{j+1}}b^*\Big[(x_0+\sqrt{t_1-t_0}x_1+\cdots+\sqrt{s-t_{j-1}}x_s)\\
&\quad -\frac{s-t_{j-1}}{t_j-t_{j-1}}(x_0+\sqrt{t_1-t_0}x_1+\cdots+\sqrt{s-t_{j-1}}x_s+\sqrt{t_j-s}x_j)\\
&\quad -\frac{t_j-s}{t_j-t_{j-1}}(x_0+\sqrt{t_1-t_0}x_1+\cdots+\sqrt{t_{j-1}-t_{j-2}}x_{j-1})\\
&\quad +\frac{s-t_{j-1}}{t_j-t_{j-1}}u_j+\frac{t_j-s}{t_j-t_{j-1}}u_{j-1}\Big]d\varphi(x_0)
\end{aligned}
$$

$$
\begin{aligned}
& d(\prod_{i=1}^{j+1}\omega)(x_1,x_2,\ldots,x_{j-1},x_s,x_j)dP_{J_{\vec{t}}}(\vec{u}) \\
= \; & \int_E\int_{\mathbb{B}}\int_{\mathbb{B}}\int_{\mathbb{B}}\Big[\frac{(t_j-s)(s-t_{j-1})^{\frac{1}{2}}}{t_j-t_{j-1}}b^*(x_s)+\frac{(s-t_{j-1})(t_j-s)^{\frac{1}{2}}}{t_j-t_{j-1}}b^*(x_j) \\
& +\frac{s-t_{j-1}}{t_j-t_{j-1}}b^*(u_j)+\frac{t_j-s}{t_j-t_{j-1}}b^*(u_{j-1})\Big]d\varphi(x_0) \\
& d(\prod_{j=1}^{2}\omega)(x_s,x_j)dP_{J_{\vec{t}}}(\vec{u}) \\
= \; & \int_E\Big(\frac{s-t_{j-1}}{t_j-t_{j-1}}b^*(u_j)+\frac{t_j-s}{t_j-t_{j-1}}b^*(u_{j-1})\Big)dP_{J_{\vec{t}}}(\vec{u}),
\end{aligned}
$$

so the example is proved.

Using Example 4.4 in above, we have the following theorem.

Example 4.5. Under the hypothesis in Example 4.4, if

$$F(y)=\int_{[a,b]}b^*(y(s))dm_L(s),$$

then F is $V_\varphi^{J_{\vec{t}}}$-integrable and from (40),

$$
\begin{aligned}
(41) \qquad & \Big[(Ba)-\int_{C(\mathbb{B})}F(y)dV_\varphi^{J_{\vec{t}}}(y)\Big](E) \\
= \; & \int_E\sum_{k=1}^{n}\Big[\frac{1}{2}(t_k-t_{k-1})b^*(u_k)+\frac{1}{2}(t_k-t_{k-1})b^*(u_{k-1})\Big]dP_{J_{\vec{t}}}(\vec{u})
\end{aligned}
$$

for E in $\mathcal{B}(\mathbb{B}^{n+1})$, so

$$
\begin{aligned}
& E^\varphi\Big(\int_a^b b^*(y(s))dm_L(s)|J_{\vec{t}}\Big)(u_0,u_1,\ldots,u_n) \\
= \; & \sum_{k=1}^{n}\Big[\frac{1}{2}(t_k-t_{k-1})b^*(u_k)+\frac{1}{2}(t_k-t_{k-1})b^*(u_{k-1})\Big].
\end{aligned}
$$

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