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SOME PROPERTIES OF THE STRONG CHAIN RECURRENT SET

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Abstract. The article is devoted to exhibit some general properties of strong chain recurrent set and strong chain transitive components for a continuous map f on a compact metric space X. We investigate the relation between the weak shadowing property and strong chain transitivity. It is shown that a continuous map f from a compact metric space X onto itself with the average shadowing property is strong chain transitive.

1. Introduction

The notion of chain recurrence, introduced by Conley [4], is a way of getting at the recurrence properties of a dynamical system. It has remarkable connections to the structure of attractors. In 1977, Easton introduced another notion of recurrence which was stronger than chain recurrency [6]. He defined strong chains and strong chain-transitive sets. He gave a specific example of a Anosov transformation f of the torus T^2 which was strong chain transitive on all of T^2. However, if a dynamical system preserves a finite Borel measure, the chain recurrent set is all of X. In this case, it may be justified to consider those isolated invariant sets on which the system is strong chain transitive. So, this is a motivation for studying the strong chain transitive sets.

2. Preliminaries

Let X be a compact metric space with metric d and $\mathcal{H}(X)$ be the space of all homeomorphisms on X. We define a *complete metric* on $\mathcal{H}(X)$ by

$$\overline{d}(f,g) = \max\{d(f,g), d(f^{-1},g^{-1})\},$$

where $d(f,g) = \sup\{d(f(x),g(x)) : x \in X\}$ (Note that $\mathcal{H}(X)$ by the usual uniform metric is not a complete metric space, so we consider the above metric which is topologically equivalent to uniform metric, [1]). Recall that a subset

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is *residual*, if it contains a countable intersection of open and dense sets. We say that a property is *generic*, if it holds on a residual subset. A point y is an ω-*limit point* of x for f provided there exists a sequence $\{n_k\}$ going to infinity as k goes to infinity such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all ω-limit points of x for f is called the ω-*limit set* of x and is denoted by $\omega(x, f)$. A subset $\Lambda \subseteq X$ is said to be *invariant* provided $f(\Lambda) = \Lambda$. We say that x is ω-*recurrent* provided $x \in \omega(x)$. The closure of the set of all ω-recurrent points of f is called the *Birkhoff center* of f, denoted by $B(f)$. A point x is called *nonwandering* provided for every neighborhood U of x, there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points for f is called the *nonwandering set* and is denoted by $\Omega(f)$. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{b}$ $(0 < b \leq \infty)$ is called a δ-*pseudo orbit* of f if for each $0 \leq i < b$,

$$d(f(x_i), x_{i+1}) < \delta.$$

A sequence $\{x_i\}_{i=0}^{\infty}$ in X is said to be ϵ-*shadowed* by a point $z \in X$ if for every $i \geq 0$,

$$d(f^i(z), x_i) < \epsilon.$$

A map f is said to have the *shadowing property* (or *weak shadowing property*), if for any $\epsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo orbit of f can be ϵ-shadowed (or weak ϵ-shadowed) by some point $y \in X$, that is

$$d(f^n(y), x_n) < \epsilon, n \in \mathbb{N}$$

(or $\{x_i\}_{i=0}^{\infty} \subseteq U_\epsilon(\mathcal{O}(y, f))$, where $U_\epsilon(\mathcal{O}(y, f))$ denotes the ϵ-ball of $\mathcal{O}(y, f)$ measured in d). A sequence $\{x_i\}_{i=0}^{\infty}$ in X is called a δ-*average pseudo orbit* for f if there is a natural number N such that for all $n \geq N$, $k \geq 0$

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

A sequence $\{x_i\}_{i=0}^{\infty}$ is said to be ϵ-*shadowed in average* by a point $z \in X$, if

$$\limsup_{n\longrightarrow\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

A map f is said to have the *average shadowing property*, if for each $\epsilon > 0$ there exists $\delta > 0$ such that every average δ-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ can be ϵ-shadowed in average by some point in X. A *strong* δ-*pseudo orbit* from x to y means a sequence $\{x = x_0, x_1, \ldots, x_{n+1} = y\}$ with the property that

$$\sum_{i=0}^{n} d(f(x_i), x_{i+1}) < \delta.$$

If for each $\delta > 0$, there exists a δ-pseudo orbit from x to itself, then x is said to be *chain-recurrent*. However, if there is a strong δ-pseudo orbit from x to itself, then x is called a *strong chain-recurrent point*. We denote the sets of all chain-recurrent points and strong chain-recurrent points of f by $CR(f)$ and $SCR(f)$, respectively. It is easy to see that

$$B(f) \subseteq \Omega(f) \subseteq CR(f).$$

For each $x, y \in CR(f)$, we say that $x \sim_C y$ if and only if for any $\epsilon > 0$, there is a ϵ-pseudo orbit from x to y and conversely from y to x. Similarly, if $x, y \in SCR(f)$, then $x \sim_S y$ if and only if there exists a strong ϵ-pseudo orbit from x to y and from y to x for each $\epsilon > 0$. Clearly $\sim_C$ and $\sim_S$ define equivalence relations on $CR(f)$ and $SCR(f)$, respectively. The equivalence classes of $\sim_C$ and $\sim_S$ are called *chain transitive components* and *strong chain transitive components* of f. An invariant set Λ is said to be *chain-transitive*, if for each $p, q \in \Lambda$ and $\delta > 0$, there exists a δ-pseudo orbit from p to q. Also, Λ is called *strong chain transitive* if given $p, q \in \Lambda$ and $\delta > 0$, there exists a strong δ-pseudo orbit from p to q. A map f is said to be *topologically transitive* if and only if there is an orbit of f which is dense in X.

3. Strong chain transitive components

Let $\epsilon > 0$. Take $SCR_\epsilon(x, f)$ the set of all points $y \in X$ so that there is a strong ϵ-pseudo orbit from x to y. We denote

$$SCR(x, f) = \bigcap_{\epsilon > 0} SCR_\epsilon(x, f).$$

If $0 < \delta < \epsilon$, then it is easy to verify that $\overline{SCR_\delta(x, f)} \subset SCR_\epsilon(x, f)$, from which it follows that $SCR(x, f)$ is a closed subset of X. Moreover, if $f \in \mathcal{H}(X)$ and S is a strong chain transitive component of f, then S is a closed invariant subset of X. In particular, if $x \in S$, then $\omega(x, f) \subseteq S$ (for more details see [8] and [7]). Furthermore, $\Omega(f)$, $SCR(f)$ and $CR(f)$ are related by the following inclusions:

$$\Omega(f) \subseteq SCR(f) \subset CR(f),$$

see [8] and [7]. We note that each strong chain transitive set is chain transitive. However the converse is not true, generically.

Example 3.1. Let X be a compact metric space. It is known that the chain transitive components of the mapping 1_X are connected components of X exactly [2]. Now, suppose that $X = [0, 1]$. Then 1_X has only one chain transitive component which is the entire of [0,1]. Let x, y be two points of $[0, 1]$ with $x \neq y$. Suppose $0 < \epsilon < |x - y|$ is given. If $\{z_0 = x, z_1, \ldots, z_k = y\}$ is any sequence in $[0, 1]$, then

$$\sum_{i=0}^{k-1} d(1_X(z_i), z_{i+1}) = \sum_{i=0}^{k-1} d(z_i, z_{i+1}) \geq d(z_o, z_k) > \epsilon.$$

So there is no any strong ϵ-pseudo orbit from x to y. Hence, each strong chain transitive component is a singleton, which implies that strong chain transitive components of $1_{[0,1]}$ are not equal to its chain transitive component.

Example 3.2. Let $f : [0,1] \longrightarrow [0,1]$ be the tent map which is defined by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ -2x+2 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

By [3], f is topologically transitive. It is easy to see that each topologically transitive map is strong chain transitive. So, f has only one strong chain transitive component which is the entire of $[0,1]$.

In below, we discuss some properties of strong chain transitive components. First, we investigate the relation between the strong chain recurrent set of $f^n, n \in \mathbb{N}$, and the strong chain recurrent set of f. However, it is well known $CR(f^n) = CR(f)$ by a result in topological dynamics [2].

Proposition 3.3. *For any $n \in \mathbb{N}$, $SCR(f^n) \subseteq SCR(f)$. Moreover, if S_0 is a strong chain transitive component of f^n, then $S = \cup_{i=0}^{n-1} f^i(S_0)$ is the strong chain transitive component of f containing S_0.*

Proof. It is obvious that, to each strong ϵ-pseudo orbit for f^n corresponds one for f with the same beginning and end. Hence, $SCR(f^n) \subseteq SCR(f)$. This implies that, if S_0 is a strong chain transitive component of f^n, then S_0 lies in a single strong chain transitive component of f, say $S^{'}$. Strong chain transitive components are invariant, so $S^{'}$ contains S. Now, let $s \in S_0$ and suppose that $z \in S^{'}$. For each $k > 0$, there is a strong $\frac{1}{kn}$-pseudo orbit for f, γ_k of length L_k, that begins and ends at s and contain z. By concatenating such a strong pseudo orbit with itself n number of times, we obtain a strong $\frac{1}{k}$-pseudo orbit whose length is a multiple of n for f. So, we may assume that, γ_k is a strong $\frac{1}{k}$-pseudo orbit of length Γ_k such that Γ_k is a multiple of n. Let M_k denote the length of some initial segment of γ_k that ends at z; the concatenation argument allows us to assume that L_k is much larger than M_k, which in turn is much larger than n. For some j with $1 \leq j \leq n$, there are infinitely many k's for which M_k is equivalent to j modulo n. Let y_k denote the points in γ_k that occur in position $M_k - j$. Let y_k converge to some point y. By continuity, $f^j(y) = z$ and by the argument of the last paragraph, we see that y is in the same strong chain transitive component of f^n as s, namely S_0. This implies that $z \in f^j(S_0) \subseteq S$. Therefore, $S = S^{'}$ and we are done. □

In the following, we assert that $SCR(f) = SCR(f^n)$ for each $n > 1$, provided f is a recurrent homeomorphism, that is $B(f) = X$.

Proposition 3.4. *If $f \in \mathcal{H}(X)$ is a recurrent homeomorphism, then for any $n \in \mathbb{N}$, $SCR(f^n) = SCR(f)$.*

Proof. By the Proposition 3.3, $SCR(f^n) \subseteq SCR(f) \subseteq CR(f)$. Now, let $x \in B(f)$. We claim that $x \in \omega(x, f^n)$. For, let $n \in \mathbb{N}$ be given. We find $m \in \{0, 1, \dots, n-1\}$ such that $f^m(x) \in \omega(x, f^n)$. Let $\{n_k\}_{k \in \mathbb{N}}$ be a sequence of positive integers such that $n_k \to \infty$ and $f^{n_k}(x) \to x$, as $k \to \infty$. Now, for each k, we choose an integer $0 \leq r_k \leq n-1$ such that n divides $n_k + r_k$.

The limit set of $\{f^{n_k+r_k}(x) : k \in \mathbb{N}\}$ is contained in $\{x, \ldots, f^{n-1}(x)\}$. So for some $0 \leq m \leq n-1$, $f^m(x)$ is a limit point of $\{f^{n_k+r_k}(x) : k \in \mathbb{N}\}$. Thus $f^m(x) \in \omega(x, f^n)$. If $m = 0$ the claim is done. Otherwise, we prove that for any $k \geq 1$, $f^{km}(x) \in \omega(x, f^n)$. By induction, let $f^{km}(x) \in \omega(x, f^n)$. We have

$$f^{(k+1)m}(x) = f^m(f^{km}(x)) \in f^m(\omega(x, f^n)) = \omega(f^m(x), f^n).$$

By $f^m(x) \in \omega(x, f^n)$ it is obvious $f^{(k+1)m}(x) \in \omega(x, f^n)$. In particular, for $k = n$, $f^{mn}(x) \in \omega(x, f^n)$. Hence for any $n \in \mathbb{N}$, there is an $m \in \{0, 1, \ldots, n-1\}$ and a sequence $\{k_l\}$ of positive integers such that $f^{nk_l}(x) \to f^{mn}(x)$. This implies that $f^{n(k_l-m)}(x) \to x$ and the claim is done. Now, for any $\epsilon > 0$, choose $k \in \mathbb{N}$ such that $d(f^{kn}(x), x) < \epsilon$. Clearly, the sequence $\{x, f^n(x), \ldots, f^{(k-1)n}(x), x\}$ is a strong ϵ-pseudo orbit for f^n from x to x and therefore, $x \in SCR(f^n)$. We deduce that if f is recurrent, then

$$B(f) = SCR(f^n) = SCR(f) = CR(f).$$

In particular, $SCR(f) = SCR(f^n)$ for each $n \in \mathbb{N}$. □

4. Shadowing and strong chain transitivity

First, we investigate strong chain transitive properties for a map with the weak shadowing property. For any $x \in X$, the *first prolongation* of f in x defined by

$$Q(x, f) = \bigcap_{\epsilon>0} \overline{\bigcup_{y \in N_\epsilon(x)} \mathcal{O}^+(y, f)}.$$

Proposition 4.1. *If f has the weak shadowing property, then each chain transitive set is strong chain transitive. In particular, the chain transitive components of f are precisely the strong chain transitive components of f.*

Proof. It is enough to verify that for any $x \in X$,

$$SCR(x, f) = CR(x, f) = Q(x, f).$$

It is well known that for homeomorphism with the weak shadowing property, $CR(x, f) \subseteq Q(x, f)$ for any $x \in X$ [5]. So, we only need to prove

$$Q(x, f) \subseteq SCR(x, f).$$

Let $y \in Q(x, f)$ and $\epsilon > 0$ be given. Choose $\eta > 0$ such that $d(x, z) < \eta$ implies $d(f(x), f(z)) < \epsilon/2$. By the definition, we can find $z \in N_\eta(x)$ and positive integer s such that $d(y, f^s(z)) < \epsilon/2$. Then

$$\{x, f(z), \ldots, f^{s-1}(z), y\}$$

is a strong ϵ-pseudo orbit from x to y and so $y \in SCR(x, f)$. □

The following result is immediate.

Corollary 4.2. *If f has the weak shadowing property, then $SCR(f) = CR(f)$.*

The conclusion that the shadowing property implies the weak shadowing property leads to the following assertion [5].

Corollary 4.3. *Each chain transitive set of f is strong chain transitive provided f satisfies the shadowing property.*

We note that Corless and Pilyugin in [5] proved C^0-genericity of the shadowing on a compact smooth manifold (without boundary) and then Mazur [9], proved the same result for generalized homogeneous compact metric spaces with no isolated point. We recall a compact metric space X is said to be generalized homogeneous if for every $\epsilon > 0$ there exists $\epsilon_1 > 0$ such that for any positive integer n if $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ are two lists of n distinct points of X such that $d(y_i, x_i) < \epsilon_1$ for $i = 1, \dots, n$, then there is a homeomorphism h on X such that $\overline{d}(h, 1_X) < \epsilon$ and $h(y_i) = x_i$, $i = 1, \dots, n$.

Corollary 4.4. *If X is a compact smooth manifold (without boundary) or a generalized homogeneous compact metric spaces with no isolated point, then for generic homeomorphism f on X and any natural number k, $SCR(f) = SCR(f^k)$.*

Proof. By the above note, the shadowing property is generic in $\mathcal{H}(X)$ [5]. Now, let $\mathcal{R}$ be the residual subset of $\mathcal{H}(X)$ such that each element of $\mathcal{R}$ has the shadowing property. Choose $f \in \mathcal{R}$. Then f and any f^k with $k \in \mathbb{N}$ have the shadowing property [1]. By Corollaries 4.2 and 4.3, $SCR(f^k) = CR(f^k)$. On the other hand, $CR(f^k) = CR(f)$, by a classical result in topological dynamics [2]. Thus for any natural number k,

$$SCR(f) = SCR(f^k) = CR(f^k) = CR(f). \qquad \square$$

We know from [10] that a continuous map f from a compact metric space X onto itself with the average shadowing property is chain transitive.

Theorem 4.5. *Let (X, d) be a compact metric space and f be a continuous map from X onto itself. If f has the average shadowing property, then f is strong chain transitive, that is f has only one strong chain transitive component which is the entire space.*

Proof. Suppose x, y are two distinct points in X and $\epsilon > 0$ is given. It is sufficient to prove that there is a strong ϵ-pseudo orbit from x to y. By continuity of f, there is an $0 < \epsilon_0 < \frac{\epsilon}{3}$ such that $d(u, v) < 2\epsilon_0$ implies $d(f(u), f(v)) < \frac{\epsilon}{3}$ for all $u, v \in X$. Since f has the average shadowing property, there is a $\delta > 0$ such that each average δ-pseudo orbit of f can be ϵ_0-shadowed in average by some point in X. Take $N_0 > 0$ such that $\frac{3D}{N_0} < \delta$, where D is the diameter of X. Now, we construct a sequence $\{w_i\}_{i=0}^{\infty}$ as follows:

$$w_i = \begin{cases} f^{[i \bmod 2N_0]}(x) & \text{if } [i \bmod 2N_0] \in [0, N_0], \\ y_{2N_0 - ([i \bmod 2N_0]+1)} & \text{if } [i \bmod 2N_0] \in [N_0 + 1, 2N_0 - 1], \end{cases}$$

where $f(y_{j-1}) = y_j$ for every $j > 0$ and $y_0 = y$. Then

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} d(f(w_{i+k}), w_{i+k+1}) &< \frac{1}{n}\cdot\frac{n}{N_0}\cdot 3D \\ &\leq \frac{3D}{N_0} < \delta\end{aligned}$$

for each $n \geq N_0$ and $k > 0$. Therefore, $\{w_i : i \geq 0\}$ is an average δ-pseudo orbit and so ϵ_0-shadowed in average by a point $z \in X$. Now, we have the following claims.

• **Claim1**. There exists infinity many positive integers j with $n_j < n_{j+1}$ such that for each n_j,

$$w_{n_j} \in \{x, f(x), \ldots, f^{N_0}(x)\} \quad \text{with} \quad d(f^{n_j}(z), w_{n_j}) < 2\epsilon_0.$$

• **Claim2**. There exists infinity many positive integers l with $n_l < n_{l+1}$ such that for each n_l,

$$w_{n_l} \in \{y_{N_0-2}, \ldots, y_1, y\} \quad \text{with} \quad d(f^{n_l}(z), w_{n_l}) < 2\epsilon_0.$$

Proof of Claim. Without loss of generality we prove only the Claim 1. Suppose, on the contrary that there is a positive integer N such that for all integers $i > N$, whenever

$$w_i \in \{x, f(x), \ldots, f^{N_0}(x)\},$$

it is obtained that $d(f^i(z), w_i) \geq 2\epsilon_0$. Then it would be obtained that

$$\liminf_{n\longrightarrow\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), w_i) \geq \epsilon_0,$$

which is a contradiction to this fact the average δ-pseudo orbit $\{w_i : i \geq 0\}$ is ϵ_0-shadowed in average by z.

By the claim, we can choose two positive integers j_0 and l_0 such that $n_{j_0} < n_{l_0}$ and

$$w_{n_{j_0}} \in \{x, f(x), \ldots, f^{N_0}(x)\} \text{ and } d(f^{n_{j_0}}, w_{n_{j_0}}) < 2\epsilon_0;$$

$$w_{n_{l_0}} \in \{y_{N_0-2}, \ldots, y_1, y\} \text{ and } d(f^{n_{l_0}}(z), w_{n_{l_0}}) < 2\epsilon_0.$$

It may be assumed

$$\begin{aligned} w_{n_{j_0}} &= f^{j_1}(x), \\ w_{n_{l_0}} &= y_{l_1} \end{aligned}$$

for some $j_1 > 0$ and $l_1 > 0$. This gives a strong ϵ-pseudo orbit from x to y:

$$x, f(x), \ldots, f^{j_1}(x) = w_{n_{j_0}}, f^{n_{j_0}+1}(z), \ldots, f^{n_{l_0}-1}(z), w_{n_{l_0}} = y_{l_1}, \ldots, y. \quad \square$$

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