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β-PRECONVEX SETS ON PRECONVEXITY SPACES

Won Keun Min

Abstract. In this paper, we introduce the concept of β-preconvex sets on preconvexity spaces. We study some properties for β-preconvex sets by using the co-convexity hull and the convexity hull. Also we introduce and study the concepts of βc-convex function and $\beta^* c$-convex function.

1. Introduction

In [1], Guay introduced the concept of preconvexity spaces defined by a binary relation on the power set $P(X)$ of a nonempty set X and investigated some properties. He showed that a preconvexity on a nonempty set yields a convexity space in the same manner as a proximity [5] yields a topological space. In [3], we introduced the concepts of co-convexity hull and co-convex sets on preconvexity spaces. And we characterized c-convex functions and c-concave functions by using the co-convexity hull and the convexity hull. In [4] we introduced the semi-preconvex set defined by the co-convexity hull on a preconvexity space and study some basic properties. And we introduced the concepts of sc-convex functions and s^*c-convex functions which are defined by the semi-preconvex sets. In this paper, we introduce the concept of β-preconvex set on a preconvexity space and study some basic properties. And we introduce the concepts of βc-convex functions and $\beta^* c$-convex functions which are defined by the β-preconvex sets. Finally, some properties of βc-convex functions and $\beta^* c$-convex functions are discussed. In particular,

(a) a function $f : (X, \sigma) \to (Y, \mu)$ on two preconvexity spaces is βc-convex if and only if for each $A \subset Y$, $f^{-1}(I(A))\sigma I(G(f^{-1}(A)))$;

(b) if a function $f : (X, \sigma) \to (Y, \mu)$ on two preconvexity spaces is c-concave and βc-convex, then f is $\beta^* c$-convex.

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2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty set. A binary relation σ on $P(X)$ is called a preconvexity on X if the relation satisfies the following properties; we write $x\sigma A$ for $\{x\}\sigma A$:

(1) If $A \subset B$, then $A\sigma B$.
(2) If $A\sigma B$ and $B = \emptyset$, then $A = \emptyset$.
(3) If $A\sigma B$ and $b\sigma C$ for all $b \in B$, then $A\sigma C$.
(4) If $A\sigma B$ and $x \in A$, then $x\sigma B$.

The pair (X, σ) is called a preconvexity space. Let (X, σ) be a preconvexity space and $A \subset X$. $G(A) = \{x \in X : x\sigma A\}$ is called the convexity hull of a subset A. A is called convex [1] if $G_\sigma(A) = A$ (simply, $G(A)$).

$I_\sigma(A) = \{x \in A : x\not\sigma(X - A)\}$ (simply, $I(A)$) is called the co-convexity hull [3] of a subset A. And A is called a co-convex set if $I(A) = A$ [3]. Let $\mathcal{I}(X) = \{A \subset X : I(A) = A\}$ and $\mathcal{G}(X) = \{A \subset X : G(A) = A\}$.

Theorem 2.2 ([1, 3]). *For a preconvexity space (X, σ),*

(1) $G(\emptyset) = \emptyset$, $I(X) = X$.
(2) $A \subset G(A)$, $I(A) \subset A$ *for all* $A \subset X$.
(3) *If* $A \subset B$, *then* $G(A) \subset G(B)$, $I(A) \subset I(B)$.
(4) $G(G(A)) = G(A)$, $I(I(A)) = I(A)$ *for* $A \subset X$.
(5) $I(A) = X - G(X - A)$ *and* $G(A) = X - I(X - A)$.

Theorem 2.3 ([1, 3]). *Let σ be a preconvexity on X and $A, B \subset X$. Then*

(1) $A\sigma B$ *if and only if* $A \subset G(B)$ *if and only if* $I(X - B) \subset X - A$.
(2) $A\sigma B$ *if and only if* $G(A)\sigma G(B)$ *if and only if* $I(X - B)\sigma I(X - A)$.

Definition 2.4 ([4]). Let (X, σ) be a preconvexity space and $A \subset X$. A is called a semi-preconvex set if $A\sigma I(A)$. And A is called a cosemi-preconvex set if the complement of A is a semi-preconvex set.

Let $\mathcal{S}_\sigma(X)$ (resp., $\mathcal{S}C_\sigma(X)$) denote the set of all semi-preconvex sets (resp., cosemi-preconvex sets) in a preconvexity space (X, σ).

Theorem 2.5 ([4]). *Let (X, σ) be a preconvexity space and $A \subset X$. Then*

(1) *A is a semi-preconvex set if and only if* $A \subset G(I(A))$.
(2) *A is a cosemi-preconvex set if and only if* $I(G(A)) \subset A$.

We recall that the notions of c-convex function and c-concave function: Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be c-concave [2] if for $C, D \subset Y$ whenever $C\mu D$, $f^{-1}(C)\sigma f^{-1}(D)$. A function $f : X \to Y$ is said to be c-convex [1] if $A\sigma B$ implies $f(A)\mu f(B)$. And f is c-convex if and only if for each $U \in \mathcal{I}(Y)$, $f^{-1}(U) \in \mathcal{I}(X)$ [3].

3. β-preconvex sets

Definition 3.1. Let (X, σ) be a preconvexity space and $A \subset X$. A is called a β-preconvex set if $A\sigma I(G(A))$. And A is called a coβ-preconvex set if the complement of A is a β-preconvex set.

Let $\beta_\sigma(X)$ (resp., $\beta C_\sigma(X)$) denote the set of all β-preconvex sets (resp., coβ-preconvex sets) in a preconvexity space (X, σ).

Now we get the following implications but the converses are not true in general as shown in the next example:

co-convex $\Rightarrow$ semi-convex $\Rightarrow$ β-convex

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{b, c\}\}$. Define $A\sigma B$ to mean $A \subset cl(B)$, the closure of B in X. Then σ is a preconvexity on X. In the preconvexity space (X, σ), $\mathcal{G}(X) = \{\emptyset, X, \{a\}\}$, $\mathcal{I}(X) = \{\emptyset, X, \{b, c\}\}$, $\beta_\sigma(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and $\mathcal{S}_\sigma(X) = \{\emptyset, X, \{b, c\}\}$. Hence we know that a β-convex set $\{a, b\}$ is neither co-convex nor semi-convex.

From Theorem 2.2 and Theorem 2.3, we get the following:

Theorem 3.3. *Let (X, σ) be a preconvexity space and $A \subset X$. Then*

(1) *A is a β-preconvex set if and only if $A \subset G(I(G(A)))$.*
(2) *A is a coβ-preconvex set if and only if $I(G(I(A))) \subset A$.*

Theorem 3.4. *Every semi-preconvex set is a β-preconvex set in a preconvexity space (X, σ).*

Proof. Let A be a semi-preconvex set; then by definition of semi-preconvex sets, $A\sigma I(A)$. By Theorem 2.2 and the transitive property of preconvexity, $A\sigma I(G(A))$. □

Corollary 3.5. *Every cosemi-preconvex set is coβ-preconvex in a preconvexity space (X, σ).*

Proof. Obvious. □

Theorem 3.6. *In a preconvexity space (X, σ), X and $\emptyset$ are both β-preconvex and coβ-preconvex.*

Proof. By Theorem 2.2, it is obvious. □

Theorem 3.7. *Let (X, σ) be a preconvexity space. Then the arbitrary union of β-preconvex sets is a β-preconvex set.*

Proof. Let $\mathfrak{F} = \{A_\alpha : A_\alpha \in \beta_\sigma(\mathbf{X})\}$ be any subfamily of $\beta_\sigma(\mathbf{X})$ and $x \in \cup\mathfrak{F}$. Then there exists a β-preconvex set A_α containing x such that $x \in A_\alpha \sigma I(G(A_\alpha))$ and so from Definition 2.1(4), $x\sigma I(G(A_\alpha))$. And from Theorem 2.2 and $A_\alpha \subset \cup\mathfrak{F}$, it follows $I(G(A_\alpha)) \subset I(G(\cup\mathfrak{F}))$. So by the transitive property of preconvexity, we have $x\sigma I(G(\cup\mathfrak{F}))$. Finally, by Definition 2.1(3), $\cup\mathfrak{F}\sigma I(G(\cup\mathfrak{F}))$. □

Theorem 3.8. *In a preconvexity space (X, σ), the arbitrary intersection of coβ-preconvex sets is a coβ-preconvex set.*

Proof. From Theorem 2.2 and Theorem 3.7, it is obvious. □

Definition 3.9. Let (X, σ) be a preconvexity space and $A \subset X$.

(1) $\beta G(A) = \cap\{F : A \subset F, F^c \in \beta_\sigma(X)\}$.
(2) $\beta I(A) = \cup\{U : U \subset A, U \in \beta_\sigma(X)\}$.

Theorem 3.10. *Let (X, σ) be a preconvexity space and $A, B \subset X$.*

(1) $I(A) \subset \beta I(A) \subset A$.
(2) $A \subset \beta G(A) \subset G(A)$.
(3) *A is β-preconvex if and only if $A = \beta I(X)$.*
(4) *A is coβ-preconvex if and only if $A = \beta C(X)$.*

Proof. (1) and (2) are obvious from Theorem 3.4 and Corollary 3.5.
(3) It is obtained from Theorem 3.7.
(4) It is obtained from Theorem 3.8. □

Theorem 3.11. *Let (X, σ) be a preconvexity space and $A, B \subset X$.*

(1) $\beta I(X) = X$.
(2) $\beta I(A) \subset A$.
(3) *If $A \subset B$, then $\beta I(A) \subset \beta I(B)$.*
(4) $\beta I(\beta I(A)) = \beta I(A)$.

Proof. (1), (2) and (3) are obvious.
(4) Since $\beta I(A) \subset A$, by (3), $\beta I(\beta I(A)) \subset \beta I(A)$.
For the converse, let $x \in \beta I(A)$; then since $x \in \beta I(A) \subset \beta I(A)$ and $\beta I(A)$ is a β-preconvex set, we get $x \in \beta I(\beta I(A))$ by Definition 3.9(2). □

Theorem 3.12. *Let (X, σ) be a preconvexity space and $A, B \subseteq X$.*

(1) $\beta G(\emptyset) = \emptyset$.
(2) $A \subset \beta G(A)$.
(3) *If $A \subset B$, then $\beta G(A) \subset \beta G(B)$.*
(4) $\beta G(\beta G(A)) = \beta G(A)$.

Proof. It is similar to the proof of Theorem 3.11. □

4. βc-convex functions and $\beta^* c$-convex functions

Definition 4.1. Let (X, σ) and (Y, μ) be two preconvexity spaces. A function $f : X \to Y$ is said to be βc-convex if for each $A \in \mathcal{I}(Y)$, $f^{-1}(A) \in \beta_\sigma(X)$.

Every *sc*-convex function is βc-convex but the converse is not always true as shown the next example:

Example 4.2. In Example 3.2, consider a function $f : (X, \sigma) \to (X, \sigma)$ defined as follows $f(a) = b, f(b) = c$ and $f(c) = a$. Then f is βc-convex but not *sc*-convex because $f^{-1}(\{b, c\}) = \{a, b\}$ is not semi-preconvex.

Theorem 4.3. *For two preconvexity spaces* (X,σ) *and* (Y,μ), *a function* $f:(X,\sigma)\to(Y,\mu)$ *is* βc*-convex if and only if for each* $A\subset Y$, $f^{-1}(I(A))\sigma I(G(f^{-1}(A)))$.

Proof. Suppose that f is βc-convex and let $A\subset Y$; then since $I(A)\subset A$, by Theorem 2.2, we get $I(G(f^{-1}(I(A))))\sigma I(G(f^{-1}(A)))$. Since $I(A)\in\mathcal{I}(Y)$ and f is βc-convex, $f^{-1}(I(A))\sigma I(G(f^{-1}(I(A))))$. By the transitive property of preconvexity, we have $f^{-1}(I(A))\sigma I(G(f^{-1}(A)))$.

For the converse, let $A\in\mathcal{I}(Y)$; then since $A=I(A)$, by hypothesis, we have $f^{-1}(A)\sigma I(G(f^{-1}(A)))$. Thus $f^{-1}(A)\in\beta_\sigma(X)$. □

Theorem 4.4. *Let* (X,σ) *and* (Y,μ) *be two preconvexity spaces and* $f:X\to Y$ *a function. Then the following things are equivalent*:

(1) f *is* βc*-convex.*
(2) $f^{-1}(I(B))\subset G(I(G(f^{-1}(B))))$ *for all* $B\subset Y$.
(3) $I(G(I(f^{-1}(B))))\subset f^{-1}(G(B))$ *for all* $B\subset Y$.
(4) $f(I(G(I(A))))\subset G(f(A))$ *for all* $A\subset X$.
(5) *For each* $U\in\mathcal{G}(Y)$, $f^{-1}(U)\in\beta C_\sigma(X)$.

Proof. (1) $\Rightarrow$ (2) By Theorem 4.3 and Theorem 2.3, we get the result.

(2) $\Rightarrow$ (3) Let $B\subset Y$; then by (2), we have $X-f^{-1}(G(B))=f^{-1}(I(Y-B))\subset G(I(G(f^{-1}(Y-B))))=X-I(G(I(f^{-1}(B))))$. Hence (3) is obtained.

(3) $\Rightarrow$ (4) Let $A\subset X$; then since $f(A)\subset Y$, (4) is easily obtained by (3).

(5) $\Rightarrow$ (1) It is obvious. □

From Theorem 3.10 and Theorem 4.4, we get the following:

Theorem 4.5. *Let* (X,σ) *and* (Y,μ) *be two preconvexity spaces and* $f:X\to Y$ *a function. Then the following things are equivalent*:

(1) f *is* βc*-convex.*
(2) $f^{-1}(I(B))\subset\beta I(f^{-1}(B))$ *for all* $B\subset Y$.
(3) $\beta C(f^{-1}(B))\subset f^{-1}(G(B))$ *for all* $B\subset Y$.
(4) $f(\beta C(A))\subset G(f(A))$ *for all* $A\subset X$.

Definition 4.6. Let (X,σ) and (Y,μ) be two preconvexity spaces. A function $f:X\to Y$ is said to be β^*c-convex if for each $A\in\beta_\mu(Y)$, $f^{-1}(A)\in\beta_\sigma(X)$.

Every β^*c-convex function is βc-convex but the converse is not always true as shown in the next example:

Example 4.7. In Example 4.2, the function f is βc-convex. But f is not β^*c-convex because $f^{-1}(\{b\})=\{a\}$ is not β-preconvex for a β-preconvex set $\{b\}$.

We have the following:

$$\begin{array}{ccccc} & & \beta c\text{-convex} & \Leftarrow & \beta^* c\text{-convex} \\ & & \Uparrow & & \Uparrow \\ c\text{-convex} & \Rightarrow & sc\text{-convex} & \Leftarrow & s^* c\text{-convex} \end{array}$$

Theorem 4.8. *For two preconvexity spaces (X,σ) and (Y,μ), a function $f : (X,\sigma) \to (Y,\mu)$ is β^*c-convex if and only if for $A \subset Y$, $f^{-1}(A)\sigma I(G(f^{-1}(A)))$ whenever $A\mu I(G(A))$.*

Proof. From Definition 3.1, it is obvious. □

Theorem 4.9 ([3])**.** *Let $f : X \to Y$ be a function on two preconvexities (X,σ) and (Y,μ). Then the following things are equivalent*:

(1) *f is c-concave*
(2) *$f^{-1}(G(A)) \subset G(f^{-1}(A))$ for all $A \subset Y$.*
(3) *$I(f^{-1}(A)) \subset f^{-1}(I(A))$ for all $A \subset Y$.*

Theorem 4.10. *Let (X,σ) and (Y,μ) be two preconvexity spaces and $f : X \to Y$ a function. Then if f is c-concave and βc-convex, then f is β^*c-convex.*

Proof. Suppose f is c-concave and βc-convex. Let $A \in \beta_\mu(Y)$; then $A\mu I(G(A))$ and since f is c-concave, we have $f^{-1}(A)\sigma f^{-1}(I(G(A)))$. From Theorem 4.3 and Theorem 4.9, it follows

$$f^{-1}(I(G(A)))\sigma I(G(f^{-1}(G(A))))\sigma I(G(G(f^{-1}(A)))) = I(G(f^{-1}(A)).$$

By the transitive property of preconvexity, we have $f^{-1}(A)\sigma I(G(f^{-1}(A))$. Hence from Theorem 4.8, f is β^*c-convex. □

Theorem 4.11. *Let $f : X \to Y$ be a function on two preconvexity spaces (X,σ) and (Y,μ). Then the following things are equivalent*:

(1) *f is β^*c-convex.*
(2) *For each $U \in \beta C_\mu(Y)$, $f^{-1}(U) \in \beta C_\sigma(X)$.*
(3) *$f(\beta C(A)) \subset \beta C(f(A))$ for all $A \subset X$.*
(4) *$\beta C(f^{-1}(B)) \subset f^{-1}(\beta C(B))$ for all $B \subset Y$.*
(5) *$f^{-1}(\beta I(B)) \subset \beta I(f^{-1}(B))$ for all $B \subset Y$.*

Proof. Straightforward. □

References

[1] M. D. Guay, *An introduction to preconvexity spaces*, Acta Math. Hungar. **105** (2004), no. 3, 241–248.
[2] W. K. Min, *Some results on preconvexity spaces*, Bull. Korean Math. Soc. **45** (2008), no. 1, 39–44.
[3] ———, *A note on preconvexity spaces*, Honam Math. J. **29** (2007), no 4, 589–595.
[4] ———, *Semi-preconvex sets on preconvexity spaces*, Commun. Korean Math. Soc. **23** (2008), no. 2, 251–256.
[5] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge University Press, 1970.

Department of Mathematics
Kangwon National University
Chuncheon 200-701, Korea
E-mail address: wkmin@kangwon.ac.kr