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GEODESIC SPHERES AND BALLS OF THE HEISENBERG GROUPS

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Abstract. Let $\mathbb{H}^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group equipped with a left-invariant metric. In this paper we study the Gaussian curvatures of the geodesic spheres and the volumes of geodesic balls in $\mathbb{H}^{2n+1}$.

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$ and N its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $\langle , \rangle$ on $\mathcal{N}$. Let $\mathcal{Z}$ be the center of $\mathcal{N}$. Then $\mathcal{N}$ is represented by the direct sum of $\mathcal{Z}$ and its orthgonal complement $\mathcal{Z}^\perp$.

For each $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (adX)^* Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for all $X, Y \in \mathcal{Z}^\perp$.

A 2-step nilpotent Lie algebra $\mathcal{N}$ is said to be an algebra of Heisenberg type if

$$j(Z)^2 = -|Z|^2 \,\mathrm{id}$$

for all $Z \in \mathcal{Z}$. And a Lie group N is said to be a group of Heisenberg type if its Lie algebra $\mathcal{N}$ is of Heisenberg type. The classical Heisenberg groups are examples of Heisenberg type.

The Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ a basis of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be an one dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any $i = 1, 2, \ldots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the set of vectors $\{X_i, Y_i, Z \mid i = 1, 2, \ldots, n\}$ forms an

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orthonormal basis. Let N be the simply connected 2-step nilpotent group of Heisenberg type which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$. The group N is called the $(2n+1)$-dimensional Heisenberg group and denoted by $\mathbb{H}^{2n+1}$.

In this paper, we characterize the Gaussian curvature of the geodesic spheres and the volumes of the geodesic balls on the Heisenberg group $\mathbb{H}^{2n+1}$:

Theorem A. *Let $0 < R < 2\pi$ and $B_e(R)$ be the geodesic ball with center e (identity element) and radius R in $\mathbb{H}^{2n+1}$. Then, the following holds.*

$$Vol(B_e(R)) \leq \frac{R^{2n+1}}{2n+1} Vol(S^{2n}) \left(1 + \frac{R^2}{12}\right).$$

Theorem B. *Let $0 < R < 2\pi$ and $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$Vol(B_e(R)) = 4\pi \left(\frac{R^3}{3} + 2\sum_{n=2}^{\infty}(-1)^n \frac{R^{2n+1}}{(2n+1)!(2n-1)(2n-3)}\right).$$

Theorem C. *Let $0 < R < 2\pi$ and $S_e(R)$ be the geodesic sphere with center e and radius R in $\mathbb{H}^{2n+1}$. Let $p = \gamma(R) \in S_e(R)$, where $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$. Then, Gaussian curvature $K(p)$ in $S_e(R)$ is given as follows:*

$$\begin{aligned} K(p) = &\left(-\frac{1}{4} + \frac{|Z_0|^2(1-(1-|Z_0|^2)\cos(|Z_0|R))}{2(1-\cos(|Z_0|R)) - (1-|Z_0|^2)|Z_0|R\sin(|Z_0|R)}\right) \\ &\times \left(-\frac{1}{4}(|Z_0|^4 - |Z_0|^2 + 1) + \frac{|Z_0|^2}{2(1-\cos(|Z_0|R))}\right)^{n-1}. \end{aligned}$$

2. Preliminaries

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $\langle , \rangle$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $\langle , \rangle$ on $\mathcal{N}$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^\perp$.

Recall that for $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (\mathrm{ad}X)^*Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for $X, Y \in \mathcal{Z}^\perp$. A 2-step nilpotent Lie group N is said to be *of Heisenberg type* if

$$j(Z)^2 = -|Z|^2 \,\mathrm{id}$$

for all $Z \in \mathcal{Z}$.

Let ∇ be the unique Riemannian connection on $\mathcal{N}$. If ξ_1, ξ_2 and ξ_3 are left-invariant vector fields, then the formula of the covariant derivative

$$\begin{aligned}\langle \xi_3, \nabla_{\xi_1}\xi_2\rangle = \frac{1}{2}\{&\xi_1\langle \xi_2, \xi_3\rangle + \langle \xi_1, [\xi_3, \xi_2]\rangle + \xi_2\langle \xi_1, \xi_3\rangle \\ &+ \langle \xi_2, [\xi_3, \xi_1]\rangle - \xi_3\langle \xi_2, \xi_1\rangle - \langle \xi_3, [\xi_2, \xi_1]\rangle\}\end{aligned}$$

can be reduced to

$$\langle \xi_3, \nabla_{\xi_1}\xi_2\rangle = \frac{1}{2}\{\langle \xi_1, [\xi_3, \xi_2]\rangle + \langle \xi_2, [\xi_3, \xi_1]\rangle - \langle \xi_3, [\xi_2, \xi_1]\rangle\}.$$

Using this, the covariant derivatives on $\mathcal{N}$ are given as follows:

Lemma 2.1 ([3]). *For a 2-step nilpotent Lie group N with a left invariant metric, the following hold.*

(1) $\nabla_X Y = \frac{1}{2}[X, Y]$ *for* $X, Y \in \mathcal{Z}^\perp$.
(2) $\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X$ *for* $X \in \mathcal{Z}^\perp$ *and* $Z \in \mathcal{Z}$.
(3) $\nabla_Z Z^* = 0$ *for* $Z, Z^* \in \mathcal{Z}$.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism ([10]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$\begin{aligned} X(t) \in \mathcal{Z}^\perp, \quad X'(0) = X_0, \quad X(0) = 0 \\ Z(t) \in \mathcal{Z}, \quad Z'(0) = Z_0, \quad Z(0) = 0. \end{aligned}$$

A. Kaplan ([7]) shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{aligned} X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0. \end{aligned}$$

The following lemma is useful in the later.

Lemma 2.2 ([3]). *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Then, one has*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), \ t \in R,$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via left translation. So, in above lemma, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

3. Main results

Let $\mathbb{H}^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group with a left invariant metric and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be a unit speed geodesic on $\mathbb{H}^{2n+1}$ with $\gamma(0) = e$ (the identity element of $\mathbb{H}^{2n+1}$) and $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. Since

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}$$

is an ortonormal set in $\mathcal{N}$, we can obtain an orthonormal basis

$$\mathcal{B} = \{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0, Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^\perp, \\ k = 1, 2, \ldots, n-1\}$$

by adding

$$\{Y_k, \frac{1}{|Z_0|}j(Z_0)Y_k | Y_k \in \mathcal{Z}^\perp, k = 1, 2, \ldots, n-1\}$$

to

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}.$$

Then, it is easy to show that $[X_0, Y_k] = [X_0, j(Z_0)Y_k] = 0$ for each $k = 1, 2, \ldots, n-1$.

Proposition 3.1 ([1], [6]). *Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. If $J(t)$ is a normal Jacobi field along γ in $\mathbb{H}^{2n+1}$ with $J(0) = 0$, then*

$$\begin{aligned} J(t) =& (c_1(\sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t) + c_2(1 - \cos(|Z_0|t)))e_1(t) \\ &+ (c_1|Z_0|(\cos(|Z_0|t) - 1) + c_2|Z_0|\sin(|Z_0|t)e_2(t) \\ &+ \sum_{k=2}^{n}[\{\frac{c_{2k-1}}{|Z_0|}\sin(|Z_0|t) + \frac{c_{2k}}{|Z_0|}(1 - \cos(|Z_0|t))\}e_{2k-1}(t) \\ &\quad + \{c_{2k-1}|Z_0|(\cos(|Z_0|t) - 1) + c_{2k}|Z_0|\sin(|Z_0|t)\}e_{2k}(t)], \end{aligned}$$

where $c_{2k-1}, c_{2k}, k = 1, 2, \ldots, n$ are arbitrary constants and $e_{2k-1}(t), e_{2k}(t), k = 1, 2, \ldots, n$ are given in Lemma 3.2.

Lemma 3.2. *Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. Let*

$$e_1(t) = \frac{|Z_0|}{|X_0|}X'(t) - \frac{|X_0|}{|Z_0|}Z_0,$$
$$e_2(t) = \frac{1}{|Z_0||X_0|}j(Z_0)X'(t)$$

and let

$$e_{2k-1}(t) = e^{tj(Z_0)}Y_k,$$
$$e_{2k}(t) = \frac{1}{|Z_0|}e^{tj(Z_0)}j(Z_0)Y_k \quad \textit{for each } k = 2, 3, \ldots, n.$$

Then, $\{\gamma'(t), e_{2k-1}(t), e_{2k}(t) | k = 1, 2, \ldots, n\}$ *is an orthonormal frame along* $\gamma(t)$ *on* $\mathbb{H}^{2n+1}$ *such that*

(1) $\nabla_{\gamma'(t)} e_1(t) = \frac{1}{2} e_2(t)$ *and* $\nabla_{\gamma'(t)} e_2(t) = -\frac{1}{2} e_1(t)$

(2) $\nabla_{\gamma'(t)} e_{2k-1}(t) = \frac{|Z_0|}{2} e_{2k}(t)$ *and* $\nabla_{\gamma'(t)} e_{2k}(t) = -\frac{|Z_0|}{2} e_{2k-1}(t)$ *for each* $k = 2, 3, \ldots, n$.

Or simply,

$$\begin{bmatrix} e_1'(t) \\ e_2'(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} e_{2k-1}'(t) \\ e_{2k}'(t) \end{bmatrix} = \frac{|Z_0|}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix}$$

for $k = 2, 3, \ldots, n$.

The following proposition is a slight modification of Proposition 3.1, which is useful.

Proposition 3.3. *For each* $k = 1, 2, \ldots, n$, *let* $J_{2k-1}(t)$ *and* $J_{2k}(t)$ *be the Jacobi fields with* $J_{2k-1}(0) = J_{2k}(0) = 0, J_{2k-1}'(0) = e_{2k-1}(0)$ *and* $J_{2k}'(0) = e_{2k}(0)$. *Then, we have that*

(1) *for* $k = 1$,

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

where

$$B_1(t) = \frac{1}{|Z_0|^3} \begin{bmatrix} \sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t & |Z_0|(\cos(|Z_0|t) - 1) \\ |Z_0|(1 - \cos(|Z_0|t)) & |Z_0|^2 \sin(|Z_0|t) \end{bmatrix},$$

(2) *for* $k = 2, 3, \ldots, n$

$$\begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix} = B_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix},$$

where

$$B_k(t) = \begin{bmatrix} \frac{1}{|Z_0|} \sin(|Z_0|t) & |Z_0|(\cos(|Z_0|t) - 1) \\ \frac{1}{|Z_0|^3}(1 - \cos(|Z_0|t)) & \frac{1}{|Z_0|} \sin(|Z_0|t) \end{bmatrix}.$$

Proof. Let $J(t)$ be a normal Jacobi field along $\gamma(t)$ with $J(0) = 0$. Then, by Proposition 3.1 and Lemma 3.2, we can represent $J(t)$ as follow.

$$J(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{2n-1} & c_{2n} \end{bmatrix} \begin{bmatrix} B_1(t) & 0 & \cdots & 0 \\ 0 & B_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n(t) \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_{2n-1}(t) \\ e_{2n}(t) \end{bmatrix}.$$

Since $B_k(0) = 0$ and $B'_k(0) = I$ for each $k = 1, 2, \ldots, n$, we have that

$$J'(0) = c_l e_1(0) + c_2 e_2(0) + \cdots + c_{2n-1} e_{2n-1}(0) + c_{2n} e_{2n}(0).$$

Just letting $J'(0) = e_k(0)$ for each $k = 1, 2, \ldots, 2n$, we completes the proof. $\square$

Corollary 3.4 ([1], [6]). *Let $\mathbb{H}^{2n+1}$ be the $(2n+1)$-dimensional Heisenberg group and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be a unit speed geodesic in N with $\gamma(0) = e$ (the identity element of N) and $\gamma'(0) = X_0 + Z_0$, where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$.*

(1) *If $Z_0 \neq 0$, then all the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$ where*

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \ldots\}$$

and

$$\mathbb{A} = \{t \in \mathbb{R} - \{0\} | (1 - |Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2}\}.$$

In particular, $\frac{2\pi}{|Z_0|}$ is the first conjugate point of e along γ.

(2) *If $Z_0 = 0$, then there are no conjugate points along γ.*

G. Walschap ([11]) showed that the first conjugate loci and the cut loci are equal in the case of the groups of Heisenberg type or the 2-step nilpotent groups with one-dimensional center.

So, we see that the geodesic sphere $S_e(r)$ with center e and radius r is defined if and only if $r \leq 2\pi$. So, we consider the geodesic spheres $S_e(r)$ and the geodesic balls $B_e(r)$ with the radius $r \leq 2\pi$.

Note that

$$\det(B_1(t)) = \frac{1}{|Z_0|^4}\{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\}$$

and

$$\det(B_k(t)) = \frac{2}{|Z_0|^2}(1 - \cos(|Z_0|t))$$

for each $k = 2, 3, \ldots, n$.

Lemma 3.5 ([9]). *For $t \geq 0$, the following holds.*

$$\det(B_1(t)) = \frac{1}{|Z_0|^4}\{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t \sin(|Z_0|t)\} \geq 0.$$

Lemma 3.6. *For $x > 0$, the followings are hold.*

(1) $\frac{\sin x}{x} \leq 1$.
(2) $\frac{1 - \cos x}{x^2} \leq \frac{1}{2}$.
(3) $\frac{2(1 - \cos x) - x \sin x}{x^4} \leq \frac{1}{12}$.

Proof. We give only the proof of (3) since others are easy. Let

$$f(t) = \frac{1}{12}x^4 - \{2(1 - \cos x) - x \sin x\}.$$

Then, we have that

$$f'(x) = \frac{1}{3}x^3 - (\sin x - x \cos x)$$

and

$$f''(x) = x(x - \sin x).$$

Since $f''(x) > 0$ for $x > 0$ and $f'(0) = 0$, we see that $f'(x) > 0$. Since $f(0) = 0$, we get that $f(x) > 0$ for $x > 0$. □

Let M be a Riemannian manifold with a metric g and $p \in M$. Take an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of T_pM and let $(x_1, x_2, \ldots, x_n)$ be the coordinates determined by $\{u_1, u_2, \ldots, u_n\}$. This local coordinate system is called the normal coordinate system at p. It is easy to show that

$$\frac{\partial}{\partial x_i}\bigg|_m = (d\exp_p)_{\sum_{i=1}^n x_i u_i}(u_i),$$

where $m = \exp_p(\sum_{i=1}^n x_i u_i)$. Then, the volume form v_g on U_p is given by

$$v_g = \sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)}dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n,$$

where g_{ij} is the metric coefficients of g in U_p. Therefore, the volume of the geodesic ball $B_p(r)$ is given by

$$Vol(B_p(r)) = \int_{\exp_p^{-1}(B_p(r))} \exp_p^* v_g.$$

Let $\gamma(t)$ be the unit speed geodesic in M with $\gamma(0) = p$, $\gamma'(0) = u_1$ and let $J_i(t)$ be the Jacobi field with $J_i(0) = 0$ and $J_i'(0) = u_i$ for each $i = 2, 3, \ldots, n$. Then we know that

$$(d\exp_p)_{tu_1}u_1 = \gamma'(t)$$

and

$$(d\exp_p)_{tu_1}u_i = \frac{1}{t}J_i(t)$$

for each $i = 2, 3, \ldots, n$. So, we see that

$$\sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)} = t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}.$$

Hence, we have that

$$\begin{aligned}\exp_p^* v_g &= t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}dx_1 dx_2 \cdots dx_n \\ &= \sqrt{\det(g(J_i(t), J_j(t)))}dt du,\end{aligned}$$

where du denote the canonical measure of the unit sphere S^{n-1}. Therefore, by Fubini's Theorem we get that

$$Vol(B_p(r)) = \int_{S^{n-1}} \int_0^r \sqrt{\det(g(J_i(t), J_j(t)))} dt du.$$

Now we are ready to prove the following proposition which is concerned to the volume of geodesic ball in the Heisenberg group $\mathbb{H}^{2n+1}$ with a left invariant metric.

Theorem 3.7. *Let $0 < R < 2\pi$ and $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^{2n+1}$. Then, the following holds.*

$$Vol(B_e(R)) \leq \frac{R^{2n+1}}{2n+1} Vol(S^{2n}) \left(1 + \frac{R^2}{12}\right).$$

Proof. Using Propsition 3.3, we obtain that

$$\begin{aligned}
&\det\left(\langle J_i(t), J_j(t)\rangle\right) \\
=& \det\left(J_i(t) \cdot J_j(t)\right) \\
=& \det\left(\begin{bmatrix} J_1(t) \\ J_2(t) \\ \vdots \\ J_{2n-1}(t) \\ J_{2n}(t) \end{bmatrix} \begin{bmatrix} J_1(t) & J_2(t) & \cdots & J_{2n-1}(t) & J_{2n}(t) \end{bmatrix} \right) \\
=& \prod_{k=1}^{n} \det\left(B_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} \cdot^t \left(B_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} \right) \right) \\
=& \prod_{k=1}^{n} \det\left(B_k(t) \cdot^t (B_k(t)) \right) \\
=& \left(\frac{1}{|Z_0|^4} \left\{ 2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0| t \sin(|Z_0|t) \right\} \left\{ \frac{2}{|Z_0|^2}(1 - \cos(|Z_0|t)) \right\}^{n-1} \right)^2.
\end{aligned}$$

So, by Lemma 3.5 and Lemma 3.6, we have that

$$\begin{aligned}
&\sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right))} \\
=& \frac{1}{|Z_0|^4} \{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0| t \sin(|Z_0|t)\} \left\{ \frac{2}{|Z_0|^2}(1 - \cos(|Z_0|t)) \right\}^{n-1} \\
=& \left\{ \frac{2(1 - \cos(|Z_0|t)) - |Z_0| t \sin(|Z_0|t)}{(|Z_0|t)^4} t^4 + \frac{\sin(|Z_0|t)}{|Z_0|t} t^2 \right\} \left\{ \frac{2(1 - \cos(|Z_0|t))}{(|Z_0|t)^2} t^2 \right\}^{n-1} \\
\leq& \left(\frac{1}{12} t^4 + t^2 \right) t^{2n-2}.
\end{aligned}$$

Hence, we get that

$$\begin{aligned}
Vol(B_e(R)) &= \int_{S^{2n}} \int_0^R \sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right)} dt du \\
&\le Vol(S^{2n}) \int_0^R \left(\frac{1}{12}t^4 + t^2\right) t^{2n-2} dt \\
&= Vol(S^{2n}) \left(\frac{R^{2n+3}}{12(2n+3)} + \frac{R^{2n+1}}{2n+1}\right) \\
&= \frac{R^{2n+1}}{2n+1} Vol(S^{2n}) \left(1 + \frac{2n+1}{12(2n+3)} R^2\right) \\
&\le \frac{R^{2n+1}}{2n+1} Vol(S^{2n}) \left(1 + \frac{R^2}{12}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let $0 < R < 2\pi$ and $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$Vol(B_e(R)) = 4\pi \left(\frac{R^3}{3} + 2\sum_{n=2}^{\infty} (-1)^n \frac{R^{2n+1}}{(2n+1)!(2n-1)(2n-3)}\right).$$

Proof. Let $u = (x_1, x_2, x_3) \in S^2$ and

$$\begin{aligned}
f(x_3, t) &= \sqrt{\det\left(\langle J_i(t), J_j(t)\rangle\right)} \\
&= \frac{2(1-\cos(x_3 t)) - x_3 t \sin(x_3 t)}{(x_3 t)^4} t^4 + \frac{\sin(x_3 t)}{x_3 t} t^2.
\end{aligned}$$

Then, we see that

$$Vol(B_e(R)) = \int_{S^2} \int_0^R f(x_3, t) dt du.$$

Let $D = \{(x_1, x_2) | x_1^2 + x_2^2 \le 1\}$, then since area element du on the sphere S^2 is given by

$$du = \frac{1}{\sqrt{1-(x_1^2+x_2^2)}} dx_1 dx_2,$$

we have that

$$Vol(B_e(R)) = 2\int_D \int_0^R f(\sqrt{1-(x_1^2+x_2^2)}, t) \frac{1}{\sqrt{1-(x_1^2+x_2^2)}} dt dx_1 dx_2.$$

Changing the coordinates on D to polar coordinates, we have

$$Vol(B_e(R)) = 4\pi \int_0^1 \int_0^R f(\sqrt{1-r^2}, t) \frac{r}{\sqrt{1-r^2}} dt dr.$$

Replacing $x = \sqrt{1-r^2}$, we see that

$$Vol(B_e(R)) = 4\pi \int_0^1 \int_0^R f(x,t)dtdx,$$

where

$$f(x,t) = \frac{2(1-\cos(xt)) - xt\sin(xt)}{(xt)^4} t^4 + \frac{\sin(xt)}{xt} t^2.$$

Since

$$\frac{\sin(xt)}{xt} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(xt)^{2n-2}}{(2n-1)!}$$

and

$$\begin{aligned} &\frac{2(1-\cos(xt)) - xt\sin(xt)}{(xt)^4} \\ &= \frac{1}{(xt)^4} \left(2\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(xt)^{2n}}{(2n)!} - xt \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(xt)^{2n-1}}{(2n-1)!} \right) \\ &= \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{2}{(2n)!} - \frac{1}{(2n-1)!} \right) (xt)^{2n-4}, \end{aligned}$$

we see that

$$\int_0^1 \frac{\sin(xt)}{xt} dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-2}}{(2n-1)!(2n-1)}$$

and

$$\int_0^1 \frac{2(1-\cos(xt)) - xt\sin(xt)}{(xt)^4} dx = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2n-3} \left(\frac{2}{(2n)!} - \frac{1}{(2n-1)!} \right) t^{2n-4}.$$

Hence, we have that

$$\begin{aligned} &\int_0^1 f(x,t)dx \\ &= t^4 \int_0^1 \frac{2(1-\cos(xt)) - xt\sin(xt)}{(xt)^4} dx + t^2 \int_0^1 \frac{\sin(xt)}{xt} dx \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!(2n-1)(2n-3)} t^{2n}. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} Vol(B_e(R)) &= 4\pi \int_0^R \int_0^1 f(x,t)dxdt \\ &= 8\pi \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!(2n-1)(2n-3)} R^{2n+1}. \end{aligned}$$

Or,

$$Vol(B_e(R)) = 4\pi\left(\frac{R^3}{3} + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n+1)!(2n-1)(2n-3)}R^{2n+1}\right). \qquad \square$$

Let $\bar{M}$ be a Riemannian manifold, M its Riemannian submanifold with codimension 1, $p \in M$ and a normal vector η to $T_p(M)$. The shape operator

$$S_p : T_p(M) \to T_p(M)$$

is defined by

$$S_p(x) = -(\bar{\nabla}_x N)^T \text{ for any } x \in T_p(M),$$

where N is a local extension of η normal to M and ^{T} denotes the tangential component to $T_p(M)$. It is easy to show that if η and its extension N are unit vector and unit vector field, then the shape operator is given by

$$S_p(x) = -\bar{\nabla}_x N \text{ for any } x \in T_p(M).$$

The shape operator S_p is symmetric, so there exists an orthonormal basis of eigenvectors $\{e_1, e_2, \ldots, e_n\}$ with real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We say that the e_i are principal directions and λ_i are principal curvatures of M at p. The determinant of shape operator

$$\det(S_p) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$$

is called the Gaussian curvature.

Lemma 3.9 ([12]). *Let $\bar{M}$ be a Riemannian manifold, $m \in \bar{M}$ and M the geodesic sphere with center m and radius $r > 0$ and $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = m$. Let $J(t)$ be a Jacobi vector fields along γ with $J(0) = 0$, which is normal to γ and S_p the shape operator of M at $p = \gamma(r)$. Then, we have that $S_p(J(r)) = -J'(r)$.*

Using this lemma, we characterize the Gaussian curvatures on the geodesic spheres of the Heisenberg group $\mathbb{H}^{2n+1}$.

Theorem 3.10. *Let $0 < R < 2\pi$ and $S_e(R)$ be the geodesic sphere with center e and radius R in $\mathbb{H}^{2n+1}$. Let $p = \gamma(R) \in S_e(R)$, where $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$. Then, Gaussian curvature $K(p)$ in $S_e(R)$ is given as follows*:

$$\begin{aligned} K(p) = &\left(-\frac{1}{4} + \frac{|Z_0|^2(1-(1-|Z_0|^2)\cos(|Z_0|R))}{2(1-\cos(|Z_0|R)) - (1-|Z_0|^2)|Z_0|R\sin(|Z_0|R)}\right) \\ &\times\left(-\frac{1}{4}(|Z_0|^4 - |Z_0|^2 + 1) + \frac{|Z_0|^2}{2(1-\cos(|Z_0|R))}\right)^{n-1}. \end{aligned}$$

Proof. By Proposition 3.3, we know that

$$\begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix} = B_k(t)\begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix}$$

for each $k = 1, 2, \ldots, n$.

Since

$$\begin{bmatrix} J'_{2k-1}(t) \\ J'_{2k}(t) \end{bmatrix} = B'_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} + B_k(t) \begin{bmatrix} e'_{2k-1}(t) \\ e'_{2k}(t) \end{bmatrix},$$

by Lemma 3.2, we have that

$$\begin{aligned} \begin{bmatrix} J'_1(t) \\ J'_2(t) \end{bmatrix} &= B'_1(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + B_1(t) \begin{bmatrix} e'_1(t) \\ e'_2(t) \end{bmatrix} \\ &= \left(B'_1(t) + \frac{1}{2} B_1(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} J'_{2k-1}(t) \\ J'_{2k}(t) \end{bmatrix} &= B'_k(t) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} + B_k(t) \begin{bmatrix} e'_{2k-1}(t) \\ e'_{2k}(t) \end{bmatrix} \\ &= \left(B'_k(t) + \frac{|Z_0|}{2} B_k(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} \end{aligned}$$

for each $k = 2, 3, \ldots, n$.

Since

$$\begin{bmatrix} e_{2k-1}(t) \\ e_{2k}(t) \end{bmatrix} = B_k(t)^{-1} \begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix}$$

for eack $k = 1, 2, \ldots, n$, we have that

$$\begin{bmatrix} J'_1(t) \\ J'_2(t) \end{bmatrix} = \left(B'_1(t) + \frac{1}{2} B_1(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) B_1(t)^{-1} \begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} J'_{2k-1}(t) \\ J'_{2k}(t) \end{bmatrix} = \left(B'_k(t) + \frac{|Z_0|}{2} B_k(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) B_k(t)^{-1} \begin{bmatrix} J_{2k-1}(t) \\ J_{2k}(t) \end{bmatrix}$$

for each $k = 2, 3, \ldots, n$.

Since the shape operator $S_{\gamma(t)}$ of the geodesic sphere $S_e(t)$ is given by

$$S_{\gamma(t)}(J(t)) = -J'(t),$$

the Gaussian curvature $K_{\gamma(t)}$ is

$$\begin{aligned} &K_{\gamma(t)} \\ =\ &\det(S_{\gamma(t)}) \\ =\ &\frac{\det\left(B'_1(t) + \frac{1}{2} B_1(t) \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right)}{\det(B_1(t))} \times \prod_{k=2}^{n} \frac{\det\left(B'_k(t) + \frac{|Z_0|}{2} B_k(t) \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right)}{\det(B_k(t))}. \end{aligned}$$

Direct calculations give that

$$\frac{\det\left(B'_1(t) + \frac{1}{2} B_1(t) \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\right)}{\det(B_1(t))} = -\frac{1}{4} + \frac{|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|t))}{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t\sin(|Z_0|t)}$$

and

$$\frac{\det\left(B_k'(t)+\frac{|Z_0|}{2}B_k(t)\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\right)}{\det(B_k(t))}=-\frac{1}{4}(|Z_0|^4-|Z_0|^2+1)+\frac{|Z_0|^2}{2(1-\cos(|Z_0|t))}.$$

Hence, the Gaussian curvatue $K_p, p=\gamma(R)$ on the geodesic sphere $S_e(R)$ is given by

$$\begin{aligned}K(p)=&\left(-\frac{1}{4}+\frac{|Z_0|^2(1-(1-|Z_0|^2)\cos(|Z_0|R))}{2(1-\cos(|Z_0|R))-(1-|Z_0|^2)|Z_0|R\sin(|Z_0|R)}\right)\\&\times\left(-\frac{1}{4}(|Z_0|^4-|Z_0|^2+1)+\frac{|Z_0|^2}{2(1-\cos(|Z_0|R))}\right)^{n-1}.\end{aligned}$$ □

By Lemma 3.5, we have that

Corollary 3.11 ([9]). *Let $0<R<2\pi$ and $S_e(R)$ be the geodesic sphere with center e and radius R in $\mathbb{H}^3$. Let $p=\gamma(R)\in S_e(R)$, where $\gamma(t)$ is a unit speed geodesic with $\gamma(0)=e$ and $\gamma'(0)=X_0+Z_0$. Then, Gaussian curvature $K(p)$ of $S_e(R)$ is given as follows*:

$$K(p)=-\frac{1}{4}+\frac{|Z_0|^2(1-(1-|Z_0|^2)\cos(|Z_0|R))}{2(1-\cos(|Z_0|R))-(1-|Z_0|^2)|Z_0|R\sin(|Z_0|R)}.$$

In particular, the Gaussian curvatures of the geodesic spheres on the 3-dimensional Heisenberg groups are greater than $-\frac{1}{4}$.

Remark 3.12. If $|Z_0|=1$, then $K_p=\left(-\frac{1}{4}+\frac{1}{2(1-\cos R)}\right)^n$. So, we see that K_p goes to ∞ if the radius R goes to 2π. And if $|Z_0|=0$, then since

$$\lim_{|Z_0|\to 0}\left(-\frac{1}{4}+\frac{|Z_0|^2(1-(1-|Z_0|^2)\cos(|Z_0|R))}{2(1-\cos(|Z_0|R))-(1-|Z_0|^2)|Z_0|R\sin(|Z_0|R)}\right)=-\frac{1}{4}+\frac{6(2+R^2)}{r^2(12+R^2)}$$

and

$$\lim_{|Z_0|\to 0}\frac{|Z_0|^2}{2(1-\cos(|Z_0|R))}=\frac{1}{R^2},$$

we see that

$$K_p=\left(-\frac{1}{4}+\frac{6(2+R^2)}{R^2(12+R^2)}\right)\left(-\frac{1}{4}+\frac{1}{R^2}\right)^{n-1}.$$

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