Commun. Korean Math. Soc. **25** (2010), No. 1, pp. 111–117
DOI 10.4134/CKMS.2010.25.1.111

CERTAIN SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES OF THE WEDGE OF TWO MOORE SPACES

Myung Hwa Jeong

Abstract. For a based, 1-connected, finite CW-complex X, we denote by $\mathcal{E}(X)$ the group of homotopy classes of self-homotopy equivalences of X and by ${\mathcal{E}_{\#}}^{\dim +r}(X)$ the subgroup of homotopy classes which induce the identity on the homotopy groups of X in dimensions $\leq \dim X + r$. In this paper, we calculate the subgroups ${\mathcal{E}_{\#}}^{\dim +r}(X)$ when X is a wedge of two Moore spaces determined by cyclic groups and in consecutive dimensions.

1. Introduction

For a based space X, we denote by $1 : X \to X$ the identity. Then the set $[X, X]$ be the semi-group with respect to the composition of maps having unit 1, and the subset $\mathcal{E}(X)(\subset [X, X])$ of homotopy classes of self-homotoy equivalences of X is a group.

For a finite CW-complex X, let $\mathcal{E}_*(X)$ be the subgroup of homotopy classes which induce the identity on the homology groups of X and ${\mathcal{E}_{\#}}^{\dim +r}(X)$ be the subgroup of homotopy classes which induce the identity on the homotopy groups of X in dimensions $\leq \dim X + r$. The group $\mathcal{E}(X)$ and the subgroup ${\mathcal{E}_{\#}}^{\dim +r}(X)$ have been studied extensively. For a survey of known results and applications of $\mathcal{E}(X)$, see [2], and for a list of references on the subgroups mentioned above, see [3]. In particular, Arkowitz and Maruyama examined ${\mathcal{E}_{\#}}^{\dim +r}(X)$ for Moore spaces X, see [4].

In this paper we calculate the subgroups ${\mathcal{E}_{\#}}^{\dim +r}(X)$ when X is the wedge of two Moore spaces.

We fix some notations and conventions. We shall work in the category of spaces with base points and maps preserving the base points. If $f : X \to Y$ is a map, then $f_{*n} : H_n(X) \to H_n(Y)$ and $f_{\#n} : \pi_n(X) \to \pi_n(Y)$ denote, respectively the induced homology and homotopy homomorphism in dimension n. The subscript 'n' will often be omitted. In this paper we do not distinguish notationally between a map $X \to Y$ and its homotopy class in $[X, Y]$.

Received June 4, 2009.
2000 *Mathematics Subject Classification.* 55P10.
Key words and phrases. self-homotopy equivalences, Moore spaces.

If G is an abelian group and $n \geq 3$ an integer, then the Moore space $M(G,n)$ is the space, unique up to homotopy type, characterized by

$$\widetilde{H}_i(M(G,n)) = \begin{cases} G, & i = n, \\ 0, & i \neq n. \end{cases}$$

If G is free-abelian, $M(G,n)$ is just a wedge of the n-spheres. Note that when G is finitely-generate, $M(G,n)$ is a finite CW-complex of dim n if G is free-abelian and of $\dim n+1$ if G is not free-abelian. Since $M(G,n)$ is a double suspension, the set of homotopy classes $[M(G,n), X]$ can be given abelian group structure with binary operation '+'.

Finally, if A is an abelian group, we write

$$\bigoplus^{r} A = A \oplus \cdots \oplus A \quad (r \texttt{ summands}).$$

We also use '$\oplus$' to denote cartesian product of sets.

2. Preliminaries

We begin with some results needed in this paper.

Proposition 2.1. *If X is $(k-1)$-connected and Y is $(l-1)$-connected, $k, l \geq 2$, and $\dim P < k+l-1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection*

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

Proposition 2.1 is a consequence of [5, p. 405] since the inclusion $X \vee Y \to X \times Y$ is a $(k+l-1)$-equivalence.

Next we consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2 = M(G_1, n_1) \vee M(G_2, n_2)$ and denote by $i_j : M_j \to X$ the inclusions and by $p_j : X \to M_j$ the projections, $j = 1, 2$. If $f : X \to X$, then define $f_{jk} : M_k \to M_j$ by $f_{jk} = p_j f i_k$ for $j, k = 1, 2$.

Proposition 2.2. *The function θ which assigns to each $f \in [X, X]$, the 2×2 matrix*

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, is a bijection. In addition,

(1) *$\theta(f+g) = \theta(f) + \theta(g)$, so θ is an isomorphism $[X, X] \to \bigoplus_{j,k=1,2} [M_k, M_j]$.*

(2) *$\theta(fg) = \theta(f)\theta(g)$, where fg denotes composition in $[X, X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.*

(3) *under the identification $H_r(M_1 \vee M_2) = H_r(M_1) \oplus H_r(M_2)$, we have*

$$f_{*r}(x, y) = (f_{11*r}(x) + f_{12*r}(y), f_{21*r}(x) + f_{22*r}(y))$$

for $x \in H_r(M_1)$ and $y \in H_r(M_2)$.

(4) *If* $\alpha_r : \pi_r(M_1) \oplus \pi_r(M_2) \to \pi_r(M_1 \vee M_2)$ *and* $\beta_r : \pi_r(M_1 \vee M_2) \to \pi_r(M_1) \oplus \pi_r(M_2)$ *are the homomorphisms induced by the inclusions and projections, respectively, then*

$$\beta_r f_{\#r} \alpha_r(x, y) = (f_{11\#r}(x) + f_{12\#r}(y), f_{21\#r}(x) + f_{22\#r}(y))$$

for $x \in \pi_r(M_1)$ *and* $y \in \pi_r(M_2)$.

Proof. Clearly $[X, X] \approx [M_1, X] \oplus [M_2, X]$. And $[M_j, X] \approx [M_j, M_1] \oplus [M_j, M_2]$ by Proposition 2.1 for $j = 1, 2$. Then $[X, X] \approx [M_1, M_1] \oplus [M_1, M_2] \oplus [M_2, M_1] \oplus [M_2, M_2]$. The rest of the proof is straightforward and hence omitted. □

The homotopy groups $\pi_{n+k}(M(G, n))$ and the groups of homotopy classes $[M(G, n + k), M(G, k)]$ have been determined by Araki and Toda [1] when G is the cyclic group $\mathbb{Z}_q (q > 1)$ in stable homotopy category. They obtained the following results. See [1] if you want to know that in details.

Proposition 2.3 ([1])**.** (1) $\pi_n(M(\mathbb{Z}_q, n)) \approx \mathbb{Z}_q$ *for all* q.

(2) $\pi_{n+1}(M(\mathbb{Z}_q, n)) \approx \begin{cases} 0 & \text{for } q : \text{odd} \\ \mathbb{Z}_2 & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 & \text{for } q \equiv 0 \pmod 4. \end{cases}$

(3) $\pi_{n+2}(M(\mathbb{Z}_q, n)) \approx \begin{cases} 0 & \text{for } q : \text{odd} \\ \mathbb{Z}_4 & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } q \equiv 0 \pmod 4. \end{cases}$

(4) $\pi_{n+3}(M(\mathbb{Z}_q, n)) \approx \begin{cases} \mathbb{Z}_{(q,24)} & \text{for } q : \text{odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{(q,24)} & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_{(q,24)} & \text{for } q \equiv 0 \pmod 4. \end{cases}$

Proposition 2.4 ([1])**.** (1) $[(M(\mathbb{Z}_q, n - 1)), (M(\mathbb{Z}_q, n))] \approx \mathbb{Z}_q$ *for all* q.

(2) $[(M(\mathbb{Z}_q, n)), (M(\mathbb{Z}_q, n))] \approx \begin{cases} \mathbb{Z}_q & \text{for } q : \text{odd} \\ \mathbb{Z}_{2q} & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_q \oplus \mathbb{Z}_2 & \text{for } q \equiv 0 \pmod 4. \end{cases}$

(3) $[(M(\mathbb{Z}_q, n + 1)), (M(\mathbb{Z}_q, n))] \approx \begin{cases} 0 & \text{for } q : \text{odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } q \equiv 0 \pmod 4. \end{cases}$

(4) $[(M(\mathbb{Z}_q, n+2)), (M(\mathbb{Z}_q, n))] \approx \begin{cases} \mathbb{Z}_{(q,24)} & \text{for } q : \text{odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(q,24)} & \text{for } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{(q,24)} & \text{for } q \equiv 0 \pmod 4. \end{cases}$

We also need the following theorem.

Theorem 2.5 ([4])**.** *For the Moore space* $X = M(G, n)$,

(1) $\mathcal{E}_{\#}^{\dim}(X) \approx \bigoplus^{(r+s)s} \mathbb{Z}_2$, *where* r *is the rank of* G *and* s *is the number of 2- torsion summands in* G.

(2) $\mathcal{E}_{\#}^{\dim +1}(X) = 1$ *if* $n > 3$.

3. Wedge of two Moore spaces

In this section we completely determine the group ${\mathcal{E}_{\#}}^{\dim +r}(X)$ for $X = M(\mathbb{Z}_q, n+1) \vee M(\mathbb{Z}_q, n),\ n \geq 5$.

From now on, we let $M_1 = M(\mathbb{Z}_q, n+1) = S^{n+1} \cup_q e^{n+2}$ and $M_2 = M(\mathbb{Z}_q, n) = S^n \cup_q e^{n+1},\ q > 1$. And we let $f \in [X, X]$ and use the notation of Section 2 so that $f_{jk} = p_j f i_k \in [M_k, M_j]$ for $j, k = 1, 2$. Then $f \in \mathcal{E}(X) \Leftrightarrow f_{*n}, f_{*n+1}$ are isomorphisms. By Proposition 2.2, we can identify $f \in \mathcal{E}(X)$ with the 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{11} \in \mathcal{E}(M_1), f_{12} \in [M_2, M_1], f_{21} \in [M_1, M_2], f_{22} \in \mathcal{E}(M_2)$. The group structure in $\mathcal{E}(X)$ is then given by matrix multiplication.

Lemma 3.1. *Let $f \in [X, X]$ be given by $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then*

$$f \in \mathcal{E}(X) \Leftrightarrow f_{11} \in \mathcal{E}(M_1) \text{ and } f_{22} \in \mathcal{E}(M_2).$$

Proof. $f \in \mathcal{E}(X) \Leftrightarrow f_{*n}, f_{*n+1}$ are isomorphisms $\Leftrightarrow f_{22*n}, f_{11*n+1}$ are isomorphisms $\Leftrightarrow f_{11} \in \mathcal{E}(M_1), f_{22} \in \mathcal{E}(M_2)$. □

Lemma 3.2. *$\pi_{n+k}(M_1 \vee M_2) \approx \pi_{n+k}(M_1) \oplus \pi_{n+k}(M_2)$ for $k = 0, 1, 2, 3$.*

Proof. The Moore spaces M_1 and M_2 are n-connected and $(n-1)$-connected, respectively and $n \geq 5$. By Proposition 2.1, $[S^{n+k}, M_1 \vee M_2] \approx [S^{n+k}, M_1] \oplus [S^{n+k}, M_2]$ for $k < n$. □

From Lemma 3.2, it is clear that

$$f_{\#n+k}(x, y) = \begin{pmatrix} f_{11\#n+k} & f_{12\#n+k} \\ f_{21\#n+k} & f_{22\#n+k} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\forall x \in \pi_{n+k}(M_1), \forall y \in \pi_{n+k}(M_2), k = 0, 1, 2, 3.$$

Lemma 3.3. *Let $f \in \mathcal{E}_{\#}^{\dim}(X)$ be given by $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$. Then $f_{22} = 1$.*

Proof. By Lemma 3.2 and Proposition 2.2, $f_{22} \in \mathcal{E}_{\#}^{\dim X}(M_2)$. We know that $\dim X = n+2$ and $\dim M_2 = n+1$. So $f_{22} = 1$ from Theorem 2.5(2). □

From now on, it suffices that we consider just f_{11}, f_{12}, f_{21}.

Theorem 3.4. *For the space $X = M_1 \vee M_2$,*

$${\mathcal{E}_{\#}}^{\dim}(X) \approx \begin{cases} \mathbb{Z}_q, & \text{if } q : odd \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } q \equiv 2 \pmod 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } q \equiv 0 \pmod 4). \end{cases}$$

Proof. By Proposition 2.2, $[X,X] \approx [M_1,M_1] \oplus [M_1,M_2] \oplus [M_2,M_1] \oplus [M_2,M_2]$.

First, in the case q is odd: Since $[M_1,M_2] \approx 0$, $f_{21}=0$. By Theorem 2.5(1), $\mathcal{E}_{\#}^{n+2}(M_1)=1$. So $f_{11}=1$. Since $\pi_{n+1}(M(\mathbb{Z}_q,n)) \approx 0, \pi_{n+2}(M(\mathbb{Z}_q,n)) \approx 0$, and $\pi_n(M(\mathbb{Z}_q,n+1)) \approx 0$, $f_{12\#k}=0$ for all $k \leq n+2$. So $\left(\begin{smallmatrix} 1 & f_{12} \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_{\#}{}^{\dim}(X)$ for any f in $[M_2,M_1]$. By Proposition 2.4, $[M_2,M_1] \approx \mathbb{Z}_q = \langle i\pi \rangle$. Therefore

$$\mathcal{E}_{\#}{}^{\dim}(X) \approx \left\{ \left(\begin{array}{cc} 1 & f_{12} \\ 0 & 1 \end{array} \right) | f_{12} \in \langle i\pi \rangle \right\}.$$

Therefore $\mathcal{E}_{\#}{}^{\dim}(X) \approx \mathbb{Z}_q$.

Second, in the case $q \equiv 2 \pmod 4$: By Proposition 2.4, $[M_1,M_1] \approx \mathbb{Z}_{2q} = \langle 1 \rangle$. We know that $\pi_{n+1}(M_1) \approx \mathbb{Z}_q = \langle i \rangle$ and $\pi_{n+2}(M_1) \approx \mathbb{Z}_2 = \langle i\eta \rangle$, where $i : S^{n+1} \hookrightarrow M_1$ is inclusion and η is the generator of $\pi_{n+2}(S^{n+1})$. Let $\pi : M_1 \to S^{n+2}$ be the map shrinking S^{n+1} to the base point of S^{n+2}. Then $\pi i : S^{n+1} \to S^{n+2}$ is trivial. So $(i\eta\pi)_{\#n+1}(i) = i\eta\pi i = 0$ and $(i\eta\pi)_{\#n+2}(i\eta) = i\eta\pi i\eta = 0$. Thus $\left(\begin{smallmatrix} 1+i\eta\pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_{\#}{}^{\dim}(X)$. By Theorem 2.5(1), $\mathcal{E}_{\#}{}^{\dim}(M_1) \approx \mathbb{Z}_2$. So $f_{11} \in 1 + \langle i\eta\pi \rangle$.

When q is even, there exist elements $\overline{\eta} \in [M_2, S^{n-1}]$ and $\widetilde{\eta} \in [S^{n+2}, M_2]$ such that $\overline{\eta} i = \eta$ and $\pi\widetilde{\eta} = \eta$. Consider $[M_2,M_1] \approx \mathbb{Z}_q = \langle i\pi \rangle$ and $\pi_{n+2}(M_2) \approx \mathbb{Z}_4 = \langle \widetilde{\eta} \rangle$. Since $(i\pi)_{\#n+2}(\widetilde{\eta}) = i\pi\widetilde{\eta} = i\eta \neq 0, \left(\begin{smallmatrix} 1 & i\pi \\ 0 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{E}_{\#}{}^{\dim}(X)$. So if $f \in \mathcal{E}_{\#}^{\dim}(X)$, then $f_{12}=0$.

Now let us consider f_{21}. For even q, we use the following notations $\eta_1 = i\overline{\eta}$ and $\eta_2 = \widetilde{\eta}\pi$. Then $[M_1,M_2] \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \eta_1 \rangle \oplus \langle \eta_2 \rangle$. We know that $\pi_{n+1}(M_1) \approx \mathbb{Z}_2 = \langle i \rangle$. $\eta_{1\#n+1}(i) = \eta_1 i = i\overline{\eta} i = i\eta \neq 0$. Hence $\left(\begin{smallmatrix} 1 & 0 \\ \eta_1 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{E}_{\#}{}^{\dim}(X)$. On the other hand $\eta_{2\#n+1}(i) = \eta_2 i = \widetilde{\eta}\pi i = 0$ and $\eta_{2\#n+2}(i\eta) = \eta_2 i\eta = \widetilde{\eta}\pi i\eta = 0$. So $\left(\begin{smallmatrix} 1 & 0 \\ \eta_2 & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_{\#}{}^{\dim}(X)$. Thus we conclude the following results.

$$\mathcal{E}_{\#}{}^{\dim}(X) \approx \left\{ \left(\begin{array}{cc} 1+\epsilon & 0 \\ f_{21} & 1 \end{array} \right) | \epsilon \in \langle i\eta\pi \rangle, \epsilon^2 = 1 \text{ and } f_{21} \in \langle \eta_2 \rangle, {\eta_2}^2 = 1 \right\}.$$

Therefore $\mathcal{E}_{\#}{}^{\dim}(X) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Last, in the case $q \equiv 0 \pmod 4$: By Proposition 2.4, $[M_1,M_1] \approx \mathbb{Z}_q \oplus \mathbb{Z}_2 = \langle 1 \rangle \oplus \langle i\eta\pi \rangle$. By the same manner in case $q \equiv 2 \pmod 4$, $\left(\begin{smallmatrix} 1+i\eta\pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $\mathcal{E}_{\#}{}^{\dim}(X)$.

And $[M_1,M_2] \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \eta_1 \rangle \oplus \langle \eta_2 \rangle \oplus \langle i\eta^2\pi \rangle$. Since $\eta_{1\#n+2}(i\eta) = \eta_1 i\eta = i\overline{\eta} i\eta = i\eta^2 \neq 0$, $\left(\begin{smallmatrix} 1 & 0 \\ \eta_1 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{E}_{\#}{}^{\dim}(X)$. However $\eta_{2\#n+1}(i) = \eta_2 i = \widetilde{\eta}\pi i = 0, \eta_{2\#n+2}(i\eta) = \eta_2 i\eta = \widetilde{\eta}\pi i\eta = 0, (i\eta^2\pi)_{\#n+2}(i) = 0$ and $(i\eta^2\pi)_{2\#n+2}(i\eta) = 0$. So $\left(\begin{smallmatrix} 1 & 0 \\ \eta_2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ i\eta^2\pi & 1 \end{smallmatrix}\right)$ belong to $\mathcal{E}_{\#}{}^{\dim+1}(X)$. Now $[M_2,M_1] \approx \mathbb{Z}_2 = \langle i\pi \rangle$ and $\pi_{n+2}(M_2) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \widetilde{\eta} \rangle \oplus \langle i\eta^2 \rangle$. Since $(i\pi)_{\#n+2}(\widetilde{\eta}) = i\pi\widetilde{\eta} = i\eta \neq 0$, $\left(\begin{smallmatrix} 1 & i\pi \\ 0 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{E}_{\#}{}^{\dim}(X)$. Thus we conclude the following results.

$$\mathcal{E}_{\#}{}^{\dim}(X) \approx \left\{ \begin{pmatrix} 1+\epsilon & 0 \\ f_{21} & 1 \end{pmatrix} | f_{21} \in \langle \eta_2 \rangle \oplus \langle i\eta^2\pi \rangle, \eta_2^2 = 1, (i\eta^2\pi)^2 = 1 \right\}.$$

Therefore $\mathcal{E}_{\#}{}^{\dim}(X) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Theorem 3.5. *For the space* $X = M_1 \vee M_2,\ \ n \geq 5$,

$$\mathcal{E}_{\#}{}^{\dim+1}(X) \approx \begin{cases} \mathbb{Z}_q, & \text{if } q : \text{odd} \\ 1, & \text{if } q \equiv 2 \pmod 4 \\ 1, & \text{if } q \equiv 0 \pmod 4. \end{cases}$$

Proof. First, in the case q is odd: Since $\mathcal{E}_{\#}{}^{\dim+1}(X) \subseteq \mathcal{E}_{\#}{}^{\dim}(X)$, it suffices to consider $[M_2, M_1] \approx \mathbb{Z}_q = \langle i\pi \rangle$. We know that $\pi_{n+3}(M_2) \approx \mathbb{Z}_{(q,24)} = \langle i\nu \rangle$. And $i\pi_{\# n+3}(i\nu) = i\pi i\nu = 0$. Therefore $\mathcal{E}_{\#}{}^{\dim+1}(X) \cong \mathbb{Z}_q$. That is $\mathcal{E}_{\#}{}^{\dim+1}(X) = \mathcal{E}_{\#}{}^{\dim}(X)$.

Second, in the case $q \equiv 2 \pmod 4$: By Theorem 2.5, $\mathcal{E}_{\#}{}^{\dim+1}(M_1) = 1$. Hence, it suffices to consider only η_2. $\pi_{n+3}(M_1) \approx \mathbb{Z}_4 = \langle \widetilde{\eta} \rangle$, and $(\eta_2)_{\# n+3}(\widetilde{\eta}) = \eta_2\widetilde{\eta} = \widetilde{\eta}\pi\widetilde{\eta} = \widetilde{\eta}\eta \neq 0$. Therefore $\left(\begin{smallmatrix} 1 & 0 \\ \eta_2 & 1 \end{smallmatrix}\right)$ does not belong to $\mathcal{E}_{\#}{}^{\dim+1}(X)$. Thus we can conclude $\mathcal{E}_{\#}{}^{\dim+1}(X) = 1$.

Last, in the case $q \equiv 0 \pmod 4$: It suffices to consider just two generators η_2, $i\eta^2\pi$. $\pi_{n+3}(M_1) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle \widetilde{\eta} \rangle \oplus \langle i\eta^2 \rangle$. Like the second case, $(\eta_2)_{\# n+3}(\widetilde{\eta}) \neq 0$ and $(i\eta^2\pi)_{\# n+3}(\widetilde{\eta}) = i\eta^2\pi\eta = i\eta^3 \neq 0$. Finally, we obtain the result $\mathcal{E}_{\#}{}^{\dim+1}(X) = 1$. □

We denote by $\mathcal{Z}(X)$ the subset of $[X, X]$ consisting of all homotopy classes which induces the trivial homomorphism on homotopy groups in dimensions less than or equal to $\dim X$. Consider the bijection map $T : \mathcal{E}_{\#}{}^{\dim}(X) \to \mathcal{Z}(X)$ defined by the translation by the identity map, that is, $T(f) = f - 1$.

Corollary 3.6. *For the space* $X = M_1 \vee M_2$,

(1) *if* q : *odd, then*

$$\mathcal{Z}(X) \approx \left\{ \begin{pmatrix} 0 & 0 \\ f_{21} & 0 \end{pmatrix} | f_{21} \in \langle i\pi \rangle, (i\pi)^q = 1 \right\},$$

(2) *if* $q \equiv 2 \pmod 4$, *then*

$$\mathcal{Z}(X) \approx \left\{ \begin{pmatrix} f_{11} & 0 \\ f_{21} & 0 \end{pmatrix} | f_{11} \in \langle i\eta\pi \rangle, f_{21} \in \langle \eta_2 \ \rangle, (i\eta\pi)^2 = 1, {\eta_2}^2 = 1 \right\},$$

and

(3) *if* $q \equiv 0 \pmod 4$, *then*

$$\mathcal{Z}(X) \approx \left\{ \begin{pmatrix} f_{11} & 0 \\ f_{21} & 0 \end{pmatrix} | f_{11} \in \langle i\eta\pi \rangle, f_{21} \in \langle \eta^2 \rangle \oplus \langle i\eta^2\pi \rangle, (i\eta^2\pi)^2 = 1 \right\}.$$

Acknowledgement. The author would like to thank Professor Maruyama for many helpful advice.

References

[1] S. Araki and Toda, *Multiplicative structure in mod q cohomology theories I*, Osaka J. Math. **2** (1965), 71–115.

[2] M. Arkowitz, *The group of self-homotopy equivalences* - a survey, in: R. Piccinini, ed., Lecture Notes in Math. 1425 (Springer, New York, 1990), 170–203.

[3] M. Arkowitz and G. Lupton, *On finiteness of subgroups of self-homotopy equivalences*, in: Contemporary Mathematics, Vol. 181 (American Mathematical Society, 1995), 1–25.

[4] M. Arkowitz and K. Maruyama, *Self-homotopy equivalences which induce the identity on homology, cohomology or homotopy groups*, Topology Appl. **87** (1998), 133–154.

[5] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1996.

Division of General Education
Ajou University
Suwon 442-749, Korea
E-mail address: mhjeong@ajou.ac.kr