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SOME RESULTS ON ASYMPTOTIC BEHAVIORS OF RANDOM SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

Tran Loc Hung and Tran Thien Thanh

Abstract. Let $\{X_n,\ n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables (r.vs.), defined on a probability space $(\Omega, \mathcal{A}, P)$, and let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued r.vs., defined on the same probability space $(\Omega, \mathcal{A}, P)$. Furthermore, we assume that the r.vs. $N_n,\ n \geq 1$ are independent of all r.vs. $X_n,\ n \geq 1$. In present paper we are interested in asymptotic behaviors of the random sum

$$S_{N_n} = X_1 + X_2 + \cdots + X_{N_n}, \quad S_0 = 0,$$

where the r.vs. $N_n,\ n \geq 1$ obey some defined probability laws. Since the appearance of the Robbins's results in 1948 ([8]), the random sums S_{N_n} have been investigated in the theory probability and stochastic processes for quite some time (see [1], [4], [2], [3], [5]).

Recently, the random sum approach is used in some applied problems of stochastic processes, stochastic modeling, random walk, queue theory, theory of network or theory of estimation (see [10], [12]).

The main aim of this paper is to establish some results related to the asymptotic behaviors of the random sum S_{N_n}, in cases when the $N_n, n \geq 1$ are assumed to follow concrete probability laws as Poisson, Bernoulli, binomial or geometry.

1. Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables (r.vs.), defined on a probability space $(\Omega, \mathcal{A}, P)$ and let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued r.vs., defined on the same probability space $(\Omega, \mathcal{A}, P)$. Furthermore, we assume that the r.vs. $N_n,\ n \geq 1$ are independent of all i.i.d.r.vs. $X_n,\ n \geq 1$. From now on, the random sum is defined by

$$S_{N_n} = X_1 + X_2 + \cdots + X_{N_n}, \quad S_0 = 0. \tag{1}$$

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Since the appearance of the Robbins's results in 1948 (see [8] for more details), the random sums S_{N_n} have been investigated in the theory probability and stochastic processes for quite some time (see [5], [1], [2] and [12] for the complete bibliography).

In the classical theory of limit theorems we can consider the non-random index n in the sum $S_n = X_1 + X_2 + \cdots + X_n$ as a random variable degenerated at a point n. Therefore, the replacement of the number n of the sum S_n by the positive integer-valued r.vs. is natural. In simple terms, a random sum is a sum of a random number of r.vs.. The number of the terms $N_n,\ n \geq 1$ in the sum, as well as the individual terms, can obey various probability laws. In stochastic theory, $N_n,\ n \geq 1$ are often assumed to follow Poisson law or geometric law. In general, the r.vs. $N_n,\ n \geq 1$ should satisfy any conditions. To illustrate this, we can recall three types of classical conditions for the r.vs. N_n as follows

$$E(N_n) \to +\infty \quad \text{as} \quad n \to \infty, \tag{2}$$

$$\frac{N_n}{n} \xrightarrow{\mathbb{P}} 1 \quad \text{as} \quad n \to \infty, \tag{3}$$

or

$$N_n \xrightarrow{\mathbb{P}} \infty \quad \text{as} \quad n \to \infty. \tag{4}$$

The condition (2) was used in well-known results of H. Robbins's in 1948 (see details in [8]). The condition (3) was applied in Feller's theorem for random sums (cf. [1]), while the last condition in (4) was presented in various papers like [4], [5], [9], [6], [7],.... It is to be noticed that the conditions (2), (3) and (4) are in following relationship

$$(3) \Rightarrow (4) \Rightarrow (2).$$

This paper is organized as follows. In Section 2 we present the main results related to the asymptotic behaviors of random sums in (1), when the r.vs. $N_n, n \geq 1$ belong to some discrete probability laws. The proofs of these main results are presented in Section 3.

2. Main results

From now on, the random variable of standard normal law $\mathcal{N}(0,1)$ will be denoted by X^*, the notation $\xrightarrow{d}$ will mean the convergence in distribution and $\xrightarrow{\mathbb{P}}$ will denote the convergence in probability.

Theorem 2.1. *Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d.r.vs., $X_j \sim Bernoulli(p)$, $p \in (0,1),\ j = 1, 2, \ldots, n$. Moreover, suppose that $\{N_n,\ n \geq 1\}$ is a sequence of positive integer-valued r.vs. independent on all $X_j,\ j = 1, 2, \ldots, n$. Then,*

(i) $S_{N_n} \sim Poisson(\lambda p)$, *if* $N_n \sim Poisson(\lambda),\ \lambda > 0,\ n \geq 1$.

(ii) $S_{N_n} \sim Geometry\left(\frac{q}{p+q-pq}\right)$, *if* $N_n \sim Geometry(q),\ q \in (0,1),\ p + q = 1,\ n \geq 1$ *with* $\mathbb{P}(N_n = k) = q(1-q)^k,\ k = 0, 1, \ldots$

(iii) $S_{N_n} \sim Binomial(n, pq)$, *if* $N_n \sim Binomial(n, q)$, $q \in (0,1)$, $p + q = 1$, $n \geq 1$.

It is well known that the sum of independent r.vs. from Bernoulli law will belong to the same law. But in cases (i) and (ii) the random sums of independent r.vs. of Bernoulli law will not obey the Bernoulli. The final distributions of the random sums S_{N_n} depend on the distribution of r.vs. N_n, $n \geq 1$. The same concludes would be found in B. Gnedenko's or V. Kruglov's and V. Korolev's papers, in cases when the r.vs. X_j, $j = 1, 2, \ldots, n$ were independent standard normal distributed while the r.vs. N_n, $n \geq 1$ were uniformly distributed (see for more details in [2], [3] and [5]). The conclusion in (iii) is a very interesting result when the random sum S_{N_n} and the r.vs. $N_n, n \geq 1$ are identically distributed. Furthermore, we can receive the Poisson's Approximation Theorem by using the Theorem 2.1(i) as follows

Theorem 2.2. *Assume that for each* $n = 1, 2, \ldots,$ $\{X_{nk},\ k = 1, 2, \ldots, n\}$ *be a sequence of independent and identically Bernoulli distributed r.vs. with parameter* $p_n \in (0,1)$ *and* $p_n \to 0$, $np_n \to \lambda$ $(\lambda > 0)$ *as* $n \to \infty$. *Then, as* $n \to \infty$,

$$S_n = \sum_{k=1}^{n} X_{nk} \xrightarrow{d} Poisson(\lambda).$$

Theorem 2.3. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of i.i.d.r.vs.. Suppose that* $\{N_n,\ n \geq 1\}$ *is a sequence of positive integer-valued r.vs. independent on all* X_j, $j = 1, 2, \ldots, n$. *Furthermore, assume that* $N_n \sim Geometry(p)$, $n \geq 1$. *Then, we have*

(i) $S_{N_n} \sim Exp(\lambda p)$, *when* $X_j \sim Exp(\lambda)$, $j = 1, 2, \ldots, n$.

(ii) $S_{N_n} \sim Geometry(pq)$, *when* $X_j \sim Geometry(q)$, $j = 1, 2, \ldots, n$.

Theorem 2.4. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of i.i.d.r.vs. such that* $E(X_1) = 0$, $Var(X_1) = 1$. *Moreover, suppose that* $\{N_n,\ n \geq 1\}$ *is a sequence of r.vs. belonging to Poisson law* $Poisson(\lambda_n)$ *and independent of all* X_n, $n \geq 1$. *If*

$$\frac{\lambda_n}{n} \to 1 \quad as \quad n \to \infty,$$

then

$$\frac{S_{N_n}}{\sqrt{n}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Theorem 2.5. *Let* $\{X_n,\ n \geq 1\}$ *be a sequence of i.i.d.r.vs. such that* $E(X_1) = \mu_1$, $E(X_1^2) = \mu_2$. *Suppose that* $\{N_n,\ n \geq 1\}$ *is a sequence of r.vs. from Poisson law* $Poisson(\lambda_n)$ *and independent of all* X_n, $n \geq 1$. *Assume that*

$$\frac{\lambda_n}{n} \to 1 \quad as \quad n \to \infty$$

and

$$\sqrt{n}\left(\frac{\lambda_n}{n} - 1\right) \to 0 \quad as \quad n \to \infty.$$

Then

$$\frac{S_{N_n} - n\mu_1}{\sqrt{n\mu_2}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Theorem 2.6. *Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d.r.vs. such that $E(X_1) < \infty,\ 0 < E(X_1^2) < \infty$. Furthermore, assume that the $\{N_n,\ n \geq 1\}$ is a sequence of r.vs. from Poisson law Poisson(λ_n) and independent of all $X_n,\ n \geq 1$. If*

$$\lambda_n \to +\infty \quad as \quad n \to \infty,$$

then

$$\frac{S_{N_n} - \lambda_n E(X_1)}{\sqrt{\lambda_n E(X_1^2)}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Theorem 2.7. *Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d.r.vs. such that $E(X_1) = 0,\ Var(X_1) = 1$. Moreover, suppose that the $\{N_n, n \geq 1\}$ is a sequence of r.vs. from Binomial law $B(n,p)$ and they are independent of all $X_n,\ n \geq 1$. Then*

$$\frac{S_{N_n}}{\sqrt{E(N_n)}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Theorem 2.8. *Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d.r.vs. with finite moments. Suppose that $\{N_n,\ n \geq 1\}$ is a sequence of r.vs. from Binomial law $B(n,p)$ and they are independent of all $X_n,\ n \geq 1$. Then*

$$\frac{S_{N_n} - E(S_{N_n})}{\sqrt{Var(S_{N_n})}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Theorem 2.9. *Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d.r.vs. such that $E(X) = 0$ and $Var(X) = 1$. We assume that $\{N_n, n \geq 1\}$ is a sequence of random variables of Geometry law Geometry(p_n) and they are independent of all $X_n,\ n \geq 1$. If*

$$p_n \to 0 \quad as \quad n \to \infty,$$

then

$$\frac{S_{N_n}}{\sqrt{E(N_n)}} \xrightarrow{d} Z \quad as \quad n \to \infty,$$

where Z does not belong to $\mathcal{N}(0,1)$.

Remark 1. It is worth pointing out that from Theorem 2.9, we have $E(N_n) = 1/p_n \to \infty$, as $n \to \infty$ but the random sum S_{N_n} does not obey central limit theorem. Therefore, the condition $E(N_n) \to \infty$, as $n \to \infty$ is not sufficient for satisfying the central limit theorem for random sum S_N.

Theorem 2.10. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent standard normal distributed r.vs.. Suppose that $\{N_n,\ n \geq 1\}$ is a sequence of positive integer-valued r.vs. such that the condition* (2) *is true, and*

$$\frac{E|N_n - E(N_n)|}{E(N_n)} \to 0 \quad as \quad n \to \infty. \tag{5}$$

Then

$$\frac{S_{N_n}}{\sqrt{E(N_n)}} \xrightarrow{d} X^* \quad as \quad n \to \infty.$$

Remark 2. Based on the fact that $E|N_n - E(N_n)| \le \sqrt{Var(N_n)}$ we can conclude that the condition $\frac{Var(N_n)}{E(N_n)} \to 0$, as $n \to \infty$ in Robbins's results [8] will be stronger than the condition (5).

Theorem 2.11. *Let $\{X_n,\ n \ge 1\}$ be a sequence of i.i.d.r.vs. such that $E|X_1| < \infty,\ E(X_1) = \mu$ and suppose that $\{N_n,\ n \ge 1\}$ is a sequence of positive integer-valued r.vs. independent of all $X_n,\ n \ge 1$. Assume that the condition* (3) *true, then*

$$\frac{S_{N_n}}{n} \xrightarrow{\mathbb{P}} \mu \quad as \quad n \to \infty.$$

Theorem 2.12. *Let $\{X_n,\ n \ge 1\}$ be a sequence of i.i.d.r.vs., and assume that $N_n \sim Binomial(n, p_n),\ n \ge 1$, satisfying $np_n \to \lambda$, as $n \to \infty$. Then,*

$$S_{N_n} \xrightarrow{d} S_N \quad as \quad n \to \infty,$$

where

$$N \sim Poisson(\lambda).$$

3. Proofs

Proof of Theorem 2.1. (i) It is clear that the generating function of r.vs. N_n, $n \ge 1$ is $g(t) = e^{\lambda(t-1)}$ and characteristic function of the r.vs. $X_n,\ n \ge 1$ is $\varphi(t) = 1 + p(e^{it} - 1)$. Then, the characteristic function of random sum S_{N_n} is given by

$$\psi(t) = g(\varphi(t)) = e^{\lambda p(e^{it}-1)}.$$

Thus $S_{N_n} \sim Poisson(\lambda p)$.

(ii) Let us denote $g(t) = \frac{q}{1-(1-q)t}$ the generating function of the r.vs. N_n and let the characteristic function of the r.vs. $X_n,\ n \ge 1$ be $\varphi(t) = 1 + p(e^{it} - 1)$. Then, the characteristic function of random sum S_{N_n} can be calculated by

$$\psi(t) = g(\varphi(t)) = \frac{\frac{q}{p+q-pq}}{1 - \left(1 - \frac{q}{p+q-pq}\right)e^{it}}.$$

Therefore $S_{N_n} \sim Geometry(\frac{q}{p+q-pq})$.

(iii) Let $g(t) = [1 + q(t-1)]^n$ and $\varphi(t) = 1 + p(e^{it} - 1)$ be the generating function and characteristic function of r.vs. $N_n,\ n \ge 1$ and $X_j,\ j = 1, 2, \dots$, respectively. Then, characteristic function of random sum S_{N_n} will be defined by

$$\psi(t) = g(\varphi(t)) = [1 + pq(e^{it} - 1)]^n.$$

By this way, $S_{N_n} \sim Binomial(n, pq)$. $\square$

Proof of Theorem 2.2. Let consider $N_n \sim Poisson(n)$. According to Theorem 2.1(i) we have $S_{N_n} \sim Poisson(np_n)$. Then,

$$\mathbb{P}(S_{N_n} = k) = \frac{e^{-np_n}(np_n)^k}{k!} \to \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{as} \quad n \to \infty,$$

or $S_{N_n} \xrightarrow{d} Poisson(\lambda)$. Therefore, we must only to show that

$$\Delta_n(t) = |\varphi_{S_{N_n}}(t) - \varphi_{S_n}(t)| \to 0.$$

We contend that

$$\Delta_n(t) = |e^{np_n(e^{it}-1)} - [1 - p_n(1 - e^{it})]^n| \leq n|e^{p_n(e^{it}-1)} - 1 - p_n(e^{it} - 1)|.$$

Clearly, $\mathrm{Re}(p_n(e^{it} - 1)) \leq 0$, applying the inequality $|e^{\alpha} - 1 - \alpha| \leq \frac{|\alpha|^2}{2}$ with $\mathrm{Re}(\alpha) \leq 0$, we obtain

$$\Delta_n(t) \leq \frac{n|p_n(e^{it} - 1)|^2}{2} = np_n \frac{p_n|e^{it} - 1|^2}{2} \to 0 \quad \text{as} \quad n \to \infty.$$

This completes the proof. □

Proof of Theorem 2.3. (i) Let us denote the generating function $g(t) = \frac{pt}{1-(1-p)t}$ and the characteristic function $\varphi(t) = \frac{\lambda}{\lambda - it}$ of $N_n,\ n \geq 1$ and $X_j,\ j = 1, 2, \ldots$ respectively. Then, the characteristic function of random sum S_{N_n} is given by

$$\psi(t) = g(\varphi(t)) = \frac{\lambda p}{\lambda p - it}.$$

We conclude that $S_{N_n} \sim Exp(\lambda p)$.

(ii) Let $g(t) = \frac{pt}{1-(1-p)t}$ be the generating function of $N_n,\ n \geq 1$ and denote $\varphi(t) = \frac{qe^{it}}{1-(1-q)e^{it}}$ the characteristic function of $X_j,\ j = 1, 2, \ldots$. Then, the characteristic function of random sum $S_{N_n},\ n \geq 1$ will be given by

$$\psi(t) = g(\varphi(t)) = \frac{pqe^{it}}{1 - (1 - pq)e^{it}}.$$

Hence, $S_{N_n} \sim Geometry(pq)$. □

Proof of Theorem 2.4. Denote $g_n(t) = e^{\lambda_n(t-1)}$ the generating function of N_n, $n \geq 1$ and $\varphi,\ \psi_n$ are characteristic functions of $X_1,\ \frac{S_{N_n}}{\sqrt{n}}$, respectively. Then,

$$\psi_n(t) = \varphi_{S_{N_n}}\left(\frac{t}{\sqrt{n}}\right) = g_n\left(\varphi\left(\frac{t}{\sqrt{n}}\right)\right) = e^{\lambda_n\left[\varphi\left(\frac{t}{\sqrt{n}}\right) - 1\right]},$$

where $\varphi\left(\frac{t}{\sqrt{n}}\right) = 1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)$.

Hence

$$\ln \psi_n(t) = \lambda_n\Big[-\frac{t^2}{2n} + o\Big(\frac{1}{n}\Big)\Big] = -\frac{\lambda_n}{n} \cdot \frac{t^2}{2} + o\Big(\frac{\lambda_n}{n}\Big).$$

According to assumptions, as $n \to \infty$, we have then $\ln \psi_n(t) \to -\frac{t^2}{2}$. This finishes the proof. □

Proof of Theorem 2.5. Let $g_n(t) = e^{\lambda_n(t-1)}$ be generating function of $N_n,\ n \geq 1$ and denote $\varphi,\ \psi_n$ are characteristic functions of $X_1,\ \frac{S_{N_n}-n\mu_1}{\sqrt{n\mu_2}}$, respectively. Then,

$$\psi_n(t) = e^{-it\frac{n\mu_1}{\sqrt{n\mu_2}}} \cdot \varphi_{S_{N_n}}\left(\frac{t}{\sqrt{n\mu_2}}\right) = e^{-it\frac{n\mu_1}{\sqrt{n\mu_2}}} \cdot e^{\lambda_n\left[\varphi\left(\frac{t}{\sqrt{n\mu_2}}\right)-1\right]},$$

here $\varphi(\frac{t}{\sqrt{n\mu_2}}) = 1 + \frac{it\mu_1}{\sqrt{n\mu_2}} - \frac{t^2}{2n} + o(\frac{1}{n})$.

From this

$$\begin{aligned}\ln\psi_n(t) &= -it\frac{n\mu_1}{\sqrt{n\mu_2}} + it\frac{\lambda_n\mu_1}{\sqrt{n\mu_2}} - \frac{\lambda_n}{n}\cdot\frac{t^2}{2} + o\Big(\frac{\lambda_n}{n}\Big)\\ &= it\frac{\mu_1}{\sqrt{\mu_2}}\sqrt{n}\Big(\frac{\lambda_n}{n} - 1\Big) - \frac{\lambda_n}{n}\cdot\frac{t^2}{2} + o\Big(\frac{\lambda_n}{n}\Big).\end{aligned}$$

On account of assumptions, when $n \to \infty$, we get $\ln\psi_n(t) \to -\frac{t^2}{2}$. The proof is straightforward. □

Proof of Theorem 2.6. Putting $\mu_1 = E(X_1),\ \mu_2 = E(X_1^2)$ and let $g_n(t) = e^{\lambda_n(t-1)}$ be a generating function of $N_n,\ n \geq 1$ and $\varphi,\ \psi_n$ be are characteristic functions of $X_1,\ \frac{S_{N_n}-\lambda_n\mu_1}{\sqrt{\lambda_n\mu_2}}$, respectively. Then,

$$\psi_n(t) = e^{-it\frac{\lambda_n\mu_1}{\sqrt{\lambda_n\mu_2}}} \cdot \varphi_{S_{N_n}}\left(\frac{t}{\sqrt{\lambda_n\mu_2}}\right) = e^{-it\frac{\lambda_n\mu_1}{\sqrt{\lambda_n\mu_2}}} \cdot e^{\lambda_n\left[\varphi\left(\frac{t}{\sqrt{\lambda_n\mu_2}}\right)-1\right]},$$

where $\varphi\left(\frac{t}{\sqrt{\lambda_n\mu_2}}\right) = 1 + \frac{it\mu_1}{\sqrt{\lambda_n\mu_2}} - \frac{t^2}{2\lambda_n} + o\Big(\frac{1}{\lambda_n}\Big)$.

Hence

$$\ln\psi_n(t) = \lambda_n\Big[-\frac{t^2}{2\lambda_n} + o\Big(\frac{1}{\lambda_n}\Big)\Big] = -\frac{t^2}{2} + o(1).$$

If $n \to \infty$, then $\ln\psi_n(t) \to -\frac{t^2}{2}$. We obtain the proof. □

Proof of Theorem 2.7. It is easy to see that the $N_n,\ n \geq 1$ have the generating function $g_n(t) = [1+p(t-1)]^n$ and $E(N_n) = np$. Let $\varphi,\ \psi_n$ be are characteristic functions of X_1 and $\frac{S_{N_n}}{\sqrt{E(N_n)}}$, respectively. Then,

$$\psi_n(t) = g_n\left(\varphi\left(\frac{t}{\sqrt{np}}\right)\right) = \left(1 + p\left[\varphi\left(\frac{t}{\sqrt{np}}\right) - 1\right]\right)^n,$$

where $\varphi\left(\frac{t}{\sqrt{np}}\right) = 1 - \frac{t^2}{2np} + o\left(\frac{1}{n}\right)$.

Therefore,

$$\psi_n(t) = \Big[1 - \frac{t^2}{2n} + o\Big(\frac{1}{n}\Big)\Big]^n.$$

If $n \to \infty$, then $\psi_n(t) \to e^{-\frac{t^2}{2}}$. We have the complete proof. □

Proof of Theorem 2.8. Let us denote $\mu_1 = E(X_1);\ \mu_2 = E(X_1^2);\ \delta = \mu_2 - \mu_1^2 p$. Then,

$$E(S_{N_n}) = np\mu_1, \quad Var(S_{N_n}) = np(\mu_2 - \mu_1^2 p) = np\delta.$$

Clearly, $g_n(t) = [1 + p(t-1)]^n$ is the generating function of $N_n, n \geq 1$. We denote φ and ψ_n the characteristic functions of X_1 and $\frac{S_{N_n} - np\mu_1}{\sqrt{np\delta}}$, respectively. Then,

$$\psi_n(t) = e^{-it\mu_1\sqrt{np/\delta}} \cdot \varphi_{S_{N_n}}\left(\frac{t}{\sqrt{np\delta}}\right) = e^{-it\mu_1\sqrt{np/\delta}}\left(1 + p\left[\varphi\left(\frac{t}{\sqrt{np\delta}}\right) - 1\right]\right)^n,$$

where $\varphi\left(\frac{t}{\sqrt{np\delta}}\right) = 1 + i\mu_1\frac{t}{\sqrt{np\delta}} - \frac{t^2\mu_2}{2np\delta} + o\left(\frac{1}{n}\right)$.

Therefore

$$\begin{aligned}
\psi_n(t) &= e^{-it\mu_1\sqrt{np/\delta}}\Big[1 + it\mu_1\sqrt{\frac{p}{n\delta}} - \frac{t^2(\delta + \mu_1^2 p)}{2n\delta} + o\Big(\frac{1}{n}\Big)\Big]^n \\
&= e^{-it\mu_1\sqrt{np/\delta}}\Big[1 + it\mu_1\sqrt{\frac{p}{n\delta}} - \frac{t^2\mu_1^2 p}{2n\delta} - \frac{t^2}{2n} + o\Big(\frac{1}{n}\Big)\Big]^n \\
&= e^{-it\mu_1\sqrt{np/\delta}}\Big[e^{it\mu_1\sqrt{p/(n\delta)}} - \frac{t^2}{2n} + o\Big(\frac{1}{n}\Big)\Big]^n \\
&= \Big[1 - \frac{t^2}{2n}e^{-it\mu_1\sqrt{p/(n\delta)}} + o\Big(\frac{1}{n}\Big)\Big]^n.
\end{aligned}$$

By letting $n \to \infty$, $\psi_n(t) \to e^{-\frac{t^2}{2}}$. We have the proof. □

Proof of Theorem 2.9. Let $g_n(t) = \frac{p_n t}{1-(1-p_n)t}$ be a generating function of the random sum $N_n,\ n \geq 1$ and suppose that $E(N_n) = \frac{1}{p_n}$. We denote φ and ψ_n the characteristic functions of X_1 and $\sqrt{p_n}S_{N_n}$, respectively. Then,

$$\psi_n(t) = g_n(\varphi(t\sqrt{p_n})) = \frac{p_n\varphi(t\sqrt{p_n})}{1 - (1-p_n)\varphi(t\sqrt{p_n})},$$

where $\varphi(t\sqrt{p_n}) = 1 - p_n t^2/2 + o(p_n)$.

Therefore

$$\psi_n(t) = \frac{1 - p_n t^2/2 + o(p_n)}{1 + t^2/2 - o(1-p_n)}.$$

From the assumptions, by letting $n \to \infty$, then $\psi_n(t) \to \frac{2}{t^2+2} \neq e^{-\frac{t^2}{2}}$. We get the complete proof. □

Proof of Theorem 2.10. Suppose that $X_j,\ j = 1, 2, \ldots$ having the characteristic function $\varphi(t) = e^{-\frac{t^2}{2}}$. Let us put $g_n(t) = E(t^{N_n}),\ p_k = \mathbb{P}(N_n = k),\ a_n = E(N_n)$. Moreover, suppose that $\psi_n(t)$ is a characteristic function of $\frac{S_{N_n}}{\sqrt{a_n}}$. Then,

$$\psi_n(t) = g_n\left(\varphi\left(\frac{t}{\sqrt{a_n}}\right)\right) = \sum_{k=0}^{\infty} p_k e^{-\frac{t^2 k}{2a_n}}.$$

Therefore

$$\Big|\psi_n(t) - e^{-\frac{t^2}{2}}\Big| = \Big|\sum_{k=0}^{\infty} p_k\Big(e^{-\frac{t^2k}{2a_n}} - e^{-\frac{t^2}{2}}\Big)\Big| \leq \sum_{k=0}^{\infty} p_k\Big|e^{-\frac{t^2k}{2a_n}} - e^{-\frac{t^2}{2}}\Big|.$$

In another way, using the Lagrange's Theorem for the function $h(x) = \big(e^{-\frac{t^2}{2}}\big)^x$ continuing on $[\frac{k}{a_n}, 1]$ or $[1, \frac{k}{a_n}]$, we have

$$\begin{aligned}\Big|e^{-\frac{t^2k}{2a_n}} - e^{-\frac{t^2}{2}}\Big| &= \Big|h\left(\frac{k}{a_n}\right) - h(1)\Big| = \Big|\frac{k}{a_n} - 1\Big||h'(c)| \text{ (for any } c)\\ &= \frac{t^2}{2}\Big|\frac{k}{a_n} - 1\Big| \cdot h(c) \leq \frac{t^2}{2} \cdot \Big|\frac{k}{a_n} - 1\Big|.\\ &\text{(because of } c \geq 0,\ h(x) \text{ is decreasing)}\end{aligned}$$

Then

$$\Big|\psi_n(t) - e^{-\frac{t^2}{2}}\Big| \leq \sum_{k=0}^{\infty} p_k\Big|\frac{k}{a_n} - 1\Big|\frac{t^2}{2} = \frac{t^2}{2}\frac{E|N_n - a_n|}{a_n}.$$

According to assumptions, by letting $n \to \infty$, then $\psi_n(t) \to e^{-\frac{t^2}{2}}$. We have the proof. ☐

Proof of Theorem 2.11. According to the Weak Law of Large Numbers, we have $\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu$. Because of assumptions, using the considerations on relationships among the conditions (3) and (4), it follows $N_n \xrightarrow{\mathbb{P}} \infty$. Since, for $\epsilon > 0, \exists n_0, \forall n > n_0 : \mathbb{P}(|\frac{S_n}{n} - \mu| > \epsilon) < \epsilon$, and

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_{N_n}}{N_n} - \mu\right| > \epsilon\right) &= \sum_{k=1}^{\infty}\mathbb{P}(N_n = k)\mathbb{P}\left(\left|\frac{S_k}{k} - \mu\right| > \epsilon\right) = \sum_{k=1}^{n_0} + \sum_{k=n_0+1}^{\infty}\\ &\leq \mathbb{P}(N_n \leq n_0) + \epsilon,\end{aligned}$$

it is easy to derive $\frac{S_{N_n}}{N_n} \xrightarrow{\mathbb{P}} \mu$. Then,

$$\frac{S_{N_n}}{n} = \frac{S_{N_n}}{N_n} \cdot \frac{N_n}{n} \xrightarrow{\mathbb{P}} \mu.$$

We obtain the proof. ☐

Proof of Theorem 2.12. Let $g_n(t) = [1 + p_n(t-1)]^n$ be a generating function of N_n and suppose that φ and ψ_n are characteristic functions of X_1 and S_{N_n}, respectively. Then

$$\psi_n = g_n(\varphi(t)) = \big(1 + p_n[\varphi(t) - 1]\big)^n = \Big(1 + \frac{np_n}{n}[\varphi(t) - 1]\Big)^n.$$

By putting $n \to \infty$, then $\psi_n(t) \to e^{\lambda[\varphi(t)-1]}$. The proof is finished. ☐

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Tran Loc Hung
Department of Mathematics
Hue University
77 Nguyen Hue, Hue city, +84-54, Vietnam
E-mail address: tlhungvn@gmail.com

Tran Thien Thanh
Department of Mathematics
Hue University
77 Nguyen Hue, Hue city, +84-54, Vietnam
E-mail address: thanhmaunong@gmail.com