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ON OPTIMALITY AND DUALITY FOR GENERALIZED NONDIFFERENTIABLE FRACTIONAL OPTIMIZATION PROBLEMS

Moon Hee Kim and Gwi Soo Kim

Abstract. A generalized nondifferentiable fractional optimization problem (GFP), which consists of a maximum objective function defined by finite fractional functions with differentiable functions and support functions, and a constraint set defined by differentiable functions, is considered. Recently, Kim et al. [Journal of Optimization Theory and Applications **129** (2006), no. 1, 131–146] proved optimality theorems and duality theorems for a nondifferentiable multiobjective fractional programming problem (MFP), which consists of a vector-valued function whose components are fractional functions with differentiable functions and support functions, and a constraint set defined by differentiable functions. In fact if $\bar{x}$ is a solution of (GFP), then $\bar{x}$ is a weakly efficient solution of (MFP), but the converse may not be true. So, it seems to be not trivial that we apply the approach of Kim et al. to (GFP). However, modifying their approach, we obtain optimality conditions and duality results for (GFP).

1. Introduction

Many authors have introduced various concepts of generalized convexity and have obtained duality results for a fractional programming problem ([2]-[8], [11]). In [4], Kuk et al. defined the concept of (V,ρ)-invexity for vector-valued functions, which is generalization of the V-invexity concept; they proved the generalized Karush-Kuhn-Tucker sufficient optimality theorem as well as weak and strong duality for nonsmooth multi-objective programs under the (V,ρ)-invexity assumptions. Later, Kuk et al. [5] extended their results to nonsmooth multiobjective fractional programs.

In 1996, Mond and Schechter [10] obtained duality and optimality for nondifferentiable multiobjective programming problems in which the objective function contains a support function.

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Recently, Kim et al. [2] proved optimality theorem and duality theorem for a nondifferentiable multiobjective fractional programming problem (MFP), which consists of a vector-valued function whose components are fractional functions with differentiable functions and support functions, and a constraint set defined by differentiable functions.

Now we consider the following generalized fractional problem (GFP):

$$\text{(GFP)} \qquad \text{Minimize} \qquad \max\left\{ \frac{f_i(x) + s(x|C_i)}{g_i(x)} \mid i = 1, \dots, p \right\}$$
$$\text{subject to} \qquad h_j(x) \leq 0, \quad j = 1, \dots, m,$$

where $f := (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \dots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \dots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. We assume that $g_i(x) > 0, \ i = 1, \dots, p$. For each $i = 1, \dots, p$, C_i is a compact convex set of $\mathbb{R}^n$ and we define a support function with respect to C_i as follows:

$$s(x|C_i) := \max\{\langle x, y_i \rangle \mid y_i \in C_i\}.$$

Further let, $J(x) = \{j : h_j(x) = 0\}$, for any $x \in \mathbb{R}^n$ and let

$$k_i(x) = s(x|C_i), \quad i = 1, \dots, p.$$

Then, k_i is a convex function and we can prove that

$$\partial k_i(x) = \{w_i \in C_i \mid \langle w_i, x \rangle = s(x|C_i)\},$$

where ∂k_i is the subdifferential of k_i.

We recall the nondifferentaible multiobjective fractional programming problem (MFP) in [2]:

$$\text{(MFP)} \qquad \text{Minimize} \qquad \left(\frac{f_1(x) + s(x|C_1)}{g_1(x)}, \dots, \frac{f_p(x) + s(x|C_p)}{g_p(x)} \right)$$
$$\text{subject to} \qquad h_j(x) \leq 0, \quad j = 1, \dots, m,$$

where $f := (f_1, \dots, f_p) : \mathbb{R}^n \to \mathbb{R}^p$, $g := (g_1, \dots, g_p) : \mathbb{R}^n \to \mathbb{R}^p$ and $h := (h_1, \dots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable. We assume that $g_i(x) > 0, \ i = 1, \dots, p$. Further let, $S = \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, \ j = 1, \dots, m\}$.

Weakly efficient solution of (MFP) are defined as follows:

Definition. A point $\bar{x} \in S$ is a weakly efficient solution of (MFP) if there exist no other feasible point $x \in S$ such that $\frac{f_i(x)+s(x|C_i)}{g_i(x)} < \frac{f_i(\bar{x})+s(\bar{x}|C_i)}{g_i(\bar{x})}$ for all $i = 1, 2, \dots, p$.

Then $sol(\text{GFP}) \subset WEff(\text{MFP})$, where $sol(\text{GFP})$ is the set of all minimum of (GFP) and $WEff(\text{MFP})$ is the set of all weakly efficient solution of (MFP). But the converse may not be true.

Example 1.1. Let $f_1(x) = x, \ f_2(x) = x^2, \ C_1 = C_2 = \{0\}, \ g_1(x) = g_2(x) = 1$ and $h(x) = x$. Then $sol(\text{GFP}) = \{0\}$ and $WEff(\text{MFP}) = (-\infty, 0]$.

The above example says that the inclusion: $WEff(\text{MFP}) \subset sol(\text{GFP})$ may not be true. So, it seems to be not trivial that we apply the approach of Kim et al. [2] to (GFP). However, in this paper, we can modifying their approach, we obtain optimality conditions and duality results for (GFP).

We introduce the following definition due to Kuk et al. [4].

Definition. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be (V, ρ)-invex at $u \in \mathbb{R}^n$ with respect to the functions η and $\theta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if there exists $\alpha_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R}, i = 1, \dots, p$ such that for any $x \in \mathbb{R}^n$ and for all $i = 1, \dots, p$,

$$\alpha_i(x, u) \left[f_i(x) - f_i(u) \right] \geq \nabla f_i(u)\eta(x, u) + \rho_i \|\theta_i(x, u)\|^2.$$

Definition. A vector function $f : \mathbb{R}^n \to \mathbb{R}^p$ is said to be η-invex at $u \in \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ and for all $i = 1, \dots, p$,

$$f_i(x) - f_i(u) \geq \nabla f_i(u)\eta(x, u).$$

We recall the following theorem due to Kim et al. [2]

Theorem 1.1. *Assume that f and g are vector-valued differentiable functions defined on $\mathbb{R}^n$ and $f(x) + \langle w, x\rangle \geq 0$, $g(x) > 0$ for all $x \in \mathbb{R}^n$. If $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ are (V, ρ)-invex at x_0, then $\frac{f(\cdot)+\langle w,\cdot\rangle}{g(\cdot)}$ is (V, ρ)-invex at x_0, where*

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \left(\frac{1}{g_i(x_0)}\right)^{\frac{1}{2}} \theta(x, x_0),$$

that is, for all i,

$$\begin{aligned} &\alpha_i(x, x_0) \left[\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right] \\ &\geq \frac{g_i(x_0)}{g_i(x)} \left[\nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) \eta(x, x_0) + \rho_i \| \left(\frac{1}{g_i(x_0)} \right)^{\frac{1}{2}} \theta_i(x, x_0) \|^2 \right]. \end{aligned}$$

2. Optimality conditions

Now, we establish the Kuhn-Tucker necessary and sufficient conditions for a solution of (GFP).

Theorem 2.1 (Kuhn-Tucker Necessary Optimality Theorem). *If x_0 is a solution of (GFP), and assume that $0 \notin co\{\nabla h_j(x_0) \mid j \in J(x_0)\}$, then there exist $\lambda_i \geq 0$, $i \in I(x_0) := \{i \mid \max\left\{\frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)} \mid i = 1, \dots, p\right\}\}$,*

$\sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1,\dots,m$ *and* $w_i \in C_i,\ i \in I(x_0)$ *such that*

$$\begin{aligned}
&\sum_{i\in I(x_0)} \lambda_i \nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,\\
&\langle w_i, x_0\rangle = s(x_0|C_i),\\
&\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\end{aligned}$$

Proof. Let $\varphi_i(x) = \frac{f_i(x)+s(x|C_i)}{g_i(x)}, i = 1,\dots,p$. Let x_0 be a solution of (GFP) and let $I(x_0) = \{i \mid \max\{\varphi_i(x_0) \mid i = 1,\dots,p\}\}$. Then by Proposition 2.3.12 in [1] and Corollary 5.1.8 in [9], there exist $\mu_j \geq 0,\ j = 1,\dots,m,$

$$\begin{aligned}
&0 \in \mathrm{co}\{\partial^c \varphi_i(x_0) \mid i \in I(x_0)\} + \sum_{j=1}^{m} \mu_j \partial^c h_j(x_0)\\
&\text{and}\ \ \mu_j h_j(x_0) = 0.
\end{aligned}$$

Thus there exist $\lambda_i \geq 0,\ i \in I(x_0),\ \sum_{i\in I(x_0)} \lambda_i = 1$ such that

$$\begin{aligned}
&0 \in \sum_{i\in I(x_0)} \lambda_i \partial^c \varphi_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0)\\
&\text{and}\ \ \mu_j h_j(x_0) = 0.
\end{aligned} \tag{2.1}$$

By Proposition 2.3.14 in [1],

$$\partial^c \varphi_i(x_0) = \frac{g_i(x_0)(\nabla f_i(x_0) + \partial s(x_0|C_i)) - (f_i(x_0) + s(x_0|C_i))\nabla g_i(x_0)}{g_i^2(x_0)}.$$

Since

$$\begin{aligned}
\partial^c \varphi_i(x_0) &= \left\{ \frac{g_i(x_0)(\nabla f_i(x_0) + w_i) - (f_i(x_0) + \langle w_i, x_0\rangle)\nabla g_i(x_0)}{g_i^2(x_0)} \mid w_i \in C_i, \right.\\
&\qquad \langle w_i, x_0\rangle = s(x_0|C_i),\ i \in I(x_0)\}\\
&= \left\{ \nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) | w_i \in C_i, \langle w_i, x_0\rangle = s(x_0|C_i),\ i \in I(x_0) \right\}
\end{aligned}$$

and hence from (2.1), there exist $\lambda_i \geq 0,\ i \in I(x_0),\ \sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1,\dots,m$ and $w_i \in C_i,\ i \in I(x_0)$ such that

$$\begin{aligned}
&\sum_{i\in I(x_0)} \lambda_i \nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,\\
&\langle w_i, x_0\rangle = s(x_0|C_i),\\
&\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.
\end{aligned}$$

□

Theorem 2.2 (**Kuhn-Tucker Sufficient Optimality Theorem**). *Let x_0 be a feasible solution of (GFP). Suppose that there exist $\lambda_i \geq 0,\ i \in I(x_0)$, $\sum_{i\in I(x_0)} \lambda_i = 1,\ \mu_j \geq 0,\ j = 1, \ldots, m$ and $w_i \in C_i,\ i \in I(x_0)$ such that*

$$\sum_{i\in I(x_0)} \lambda_i \nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0, \tag{2.2}$$

$$\langle w_i, x_0\rangle = s(x_0|C_i),$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

If $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ are (V, ρ)-invex at x_0, and h is η-invex at x_0 with respect to the same η, and $\sum_{i\in I(x_0)} \lambda_i\rho_i\|\bar{\theta}_i(x, x_0)\|^2 \geq 0$, then x_0 is a solution of (GFP).

Proof. Suppose that x_0 is not a solution of (GFP). Then there exist a feasible solution x of (GFP) such that

$$\max_{1\leq i\leq p} \frac{f_i(x) + s(x|C_i)}{g_i(x)} < \max_{1\leq i\leq p} \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0)}.$$

Then

$$\frac{f_i(x) + s(x|C_i)}{g_i(x)} < \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0)} \quad \text{for all } i \in I(x_0).$$

Since $\langle w_i, x_0\rangle = s(x_0|C_i)$ and $w_i \in C_i$, we have for all $i \in I(x_0)$,

$$\begin{aligned} \frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} &\leq \frac{f_i(x) + s(x|C_i)}{g_i(x)} \\ &< \frac{f_i(x_0) + s(x_0|C_i)}{g_i(x_0)} \\ &= \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \end{aligned}$$

and hence $\bar{\alpha}_i(x, x_0) > 0$,

$$\bar{\alpha}_i(x, x_0) \left[\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right] < 0.$$

By the (V, ρ)-invexity of $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ at x_0, and by Theorem 1.1, we have

$$\nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) \eta(x, x_0) + \rho_i\|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Hence, we have

$$\sum_{i\in I(x_0)} \lambda_i \nabla \left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)} \right) \eta(x, x_0) + \sum_{i\in I(x_0)} \lambda_i\rho_i\|\bar{\theta}_i(x, x_0)\|^2 < 0.$$

Since $\sum_{i\in I(x_0)} \lambda_i\rho_i\|\bar{\theta}_i(x,x_0)\|^2 \geq 0$,

$$\sum_{i\in I(x_0)} \lambda_i \nabla\left(\frac{f_i(x_0)+\langle w_i, x_0\rangle}{g_i(x_0)}\right)\eta(x,x_0) < 0$$

and so, it follows from (2.2) that

$$\sum_{j=1}^{m} \mu_j \nabla h_j(x_0)\eta(x,x_0) > 0.$$

Then, by the η-invexity of h, we have

$$\sum_{j=1}^{m} \mu_j h_j(x) - \sum_{j=1}^{m} \mu_j h_j(x_0) > 0.$$

Since $\sum_{j=1}^{m} \mu_j h_j(x_0) = 0$, we have $\sum_{j=1}^{m} \mu_j h_j(x) > 0$, which is a contradiction since $\mu_j \geq 0,\ j=1,\dots,m$ and x is a feasible solution of (GFP). Consequently, x_0 is a solution of (GFP). $\square$

3. Duality theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):

(DGFP) Maximize $\max\left\{\frac{f_i(u)+s(u|C_i)}{g_i(u)} \mid i=1,\dots,p\right\}$

(3.1) subject to
$$\sum_{i\in I(u)} \lambda_i\nabla\left(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right) + \sum_{j=1}^{m}\mu_j\nabla h_j(u) = 0,$$
$$w_i \in C_i,\ \langle w_i,u\rangle = s(u|C_i),\ i\in I(u)$$
$$\sum_{j=1}^{m}\mu_j h_j(u) = 0,$$
$$\lambda_i \geq 0,\ i\in I(u),\ \sum_{i\in I(u)}\lambda_i = 1,\ \mu_j\geq 0,\ j=1,\dots,m.$$

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

Theorem 3.1 (Weak Duality). *Let x be a feasible for (GFP) and let (u,λ,μ, w) be feasible for (DGFP). Assume that $f(\cdot)+\langle w,\cdot\rangle$ and $-g(\cdot)$ are (V,ρ)-invex at u, and let h is η-invex at u with respect to the same η, and*

$$\sum_{i\in I(u)} \lambda_i\rho_i\|\bar{\theta}_i(x,u)\|^2 \geq 0.$$

Then the following holds:

$$\max\left\{\frac{f_i(x)+s(x|C_i)}{g_i(x)} \mid i=1,\dots,p\right\} \geq \max\left\{\frac{f_i(u)+s(u|C_i)}{g_i(u)} \mid i=1,\dots,p\right\}.$$

Proof. Let x be any feasible for (GFP) and let (u, λ, μ, w) be any feasible for (DGFP). Then we have

$$\sum_{j=1}^{m} \mu_j h_j(x) \leq 0 \leq \sum_{j=1}^{m} \mu_j h_j(u).$$

By the η-invexity of $h_j(u),\ j = 1, \ldots, m$, we have

$$\sum_{j=1}^{m} \mu_j \nabla h_j(u) \eta(x, u) \leq 0.$$

Using (3.1), we obtain

$$\sum_{i \in I(u)} \lambda_i \nabla \left(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \right) \eta(x, u) \geq 0. \tag{3.2}$$

Now suppose that

$$\max \left\{ \frac{f_i(x) + s(x|C_i)}{g_i(x)} \mid i = 1, \ldots, p \right\} < \max \left\{ \frac{f_i(u) + s(u|C_i)}{g_i(u)} \mid i = 1, \ldots, p \right\}.$$

Then

$$\frac{f_i(x) + s(x|C_i)}{g_i(x)} < \frac{f_i(u) + s(u|C_i)}{g_i(u)} \quad \text{for all } i \in I(u).$$

Since $\langle w_i, u \rangle = s(u|C_i)$, we have for all $i \in I(u)$,

$$\frac{f_i(x) + \langle w_i, x \rangle}{g_i(x)} < \frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)}.$$

By Theorem 1.1, we have,

$$\begin{aligned} 0 &> \bar{\alpha}_i(x, u) \left[\frac{f_i(x) + \langle w_i, x \rangle}{g_i(x)} - \frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \right] \\ &\geq \nabla \left(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \right) \eta(x, u) + \rho_i \|\bar{\theta}_i(x, u)\|^2. \end{aligned}$$

By using $\lambda_i \geq 0,\ i \in I(u)$, we have,

$$\sum_{i \in I(u)} \lambda_i \nabla \left(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \right) \eta(x, u) + \sum_{i \in I(u)} \lambda_i \rho_i \|\bar{\theta}_i(x, u)\|^2 < 0.$$

Since $\sum_{i \in I(u)} \lambda_i \rho_i \|\bar{\theta}_i(x, u)\|^2 \geq 0$, we have

$$\sum_{i \in I(u)} \lambda_i \nabla \left(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \right) \eta(x, u) < 0,$$

which contradicts (3.2). Hence the result holds. □

Now we give a strong duality theorem which holds between (GFP) and (DGFP).

Theorem 3.2 (**Strong Duality).** *If $\bar{x}$ be a solution of (GFP) and suppose that $0 \notin co\{\nabla h_j(\bar{x}) \mid j \in J(\bar{x})\}$. Then there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w})$ is feasible for (DGFP). Moreover if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w})$ is a solution of (DGFP).*

Proof. By Theorem 2.1, there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{\mu} \in \mathbb{R}^m$ and $\bar{w}_i \in C_i$, $i \in I(\bar{x})$, such that

$$\begin{aligned}
&\sum_{i \in I(\bar{x})} \bar{\lambda}_i \nabla \left(\frac{f_i(\bar{x}) + \langle \bar{w}_i, \bar{x} \rangle}{g_i(\bar{x})} \right) + \sum_{j=1}^{m} \bar{\mu}_j \nabla h_j(\bar{x}) = 0, \\
&\langle \bar{w}_i, \bar{x} \rangle = s(\bar{x}|C_i), \\
&\sum_{j=1}^{m} \mu_j h_j(\bar{x}) = 0, \\
&\lambda_i \geq 0, \ i \in I(\bar{x}), \ \sum_{i \in I(\bar{x})} \lambda_i = 1.
\end{aligned}$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 3.1),

$$\max \left\{ \frac{f_i(\bar{x}) + s(\bar{x}|C_i)}{g_i(\bar{x})} \mid i = 1, \ldots, p \right\} \geq \max \left\{ \frac{f_i(u) + s(u|C_i)}{g_i(u)} \mid i = 1, \ldots, p \right\}$$

for any (DGFP) feasible solution (u, λ, μ, w). Hence $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w})$ is a solution of (DGFP). □

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Moon Hee Kim
Department of Multimedia Engineering
Tongmyong University
Pusan 608-711, Korea
E-mail address: mooni@tu.ac.kr

Gwi Soo Kim
Department of Applied Mathematics
Pukyong National University
Pusan 608-737, Korea
E-mail address: gwisoo1103@hanmail.net