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AN ITERATIVE SCHEME FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF ASYMPTOTICALLY k-STRICT PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. In this paper, we propose an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of an asymptotically k-strict pseudo-contractive mapping in the setting of real Hilbert spaces. We establish some weak and strong convergence theorems of the sequences generated by our proposed scheme. Our results are more general than the known results which are given by many authors. In particular, necessary and sufficient conditions for strong convergence of our iterative scheme are obtained.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $F : C\times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality.

In addition, there are so many other problems which also can be transformed into the model of an EP, for example, the complementarity problem, fixed point problem and optimization problem. In other words, the EP provides us with a natural, novel and unified framework for studying a wide class of problems arising in economics, finance, physics, transportation, network and structural analysis, elasticity and optimization, etc. Recently, many papers have appeared in the literature on the existence of solutions of EP (see e.g., [1, 6, 8, 9] and

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references therein). Some solutions have been proposed to solve the EP (see e.g., [2, 3, 4, 5, 14, 15] and references therein).

Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C. \tag{1.2}$$

We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in D(T) : Tx = x\}$. If C is nonempty, closed and convex subset of a real Hilbert space H, then $F(T)$ is closed and convex; Further, if C is bounded, closed and convex, then $F(T)$ is nonempty, see [7] for more details.

Takahashi and Takahashi [15] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP (1.1) and the set of fixed points of a nonexpansive mapping in the setting of Hilbert spaces. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space.

Given a closed convex subset C of a real Hilbert space H. A mapping $T : C \to C$ is called k-strict pseudo-contractive mapping if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \text{for all } x, y \in C. \tag{1.3}$$

Clearly,we find T is a nonexpensive mapping if and only if T is a 0-strict pseudo-contractive mapping.

A mapping $T : C \to C$ is said to be an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \tag{1.4}$$

for all $x, y \in C$. We easily obtain that every k-strict pseudo-contractive mappings are asymptotically k-strict pseudo-contractive mappings with sequence $\{\gamma_n\}$, where $\gamma_n \equiv 0$, and $n = 1$.

Very recently, Ceng, Homidan, Ansari, and Yao [2] introduced the following iterative scheme:

Let C be a nonempty closed convex subset of H, $F : C \times C \to \mathbb{R}$ be a bifunction and $S : C \to C$ be a k-strict pseudo-contractive mapping for some $0 \leq k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \text{for all } y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)Su_n \quad \text{for } n \geq 1. \end{cases} \tag{1.5}$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(F)$ under some parameters controlling conditions.

Further more, they obtained a necessary and sufficient condition for strong convergence of iterative scheme (1.5) under some controlling conditions.

Motivated and inspired by the facts above, in this paper, we propose a new iterative scheme for finding a common element of the set of solutions of EP (1.1) and set of fixed points of an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ defined in the setting of a real Hilbert spaces. Our results are more general than the results of [2].

2. Preliminaries

We will use the notation:
1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence;
2. $\omega_w(x_n) = \{x : \exists\, x_{n_i} \rightharpoonup x\}$ denotes the weak $\omega-$limit set of $\{x_n\}$.

Throughout the paper, we consider H is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, C is a nonempty closed convex subset of H.

Let us recall the following definitions and results which will be used in the sequel.

Lemma 2.1 ([11]). *Let H be a real Hilbert space. Then the following hold*:
(a) $\|x-y\|^2 = \|x\| - \|y\|^2 - 2\langle x-y, y\rangle, \forall x, y \in H$;
(b) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x-y\|^2$, $\forall t \in [0,1]$, $\forall x, y \in H$;
(c) *If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x-y\|^2, \ \forall y \in H.$$

Lemma 2.2 ([13]). *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality*:

$$\delta_{n+1} \le \beta_n \delta_n + \gamma_n \quad \text{for all } n \in \mathbb{N} \tag{2.1}$$

if $\beta_n \ge 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \delta_n$ exists.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.2}$$

Such a P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.3 ([11]). *Let C be nonempty closed convex subset of a real Hilbert space H, given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:
(A_1) $F(x,x) = 0$ for all $x \in C$;

(A_2) F is monotone, i.e., $F(x,y)+F(y,x)\le 0$ for all $x,y\in C$;
(A_3) For each $x,y,z\in C$, $\lim_{t\to 0}F(tz+(1-t)x,y)\le F(x,y)$;
(A_4) For each $x\in C$, the function $y\mapsto F(x,y)$ is convex and lower semicontinuous.

The following lemma appeared implicitly in [1].

Lemma 2.4 (Also see [4, 15])**.** *Let C be a nonempty closed convex subset of H and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying $(A_1)-(A_4)$. Let $r>0$ and $x\in H$, then, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0 \quad \text{for all } y\in C. \tag{2.4}$$

Lemma 2.5 ([4])**.** *Assume $F: C\times C\to\mathbb{R}$ satisfies $(A_1)-(A_4)$. For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\{z\in C: F(x,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0 \quad \text{for all } y\in C\} \tag{2.5}$$

for all $z\in H$. Then, the following hold:
1. *T_r is single-valued;*
2. *T_r is firmly nonexpansive, i.e., for any $x,y\in H$,*

$$\|T_rx-T_ry\|^2\le\langle T_rx-T_ry,x-y\rangle;$$

3. *$F(T_r)=EP(F)$;*
4. *$EP(F)$ is closed and convex.*

In order to obtain the following lemma, we firstly introduced the notion of uniformly L-Lipschtizan. A mapping $T: C\to C$ is said to be uniformly L-Lipschtizan, if there exists a constant $L>0$ such that

$$\|T^nx-T^ny\|\le L\|x-y\| \quad \forall x,y\in C,\ n\ge 1. \tag{2.6}$$

Lemma 2.6 ([10])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $S: C\to C$ be a self-mapping of C.*

(i) *If S is an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$, then S satisfies the uniformly L-Lipschtizan condition*

$$\|T^nx-T^ny\|\le L\|x-y\|,$$

where $L:=\frac{k+\sqrt{1+M}}{1-k}$, $M:=\sup_n\{\gamma_n\}$.

(ii) *If S is an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$, then the mapping $I-S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n\rightharpoonup\tilde{x}\in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|(I-S^m)x_n\|\to 0$, then $(I-S)\tilde{x}=0$.*

(iii) *If S is an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$, then the fixed point set $F(S)$ of S is closed and convex.*

Proof. (i) For $x, y \in C$, we have

$$\begin{aligned}\|T^n x - T^n y\|^2 &\le (1+\gamma_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \\ &\le (1+\gamma_n)\|x-y\|^2 + k(\|x-y\| + \|T^n x - T^n y\|)^2 \\ &\le (1+k+\gamma_n)\|x-y\|^2 \\ &\quad + k(2\|x-y\|\|T^n x - T^n y\| + \|T^n x - T^n y\|^2).\end{aligned}$$

It gives us that

$$(1-k)\|T^n x - T^n y\|^2 - 2k\|x-y\|\|T^n x - T^n y\| - (1+k+\gamma_n)\|x-y\|^2 \le 0. \tag{2.7}$$

Solving this quadratic inequality, we obtain that

$$\|T^n x - T^n y\| \le \frac{k + \sqrt{1+\gamma_n - k\gamma_n}}{1-k}\|x-y\|. \tag{2.8}$$

Let $L_n = \frac{k+\sqrt{1+\gamma_n - k\gamma_n}}{1-k}$ and $M = \sup_n\{\gamma_n\}$, we have

$$\begin{aligned}\|T^n x - T^n y\| &\le \frac{k + \sqrt{1+\gamma_n - k\gamma_n}}{1-k}\|x-y\| \\ &\le \frac{k+\sqrt{1+M}}{1-k}\|x-y\|.\end{aligned}$$

Hence (i) holds.

As for (ii), (iii), T.-H. Kim and H.-K. Xu [10] have given good proof. □

Let K be a nonempty closed subset of a Banach space E. A mapping $T : K \to K$ is said to be semicompact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^\star \in K$ (as $i \to \infty$).

We propose an iterative scheme for finding a common element of the set of solutions of EP (1.1) and the set of fixed points of an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ in the setting of real Hilbert spaces. We also prove the weak and strong convergences of the sequences generated by our iterative scheme.

3. Weak convergence results

Theorem 3.1. *Let C be a nonempty closed convex subset of H, $F : C \times C \to \infty$ be a bifunction satisfying $(A_1) - (A_4)$ and $S : C \to C$ be an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ for some $0 \le k < 1$ such that $F(S) \cap EP(F) \ne \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\ y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)S^n u_n, \ \forall\ n \ge 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) *$\{\alpha_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$;*

(2) $\{\gamma_n\} \in [0,\infty)$ *with* $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n)\gamma_n < \infty$;
(3) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.
Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to an element of* $F(S) \cap EP(F)$.

Proof. We divide the proof into six steps.

Step 1. $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(S)\cap EP(F)$.

Indeed, let p be an arbitrary element of $F(S)\cap EP(F)$. Then from the definition of T_r in Lemma 2.5, we have $u_n = T_{r_n}x_n$, and therefore

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \le \|x_n - p\|, \quad \forall\ n \ge 1. \tag{3.2}$$

Since T is an asymptotically k-strict pseudo-contractive mapping, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n u_n + (1-\alpha_n)S^n u_n - p\|^2\\ &= \alpha_n\|u_n-p\|^2 + (1-\alpha_n)\|S^n u_n - p\|^2\\ &\quad - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2\\ &\le \alpha_n\|u_n-p\|^2 + (1-\alpha_n)[(1+\gamma_n)\|u_n-p\|^2 + k\|u_n - S^n u_n\|^2]\\ &\quad - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2\\ &= [1+(1-\alpha_n)\gamma_n]\|u_n-p\|^2 - (\alpha_n - k)(1-\alpha_n)\|u_n - S^n u_n\|^2.\end{aligned} \tag{3.3}$$

Since $k < \alpha \le \alpha_n \le \beta < 1$ for all $n \ge 1$, we get

$$\|x_{n+1}-p\|^2 \le [1+(1-\alpha_n)\gamma_n]\|u_n-p\|^2. \tag{3.4}$$

And since $[1+(1-\alpha_n)\gamma_n] \ge 1$, we have

$$\|x_{n+1}-p\|^2 \le [1+(1-\alpha_n)\gamma_n]\|u_n-p\|^2 \le [1+(1-\alpha_n)\gamma_n]^2\|u_n-p\|^2.$$

So we obtain the following inequality,

$$\|x_{n+1}-p\| \le [1+(1-\alpha_n)\gamma_n]\|u_n-p\|. \tag{3.5}$$

Since $\{\alpha_n\} \subset [\alpha,\beta]$ for some $\alpha,\beta \in (k,1)$, $\sum_{n=1}^{\infty}(1-\alpha_n)\gamma_n < \infty$ and by Lemma 2.2, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exists and hence $\{x_n\}$ is bounded.

Step 2. $\lim_{n\to\infty}\|u_n - S^n u_n\| = 0$.

We can suppose $\lim_{n\to\infty}\|x_n - p\| = r$ for some $r \ge 0$. It's easy to see from (3.3) that

$$\begin{aligned}(\alpha-k)(1-\beta)\|u_n - S^n u_n\|^2 &\le (\alpha_n - k)(1-\alpha_n)\|u_n - S^n u_n\|^2\\ &\le [1+(1-\alpha_n)\gamma_n]\|x_n-p\|^2 - \|x_{n+1}-p\|^2.\end{aligned} \tag{3.6}$$

This implies that $\lim_{n\to\infty}\|u_n - S^n u_n\| = 0$.

Step 3. $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$.

Note that

$$u_n - x_{n+1} = (1-\alpha_n)(u_n - S^n u_n).$$

Now, we compute

$$
\begin{aligned}
& \|x_{n+1} - S^n x_{n+1}\|^2 \\
&= \|\alpha_n(u_n - S^n x_{n+1}) + (1-\alpha_n)(S^n u_n - S^n x_{n+1})\|^2 \\
&= \alpha_n \|u_n - S^n x_{n+1}\|^2 + (1-\alpha_n)\|S^n u_n - S^n x_{n+1}\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\le \alpha_n \|(u_n - x_{n+1}) + (x_{n+1} - S^n x_{n+1})\|^2 \\
&\quad - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + (1-\alpha_n)[(1+\gamma_n)\|u_n - x_{n+1}\|^2 \\
&\quad + k\|(u_n - S^n u_n) - (x_{n+1} - S^n x_{n+1})\|^2] \\
&= \alpha_n(\|u_n - x_{n+1}\|^2 + \|x_{n+1} - S^n x_{n+1}\|^2 \\
&\quad - 2\langle u_n - x_{n+1}, x_{n+1} - S^n x_{n+1}\rangle) - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + (1-\alpha_n)[(1+\gamma_n)\|u_n - x_{n+1}\|^2 + k(\|u_n - S^n u_n\|^2 \\
&\quad + \|x_{n+1} - S^n x_{n+1}\|^2 - 2\langle u_n - S^n u_n, x_{n+1} - S^n x_{n+1}\rangle)] \\
&= [1 + (1-\alpha_n)\gamma_n]\|u_n - x_{n+1}\|^2 + \alpha_n\|x_{n+1} - S^n x_{n+1}\|^2 \\
&\quad + 2\alpha_n\langle u_n - x_{n+1}, x_{n+1} - S^n x_{n+1}\rangle - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + k(1-\alpha_n)(\|u_n - S^n u_n\|^2 + \|x_n - S^n x_{n+1}\|^2 \\
&\quad - 2\langle u_n - S^n u_n, x_{n+1} - S^n x_{n+1}\rangle) \\
&= (1-\alpha_n)[1 + (1-\alpha_n)\gamma_n]\|u_n - S^n u_n\|^2 \\
&\quad + \alpha_n\|x_{n+1} - S^n x_{n+1}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u_n - S^n u_n, x_{n+1} - S^n x_{n+1}\rangle \\
&\quad - \alpha_n(1-\alpha_n)\|u_n - S^n u_n\|^2 \\
&\quad + k(1-\alpha_n)(\|u_n - S^n u_n\|^2 + \|x_n - S^n x_{n+1}\|^2 \\
&\quad - 2\langle u_n - S^n u_n, x_{n+1} - S^n x_{n+1}\rangle) \\
&= [\alpha_n + k(1-\alpha_n)]\|x_{n+1} - S^n x_{n+1}\|^2 \\
&\quad + [1 + (1-\alpha_n)\gamma_n + k(1-\alpha_n)]\|u_n - S^n u_n\|^2 \\
&\quad + 2(\alpha_n - k)(1-\alpha_n)\langle u_n - S^n u_n, x_{n+1} - S^n x_{n+1}\rangle) \\
&\le [\alpha_n + k(1-\alpha_n)]\|x_{n+1} - S^n x_{n+1}\|^2 \\
&\quad + [1 + (1-\alpha_n)(\gamma_n + k)]\|u_n - S^n u_n\|^2 \\
&\quad + 2(\alpha_n - k)(1-\alpha_n)\|u_n - S^n u_n\|\|x_{n+1} - S^n x_{n+1}\|.
\end{aligned} \tag{3.7}
$$

Putting $a_n = \|x_{n+1} - S^n x_{n+1}\|$ and $b_n = \|u_n - S^n u_n\|$ for each $n \ge 1$, we have

$$(1-\alpha_n)(1-k)a_n^2 \le [1 + (1-\alpha_n)(\gamma_n + k)]b_n^2 + 2(\alpha_n - k)(1-\alpha_n)a_n b_n. \tag{3.8}$$

Since $1-\alpha_n > 0$ and we may assume $b_n > 0$, we can divide the last inequality by $(1-\alpha_n)b_n^2$ and also let $\zeta_n = \frac{a_n}{b_n}$ to get the quadratic inequality for ζ_n,

$$(1-k)\zeta_n^2 - 2(\alpha_n - k)\zeta_n - (\frac{1}{1-\alpha_n} + \gamma_n + k) \leq 0. \tag{3.9}$$

Solving this inequality, we obtain

$$\zeta_n \leq \frac{\alpha_n - k + \sqrt{(\alpha_n - k)^2 + (1-k)(\frac{1}{1-\alpha_n} + \gamma_n + k)}}{1-k}.$$

Since $k < \alpha \leq \alpha_n \leq \beta < 1$ for all $n \geq 1$, and $\lim_{n\to\infty} \gamma_n = 0$, therefore, there exists a constant $M > 0$ such that

$$\zeta_n \leq \frac{\alpha_n - k + \sqrt{(\alpha_n - k)^2 + (1-k)(\frac{1}{1-\alpha_n} + \gamma_n + k)}}{1-k} < M.$$

Therefore, $a_n < Mb_n$, i.e.,

$$\|x_{n+1} - S^n x_{n+1}\| < M\|u_n - S^n u_n\|, \tag{3.10}$$

since $\lim_{n\to\infty} \|u_n - S^n u_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0.$$

Step 4. We claim that $\lim_{n\to\infty} \|x_n - u_n\| = 0$.

Let p be an arbitrary element of $F(S) \cap EP(F)$. Then as above $u_n = T_r x_n$ and we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_r x_n - T_r p\|^2 \\
&\leq \langle T_r x_n - T_r p, x_n - p\rangle \\
&= \langle u_n - p, x_n - p\rangle \\
&= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|)
\end{aligned}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|.$$

So from (3.4), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq [1 + (1-\alpha_n)\gamma_n]\|u_n - p\|^2 \\
&\leq [1 + (1-\alpha_n)\gamma_n](\|x_n - p\|^2 - \|x_n - u_n\|^2)
\end{aligned}$$

and hence

$$[1 + (1-\alpha_n)\gamma_n]\|x_n - u_n\|^2 \leq [1 + (1-\alpha_n)\gamma_n]\|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

and divide the last inequality by $[1 + (1-\alpha_n)\gamma_n]$, we get

$$\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \frac{1}{1 + (1-\alpha_n)\gamma_n}\|x_{n+1} - p\|^2.$$

So, from the existence of $\lim_n \|x_n - p\|$ and $1 + (1-\alpha_n)\gamma_n \neq 0$, we have

$$\|x_n - u_n\| \to 0 \quad \text{as } n \to \infty.$$

Step 5. We claim that $\omega_w(x_n) \subset F(S) \cap EP(F)$, where

$$\omega_w(x_n) = \{x : \exists\, x_{n_i} \rightharpoonup x\}.$$

Firstly, we prove that $\omega_w(x_n) \subset EP(F)$.

Since $\{x_n\}$ is bounded and H is reflexive, $\omega_w(x_n)$ is nonempty. Let $w \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence x_{n_i} of $\{x_n\}$ converging weakly to w. Hence, from the result of Step 4, we know that $u_{n_i} \rightharpoonup w$. As $\|S^n u_n - u_n\| \to 0$, we obtain that $S^n u_{n_i} \rightharpoonup w$. Since $u_n = T_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C.$$

From (A_2), we also get

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n),$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A_4) we have

$$0 \geq F(y, w), \quad \forall\, y \in C.$$

For $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) \leq 0$. So, from (A_1) and (A_4) we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y),$$

and hence $0 \leq F(y_t, y)$. From (A_3), we have

$$0 \leq F(w, y_t), \quad \forall\, y \in C,$$

and hence $w \in EP(F)$.

Secondly, we show that $\omega_w(x_n) \subset F(S)$.

Since $\{x_n\}$ is bounded, we can define a function f on H by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|^2, \qquad x \in H.$$

By Lemma 2.1(c), the weak convergence $x_n \to w$ implies that

$$f(x) = f(w) + \|x - w\|^2 \text{ for all } x \in H.$$

In particular, for each $m \geq 1$,

$$f(S^m w) = f(w) + \|S^m w - w\|^2. \tag{3.11}$$

On the other hand, since S is an asymptotically k-strict pseudo-contractive mapping, we get

$$\begin{aligned}
f(S^m w) &= \limsup_{n\to\infty} \|x_n - S^m w\|^2 \\
&= \limsup_{n\to\infty} \|(x_n - S^m x_n) + (S^m x_n - S^m w)\|^2 \\
&= \limsup_{n\to\infty} (\|x_n - S^m x_n\|^2 + \|S^m x_n - S^m w\|^2 \\
&\quad + 2\langle x_n - S^m x_n, S^m x_n - S^m w\rangle) \\
&\le \limsup_{n\to\infty} \|x_n - S^m x_n\|(\|x_n - S^m x_n\| + 2L_m\|x_n - w\|) \\
&\quad + \limsup_{n\to\infty}[(1+\gamma_m)\|x_n - w\|^2 + k\|(x_n - S^m x_n) - (w - S^m w)\|^2].
\end{aligned}$$

Taking $\limsup_{m\to\infty}$ on both sides and observing the facts that that $\lim_{m\to\infty} \gamma_m = 0$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - S^m x_n\| = 0$, we derive that

$$\limsup_{m\to\infty} f(S^m w) \le \limsup_{n\to\infty} \|x_n - w\|^2 + k \limsup_{m\to\infty} \|w - S^m w\|. \tag{3.12}$$

Combining (3.11) and (3.12), we conclude that $\limsup_{m\to\infty} \|w - S^m w\|^2 = 0$. That is, $S^m w \to w$, hence $Sw = w$, i.e., $\omega_w(x_n) \subset F(S)$.

Step 6. We claim that $\{x_n\}$ and $\{u_n\}$ converge weakly to an element of $F(S) \cap EP(F)$.

Indeed, to verify that the assertion is valid, it is sufficient to show that $\omega_w(x_n)$ is a single-point set. We take $w_1,\ w_2 \in \omega_w(x_n)$ arbitrarily and let $\{x_{k_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{k_i} \rightharpoonup w_1$ and $x_{m_j} \rightharpoonup w_2$, respectively. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(S) \cap EP(F)$ and since $w_1, w_2 \in F(S) \cap EP(F)$, by Lemma 2.1(c), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w_1\|^2 &= \lim_{j\to\infty} \|x_{m_j} - w_1\|^2 \\
&= \lim_{j\to\infty} \|x_{m_j} - w_2\|^2 + \|w_2 - w_1\| \\
&= \lim_{i\to\infty} \|x_{k_i} - w_2\|^2 + \|w_2 - w_1\| \\
&= \lim_{i\to\infty} \|x_{k_i} - w_1\|^2 + 2\|w_2 - w_1\| \\
&= \lim_{n\to\infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|.
\end{aligned}$$

Hence $w_1 = w_2$. This shows that $\omega_w(x_n)$ is a single-point set. This completes the proof. □

By Theorem 3.1, we derive the following result.

Corollary 3.2 ([2])**.** *Let C be a nonempty closed convex subset of H, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and $S : C \to C$ be a k-strict pseudo-contractive mapping for some $0 \le k < 1$ such that $F(S) \cap EP(F) \ne \emptyset$.*

Let $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated initially by an arbitrary element* $x_1 \in H$ *and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \quad \textit{for all } y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)Su_n \quad \textit{for } n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$;

(2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to an element of* $F(S) \cap EP(F)$.

4. Strong convergence results

Theorem 4.1. *Let* C *be a nonempty closed convex subset of* H, $F : C\times C \to \infty$ *be a bifunction satisfying* $(A_1) - (A_4)$ *and* $S : C \to C$ *be an asymptotically* k*-strict pseudo-contractive mapping with sequence* $\{\gamma_n\}$ *for some* $0 \leq k < 1$ *such that* $F(S) \cap EP(F) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated initially by an arbitrary element* $x_1 \in H$ *and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\ y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)S^n u_n, \ \forall\ n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ *and* $\{r_n\}$ *satisfy the following conditions*:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$;

(2) $\{\gamma_n\} \in [0, \infty)$ *with* $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n)\gamma_n < \infty$;

(3) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to an element of* $F(S) \cap EP(F)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F(S)\cap EP(F)) = 0$, *where* $d(x_n, F(S)\cap EP(F))$ *denotes the metric distance from the point* x_n *to* $F(S) \cap EP(F)$.

Proof. From the proof of Theorem 3.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(S)\cap EP(F)$ and $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Hence $\{x_n\}$ is bounded.

The necessity is apparent, we show the sufficiency. Suppose that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Taking the infimum all $p \in F(S) \cap EP(F)$ from (3.5), we have

$$d(x_{n+1}, F(S) \cap EP(F)) \leq [1 + (1-\alpha_n)\gamma_n]d(x_n, F(S) \cap EP(F))$$

and hence $\lim_{n\to\infty} d(x_n, F(S)\cap EP(F))$ exists (the proof of this result is similar to the proof of $\lim_{n\to\infty} \|x_n - p\|$ in Theorem 3.1). Thus, we have

$$\lim_{n\to\infty} d(x_n, F(S) \cap EP(F)) = \liminf_{n\to\infty} d(x_n, F(S) \cap EP(F)) = 0.$$

Now, it follows by (3.5) that for all $p \in F(S) \cap EP(F)$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq \prod_{i=0}^{m-1}[1 + (1-\alpha_{n+i})\gamma_{n+i}]\|x_n - p\|. \tag{4.2}$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n)\gamma_n < \infty$ and $1-\beta < 1-\alpha_n < 1-\alpha$, there exists a positive constant N, such that $(1-\alpha_n)\gamma_n \leq \frac{1}{n}$ for all $n \geq N$. Thus we can obtain the following inequality by (4.2),

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq \prod_{i=0}^{m-1}[1 + \frac{1}{n+i}]\|x_n - p\|. \tag{4.3}$$

Since $\prod_{i=0}^{m-1}[1+\frac{1}{n+i}]\|x_n - p\| \leq (1+\frac{1}{n})^m\|x_n - p\|$ and $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, so we get

$$\prod_{i=0}^{m-1}[1 + (1-\alpha_{n+i})\gamma_{n+i}] \leq e.$$

Taking the infimum over all $p \in F(S) \cap EP(F)$ from (4.2), we obtain

$$\|x_{n+m} - x_n\| \leq ed(x_n, F(S) \cap EP(F)).$$

Thus $\{x_n\}$ is a Cauchy sequence. Suppose $x_n \to \hat{x} \in H$. Then

$$d(\hat{x}, F(S) \cap EP(F)) = \lim_{n\to\infty} d(x_n, F(S) \cap EP(F)) = 0.$$

As S an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$, we know from Lemma 2.6(iii) that $F(S)$ is closed and convex. Note that $EP(F)$ is closed according to Lemma 2.5. Thus $F(S) \cap EP(F)$ is closed. Consequently, $\hat{x} \in F(S) \cap EP(F)$. In view of $\lim_{n\to\infty}\|u_n - x_n\| = 0$, we conclude that both sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $\hat{x}$ of $F(S) \cap EP(F)$. □

Theorem 4.2. *Let C be a nonempty closed convex subset of H, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and $S : C \to C$ be a semicompact asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ for some $0 \leq k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)S^n u_n, \;\; \forall\, n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following conditions:
(1) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$;
(2) $\{\gamma_n\} \in [0, \infty)$ *with* $\lim_{n\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n)\gamma_n < \infty$;
(3) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.
Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(F)$.

Proof. From the proof of Theorem 3.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists for each $p \in F(S) \cap EP(F)$ and $\lim_{n\to\infty}\|x_n - S^n x_n\| = 0$. Thus $\{x_n\}$ is bounded. Then from the semicompactness of S, we conclude that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \to q \in H \;\text{ as } i \to \infty.$$

Hence, $x_{n_i} \rightharpoonup q$. Clearly, repeating the same argument as in the proof of Theorem 3.1, we must have $q \in F(S) \cap EP(F)$. This implies that $\lim_{n\to\infty} \|x_n - q\|$ exists. Consequently, we have

$$\lim_{n\to\infty} \|x_n - q\| = \lim_{i\to\infty} \|x_{n_i} - q\| = 0.$$

Since $\|u_n - x_n\| \to 0$ as $n \to \infty$, we deduce that both the sequences x_n and u_n converge strongly to a point $q \in F(S) \cap EP(F)$. □

In particular, every k-strict pseudo-contractive mappings are asymptotically k-strict pseudo-contractive mappings with sequence $\{\gamma_n\}$, where $\gamma_n \equiv 0$, and $n = 1$. According to Theorem 4.1 and Theorem 4.2, we derive the following results easily.

Corollary 4.3 ([2]). *Let C be a nonempty closed convex subset of H, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and $S : C \to C$ be a k-strict pseudo-contractive mapping for some $0 \le k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0 \quad \textit{for all } y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n \quad \textit{for } n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$;

(2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(F)$ if and only if $\liminf_{n\to\infty} d(x_n, F(S) \cap EP(F)) = 0$, where $d(x_n, F(S) \cap EP(F))$ denotes the metric distance from the point x_n to $F(S) \cap EP(F)$.

Corollary 4.4 ([2]). *Let C be a nonempty closed convex subset of H, $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A_1) - (A_4)$ and $S : C \to C$ be a semicompact k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ for some $0 \le k < 1$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated initially by an arbitrary element $x_1 \in H$ and then by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\ y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) S u_n, \ \ \forall\ n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{r_n\}$ satisfy the following conditions:

(1) $\{\alpha_n\} \subset [\alpha, \beta]$ *for some* $\alpha, \beta \in (k, 1)$;

(2) $\{r_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to an element of $F(S) \cap EP(F)$.

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