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THE LACUNARY STRONG ZWEIER CONVERGENT SEQUENCE SPACES

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Abstract. In this paper we introduce and study the lacunary strong Zweier sequence spaces $N_\theta^0[\mathcal{Z}]$, $N_\theta[\mathcal{Z}]$ consisting of all sequences $x=(x_k)$ such that (Zx) in the space N_θ and N_θ^0 respectively, which is normed. Also, prove that $N_\theta^0[\mathcal{Z}]$, $N_\theta[\mathcal{Z}]$ are linearly isomorphic to the space N_θ^0 and N_θ, respectively. And we study some connections between lacunary strong Zweier sequence and lacunary statistical Zweier convergence sequence.

1. Introduction

A sequence space λ with linear topology is called a K-space provided each of maps $p_i \to \mathbb{C}$ defined by $p_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N}=\{0,1,2,\ldots\}$. A K-space λ is called an *FK-space* provided λ is a complete linear metric space. An FK-space whose topology is normable is called a *BK-space* [2].

For a sequence space λ, the *matrix domain* λ_A of an infinite matrix A is defined by

$$\lambda_A=\{x=(x_k)\in w\ :\ Ax\in\lambda\}, \tag{1}$$

where $A=(a_{nk})$ $(n,k=0,1,2,\ldots)$ is an infinite matrix of complex numbers and $Ax=(A_n(x))$ if $A_n(x)=\sum_k a_{nk}x_k$ converges for each n.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. In this paper the intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and also the ratio $\frac{k_r}{k_{r-1}}$ will be abridged by q_r. The space of lacunary convergence sequences N_θ was defined by Freedman et al. [5], as follows:

$$N_\theta=\left\{x=(x_i)\in w: \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|x_i-\ell|=0 \text{ for some } \ell\right\}. \tag{2}$$

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The space N_θ is a BK-space with the norm

$$\|x\|_{N_\theta} = \sup_r \frac{1}{h_r} \sum_{i \in I_r} |x_i| . \tag{3}$$

N_θ^0 denotes the subset of N_θ those sequences for which $\ell = 0$ in the definition N_θ. Also; $(N_\theta^0, \|\cdot\|_{N_\theta})$ is a BK-space.

Let c and c_0 be the linear spaces of bounded, convergent and null sequences with complex terms, respectively.

In [9], Şengönül introduced $\mathcal{Z}$ and $\mathcal{Z}_0$ spaces as the set of all sequences such that Z-transforms of them are in the spaces c and c_0, respectively, i.e.,

$$\mathcal{Z} = \{x = (x_k) \in w \ : \ Zx \in c\}$$

and

$$\mathcal{Z}_0 = \{x = (x_k) \in w \ : \ Zx \in c_0\},$$

where $Z = (z_{nk})$ $(n, k = 0, 1, 2, \ldots)$ denotes by the matrix

$$z_{nk} = \begin{cases} \frac{1}{2}, & k \leq n \leq k+1 \\ 0, & \text{otherwise} \end{cases} \quad (n, k \in \mathbb{N}).$$

This matrix is called Zweier matrix.

The purpose of this paper is to introduce and study the concept of lacunary strong Zweier convergence in the same way obtaining the sequence space $\mathcal{Z}$ from the sequence space c.

2. Lacunary strong Zweier convergence

We introduce the sequence spaces $N_\theta^0[\mathcal{Z}], N_\theta[\mathcal{Z}]$ and $\ell_\infty[\mathcal{Z}]$ as the set of all sequences such that Z-transforms of them are in the N_θ^0, N_θ and N_θ^∞, respectively, that is

$$N_\theta^0[\mathcal{Z}] = \left\{ x = (x_i) \in w : \lim_r \frac{1}{h_r} \sum_{i \in I_r} \left| \frac{1}{2}(x_i + x_{i-1}) \right| = 0 \right\}, \tag{4}$$

$$N_\theta[\mathcal{Z}] = \left\{ x = (x_i) \in w : \lim_r \frac{1}{h_r} \sum_{i \in I_r} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| = 0 \right\}, \tag{5}$$

and

$$N_\theta^\infty[\mathcal{Z}] = \left\{ x = (x_i) \in w : \sup_r \frac{1}{h_r} \sum_{i \in I_r} |\frac{1}{2}(x_i + x_{i-1})| < \infty \right\}. \tag{6}$$

The $Z = (z_{nk})_{n,k \geq 0}$ matrix is well-known as a regular matrix, [1]. With the notation of (1), we may redefine the spaces $N_\theta^0[\mathcal{Z}], N_\theta^0[\mathcal{Z}]$ and $\ell_\infty[\mathcal{Z}]$ as follows:

$$N_\theta^0[\mathcal{Z}] = (N_\theta^0)_Z, N_\theta^0[\mathcal{Z}] = (N_\theta)_Z \text{ and } N_\theta^\infty[\mathcal{Z}] = (N_\theta^\infty)_Z. \tag{7}$$

Zwei mean is two in German language. So, we called lacunary strong Zweier convergent sequence spaces(or lacunary statistical Zweier convergent sequence space) are the spaces $N_\theta^0[\mathcal{Z}], N_\theta[\mathcal{Z}],\ \ell_\infty[\mathcal{Z}]$ (or $S_\theta[\mathcal{Z}]$).

Define the sequence y which will be frequently used, as Z-transform of a sequence x, i.e.,

$$y_i = \frac{1}{2}(x_i + x_{i-1});\ \ (i \in \mathbb{N}). \tag{8}$$

Theorem 2.1. *The sets $N_\theta^0[\mathcal{Z}], N_\theta[\mathcal{Z}]$ and $N_\theta^\infty[\mathcal{Z}]$ are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the BK-spaces with the norm $\|x\|_{N_\theta^0[\mathcal{Z}]} = \|x\|_{N_\theta[\mathcal{Z}]} = \|x\|_{N_\theta^\infty[\mathcal{Z}]} = \|Zx\|_{N_\theta}$.*

Proof. The first part of the theorem is a routine verification and so we omit it. Furthermore, since (7) holds and N_θ^0, N_θ are BK-spaces with respect to the norm defined by (3) [5] and the matrix $Z = (z_{nk})$ is normal, i.e., $z_{nk} \neq 0$ for $0 \leq k \leq n$ and $z_{nk} = 0$ for $k > n$ for all $n, k \in \mathbb{N}$ and also of Theorem 4.3.2 of Wilansky [10] gives the fact that the spaces $N_\theta^0[\mathcal{Z}], N_\theta^0[\mathcal{Z}]$ are the BK-spaces. □

Theorem 2.2. *The sequence spaces $N_\theta^0[\mathcal{Z}], N_\theta[\mathcal{Z}]$ and $N_\infty[\mathcal{Z}]$ are linearly isomorphic to the spaces N_θ^0, N_θ and N_∞ respectively, i.e., $N_\theta^0[\mathcal{Z}] \cong N_\theta^0$, $N_\theta[\mathcal{Z}] \cong N_\theta$ and $N_\theta^\infty[\mathcal{Z}] \cong N_\theta^\infty$.*

Proof. We consider only $N_\theta^0[\mathcal{Z}]$. We should show the existence of a linear bijection between the spaces $N_\theta^0[\mathcal{Z}]$ and N_θ^0. Consider the transformation Z define, with the notation of (8), from $N_\theta^0[\mathcal{Z}]$ to N_θ^0 by

$$\begin{aligned} Z : N_\theta^0[\mathcal{Z}] \quad &\longmapsto \quad N_\theta^0 \\ x \quad &\longmapsto \quad Zx = y, \quad y = (y_i), \quad y_i = \frac{1}{2}(x_i + x_{i-1})\ ,\ (i \in \mathbb{N}). \end{aligned}$$

The linearity of Z is clear. Further, it is trivial that $x = 0$ whenever $Zx = 0$ and hence Z is injective. Let $y \in N_\theta^0$ and define the sequence $x = (x_k)$ by

$$x_i = 2\sum_{k=0}^{i}(-1)^{i-k} y_k\ (n \in \mathbb{N}).$$

Then

$$\begin{aligned} \|x\|_{N_\theta^0[\mathcal{Z}]} &= \lim_r \frac{1}{h_r}\sum_{i \in I_r}\left|\frac{1}{2}(x_i + x_{i-1})\right| \\ &= \lim_r \frac{1}{h_r}\sum_{i \in I_r}\left|\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{i-k}y_k + 2\sum_{k=0}^{i-1}(-1)^{(i-1)-k}y_k)\right| \\ &= \sup_r \frac{1}{h_r}\sum_{i \in I_r}|y_i| \end{aligned}$$

which says us that $x \in N_\theta^0[\mathcal{Z}]$. Additionally, we observe that

$$\begin{aligned}\|x\|_{N_\theta^0[\mathcal{Z}]} &= \sup_r \frac{1}{h_r} \sum_{i\in I_r} \left|\frac{1}{2}(x_i + x_{i-1})\right| \\ &= \sup_r \frac{1}{h_r} \sum_{i\in I_r} \left|\frac{1}{2}(2\sum_{k=0}^{i}(-1)^{i-k}y_k + 2\sum_{k=0}^{i-1}(-1)^{(i-1)-k}y_k)\right| \\ &= \sup_r \frac{1}{h_r} \sum_{i\in I_r} |y_i| = \|y\|_{N_\theta^0}.\end{aligned}$$

Thus, we have that $x \in N_\theta^0[\mathcal{Z}]$ and consequently Z is surjective. Hence, Z is linear bijection which therefore says us that the spaces $N_\theta^0[\mathcal{Z}]$ and N_θ^0 are linearly isomorphic, as was desired. It is clear here that if the spaces $N_\theta^0[\mathcal{Z}]$ and N_θ^0 replaced by the spaces $N_\theta[\mathcal{Z}]$ and N_θ or $N_\theta^\infty[\mathcal{Z}]$ and N_θ^∞, respectively. Then we obtain the fact that $N_\theta[\mathcal{Z}] \cong N_\theta$ or $N_\theta^\infty[\mathcal{Z}] \cong N_\theta^\infty$. This completes proof. □

There is a relation between N_θ and the space $|\sigma_1|$ of strong Cesaro summable sequences defined by

$$|\sigma_1| = \left\{x = (x_i) \in w : \lim_n \frac{1}{n}\sum_{i=1}^{n} |x_i - \ell| = 0 \text{ for some } \ell\right\}.$$

Clearly, in special case $\theta = (2^r)$, we have $N_\theta = |\sigma_1|$ [6].

Also we see that, there are strong connection between $N_\theta[\mathcal{Z}]$ and the sequence space $w[\mathcal{Z}]$, which is defined by

$$w[\mathcal{Z}] = \left\{x = (x_i) : \lim_{n\to\infty} \frac{1}{n}\sum_{i\in I_r} |\frac{1}{2}(x_i + x_{i-1}) - \ell| = 0\right\}.$$

Clearly, in special case $\theta = (2^r)$, we have $N_\theta[\mathcal{Z}] = w[\mathcal{Z}]$. Now we give some inclusion relations between these sequence spaces.

Hereafter, we write y_i for $\frac{1}{2}(x_i + x_{i-1})$.

Theorem 2.3. *Let $\theta = (k_r)$, $(r = 1, 2, 3, \ldots)$ be a lacunary sequence. If $\liminf q_r > 1$, then $w[\mathcal{Z}] \subseteq N_\theta[\mathcal{Z}]$.*

Proof. Let $x \in w[\mathcal{Z}]$ and $\liminf q_r > 1$. Then, there exits $\gamma > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1 + \gamma$ for sufficiently large r. We can also choose a sufficiently large number r, that

$$\frac{h_r}{k_r} \geq \frac{\gamma}{\gamma + 1} \text{ and } \frac{k_r}{h_r} \geq \frac{\gamma + 1}{\gamma}.$$

Then,

$$\frac{1}{k_r}\sum_{i=1}^{k_r} |y_i - \ell| \geq \frac{1}{k_r}\sum_{i\in I_r} |y_i - \ell| = \frac{h_r}{k_r h_r}\sum_{i\in I_r} |y_i - \ell| \geq \frac{\gamma}{(1+\gamma)h_r}\sum_{i\in I_r} |y_i - \ell|$$

which yields that $x \in N_\theta[\mathcal{Z}]$. □

Theorem 2.4. *For* $\limsup q_r < \infty$, *we have* $N_\theta[\mathcal{Z}] \subseteq w[\mathcal{Z}]$.

Proof. If $\limsup q_r < \infty$, then there exits $K > 0$ such that $q_r < K$ for every r. Now suppose that $\epsilon > 0$ and $x \in N_\theta[\mathcal{Z}]$. There exits m_0 such that for every $m \geq m_0$,

$$H_m = \frac{1}{h_m} \sum_{i \in I_r} |y_i - \ell| < \epsilon.$$

We can also find $T > 0$ such that $H_m \leq T$ for all m. Let n be any integer with $k_r \geq n > k_{r-1}$. Now write

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^{n} |y_i - \ell| &\leq \frac{1}{k_r} \sum_{i=1}^{k_r} |y_i - \ell| \\
&= \frac{1}{k_{r-1}} \sum_{m=1}^{m_0} \sum_{i \in I_m} |y_i - \ell| + \frac{1}{k_{r-1}} \sum_{m=m_0+1}^{k_r} \sum_{i \in I_m} |y_i - \ell| \\
&\leq \frac{1}{k_{r-1}} \sum_{m=1}^{m_0} \sum_{i \in I_m} |y_i - \ell| + \frac{\epsilon(k_r - k_{m_0})}{k_{r-1}} \\
&\leq \frac{1}{k_{r-1}} \sum_{m=1}^{m_0} h_i H_i + \frac{\epsilon(k_r - k_{m_0})}{k_{r-1}} \\
&\leq \frac{1}{k_{r-1}} \left(\sup_{1 \leq i \leq m_0} H_i k_{m_0} \right) + \epsilon K < \frac{k_{m_0}}{k_{r-1}} T + \epsilon K
\end{aligned}$$

from which we deduce that $x \in w[\mathcal{Z}]$. □

Also; from theorem (2.3) and (2.4) follows; If $1 < \liminf q_r \leq \limsup q_r < \infty$ then $N_\theta[\mathcal{Z}] = w[\mathcal{Z}]$.

3. Lacunary statistical Zweier convergence

The idea of statistical convergence was introduced by Fast [4] and studied by various authors (see [7], [8] and [3]).

A sequence $x = (x_k)$ is said to be statistically convergent to a number s if for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $st - \lim x = \ell$ and S denotes the set of all statistical convergent sequences.

A sequence $x = (x_k)$ is said to be lacunary statistical Zweier convergent to an ℓ if for $\epsilon > 0$

$$S_\theta[\mathcal{Z}] = \left\{ x = (x_i) \in w : \lim_r \frac{1}{h_r} |ZK_\theta(\epsilon)| = 0 \right\}, \tag{9}$$

where $ZK_\theta(\epsilon) = \left\{ i \in I_r : |\frac{1}{2}(x_i + x_{i-1}) - \ell| \geq \epsilon \right\}$.

If $x \in S_\theta[\mathcal{Z}]$, then we will write $x_k \to s(S_\theta[\mathcal{Z}])$. Let

$$I_r^1 = \left\{ i \in I_r : \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| \geq \epsilon \right\} = CK_\theta(\epsilon)$$

and

$$I_r^2 = \left\{ i \in I_r : \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| < \epsilon \right\}.$$

Theorem 3.1. *If $x_i \to s(N_\theta[\mathcal{Z}])$, then $x_i \to s(S_\theta[\mathcal{Z}])$.*

Proof. If $\epsilon > 0$ and $x_i \to s(N_\theta[\mathcal{Z}])$, then we can write

$$\frac{1}{h_r} \sum_{i \in I_r} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| \geq \frac{1}{h_r} \sum_{i \in I_r^1} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| \geq \frac{1}{h_r} |ZK_\theta(\epsilon)| \, \epsilon.$$

It follows that $x_i \to s(S_\theta[\mathcal{Z}])$. □

Theorem 3.2. *If $x \in N_\infty[\mathcal{Z}]$ and $x_i \to s(N_\theta[\mathcal{Z}])$, then $x_i \to s(S_\theta[\mathcal{Z}])$.*

Proof. Suppose that $x \in N_\infty[\mathcal{Z}]$ and $x_i \to s(S_\theta[\mathcal{Z}])$. Since $\sup |y_i| < \infty$, there is a constant $T > 0$ such that $|\frac{1}{2}(x_i + x_{i-1}) - \ell| < T$ for all i. Therefore we have, for every $\epsilon > 0$, that

$$\begin{aligned} & \frac{1}{h_r} \sum_{i \in I_r} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| \\ &= \frac{1}{h_r} \sum_{i \in I_r^1} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| + \frac{1}{h_r} \sum_{i \in I_r^2} \left| \frac{1}{2}(x_i + x_{i-1}) - \ell \right| \\ &\leq \frac{T}{h_r} |ZK_\theta(\epsilon)| + \epsilon. \end{aligned}$$

Taking limit as $\epsilon \to 0$, the desired result follows. □

Theorem 3.3. *If $x \in N_\infty[\mathcal{Z}]$, then we have $S_\theta[\mathcal{Z}] = N_\theta[\mathcal{Z}]$.*

Proof. Proof follows from Theorem 3.1 and Theorem 3.2. □

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