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A SIMPLE PROOF OF THE p-ADIC VERSION OF THE SOBOLEV EMBEDDING THEOREM

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Abstract. We give a simple proof of certain mapping properties of the p-adic Riesz potential and Bessel potential, and the p-adic version of the Sobolev embedding theorem obtained in [6].

1. Introduction

For a prime number p, let $\mathbb{Q}_p$ denote the p-adic field, and let $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ and $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$. By the standard p-adic analysis [8], we see that any non-zero element $x \in \mathbb{Q}_p$ is uniquely represented in the canonical form

$$x = \sum_{j=\gamma}^{\infty} x_j \, p^j, \; \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.1}$$

where $x_j \in \mathbb{Z}_p$ and $x_\gamma \neq 0$. Here the integer $\gamma = \gamma(x)$ is called the p-adic valuation of x and we write $\gamma = \mathrm{ord}_p(x)$ with convention $\mathrm{ord}_p(0) = \infty$. Then it is well-known [1, 8] that the nonnegative function $|\cdot|_p$ on $\mathbb{Q}_p$ given by $|x|_p = p^{-\mathrm{ord}_p(x)}$ becomes a non-Archimedean norm on $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ denotes the vector space over $\mathbb{Q}_p$ which consists of all points $\mathbf{x} = (x_1, x_2, \dots, x_d)$, $x_1, x_2, \dots, x_d \in \mathbb{Q}_p$. If we define $|\mathbf{x}|_p = \max_{1\le j\le d} |x_j|_p$ for $\mathbf{x} \in \mathbb{Q}_p^d$, then it is easy to see that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$ and moreover $\mathbb{Q}_p^d$ is a locally compact Hausdorff and totally disconnected Banach space with respect to the norm $|\cdot|_p$. For $\gamma \in \mathbb{Z}$, we denote the ball $B_\gamma(\mathbf{a})$ with center $\mathbf{a} \in \mathbb{Q}_p^d$ and radius p^γ and its boundary $S_\gamma(\mathbf{a})$ by

$$B_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p \le p^\gamma\},$$
$$S_\gamma(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^d : |\mathbf{x} - \mathbf{a}|_p = p^\gamma\},$$

respectively. Since $\mathbb{Q}_p^d$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Haar measure

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$d_H\mathbf{x}$ on $\mathbb{Q}_p^d$ (up to positive constant multiple) which is translation invariant, i.e., $d_H(\mathbf{x}+\mathbf{a}) = d_H\mathbf{x}$. We normalize the measure $d_H\mathbf{x}$ so that

$$\int_{B_0(\mathbf{0})} d_H\mathbf{x} \risingdotseq |B_0(\mathbf{0})|_H = 1,$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^d$. From this integral theory, it is easy to obtain that $|B_\gamma(\mathbf{a})|_H = p^{\gamma d}$ and $|S_\gamma(\mathbf{a})|_H = p^{\gamma d}(1-p^{-d})$ for any $\mathbf{a} \in \mathbb{Q}_p^d$.

In what follows, we say that a complex-valued measurable function f defined on $\mathbb{Q}_p^d$ is in $L^q(\mathbb{Q}_p^d), 1 \le q \le \infty$, if it satisfies

$$\begin{aligned} \|f\|_{L^q(\mathbb{Q}_p^d)} &\risingdotseq \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d_H\mathbf{x}\right)^{1/q} < \infty, \ 1 \le q < \infty, \\ \|f\|_{L^\infty(\mathbb{Q}_p^d)} &\risingdotseq \inf\{\alpha : |\{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \alpha\}|_H = 0\}\} < \infty. \end{aligned} \tag{1.2}$$

Here the integral in (1.2) is defined as

$$\begin{aligned} \int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d_H\mathbf{x} &= \lim_{N\to\infty} \int_{B_N(\mathbf{0})} |f(\mathbf{x})|^q \, d_H\mathbf{x} \\ &= \lim_{N\to\infty} \sum_{-\infty<\gamma\le N} \int_{S_\gamma(\mathbf{0})} |f(\mathbf{x})|^q \, d_H\mathbf{x}, \end{aligned} \tag{1.3}$$

if the limit exists. In general, the integral of a complex-valued measurable function f on $\mathbb{Q}_p^d$ is defined as

$$\int_{\mathbb{Q}_p^d} f(\mathbf{x}) \, d_H\mathbf{x} = \lim_{N\to\infty} \int_{B_N(\mathbf{0})} f(\mathbf{x}) \, d_H\mathbf{x} = \lim_{N\to\infty} \sum_{-\infty<\gamma\le N} \int_{S_\gamma(\mathbf{0})} f(\mathbf{x}) \, d_H\mathbf{x},$$

if the limit exists. We now mention some of the previous works on harmonic analysis on the p-adic field $\mathbb{Q}_p$ as follows; Haran [2, 3] obtained the explicit formula of Riesz potentials on $\mathbb{Q}_p$ and developed an analytical potential theory on the p-adic field $\mathbb{Q}_p$.

Let $f(\mathbf{x})$ be a complex-valued function on $\mathbb{Q}_p^d$. Then we say that f is *locally-constant* if for any $\mathbf{x} \in \mathbb{Q}_p^d$ there exists some integer $\ell(\mathbf{x}) \in \mathbb{Z}$ such that

$$f(\mathbf{x}+\mathbf{x}') = f(\mathbf{x}), \ |\mathbf{x}'|_p \le p^{\ell(\mathbf{x})}.$$

We denote by $\mathcal{E}(\mathbb{Q}_p^d)$ the class of all locally-constant functions on $\mathbb{Q}_p^d$ and we denote by $\mathcal{D}(\mathbb{Q}_p^d)$ the subclass of all functions in $\mathcal{E}(\mathbb{Q}_p^d)$ with compact support. We call a function in $\mathcal{D}(\mathbb{Q}_p^d)$ a *test function* on $\mathbb{Q}_p^d$. Since any nonzero p-adic number $x \in \mathbb{Q}_p$ with $|x|_p = p^{-\gamma}$ has the unique representation as in (1.1), we may define a function χ_p on $\mathbb{Q}_p$ by

$$\chi_p(x) = \begin{cases} \prod_{j=\gamma}^{-1} \exp(2\pi i x_j p^j), & \gamma < 0, \\ 1, & \gamma \ge 0 \text{ or } x = 0. \end{cases} \tag{1.4}$$

Then it turns out (see [8]) that the function $\mathbf{x} \to \chi_p(\langle \boldsymbol{\xi}, \mathbf{x}\rangle)$ for each fixed $\boldsymbol{\xi} \in \mathbb{Q}_p^d$ is an additive character of the space $\mathbb{Q}_p^d$ and the group $B_\gamma(\mathbf{0})$, where $\langle \boldsymbol{\xi}, \mathbf{x}\rangle$ denotes the inner product of $\boldsymbol{\xi}, \mathbf{x} \in \mathbb{Q}_p^d$. For $g \in \mathcal{D}(\mathbb{Q}_p^d)$, we define the p-adic Fourier transformation $\mathfrak{F}[g] = \widetilde{g}$ of g by

$$\widetilde{g}(\boldsymbol{\xi}) = \int_{\mathbb{Q}_p^d} \chi_p(\langle \boldsymbol{\xi}, \mathbf{x}\rangle) g(\mathbf{x})\, d_H\mathbf{x},\ \boldsymbol{\xi} \in \mathbb{Q}_p^d.$$

Then $\mathfrak{F}$ is a unitary isomorphism from $\mathcal{D}(\mathbb{Q}_p^d)$ to $\mathcal{D}(\mathbb{Q}_p^d)$ with the inversion formula

$$g(\mathbf{x}) = \int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle) \widetilde{g}(\boldsymbol{\xi})\, d_H\boldsymbol{\xi},\ g \in \mathcal{D}(\mathbb{Q}_p^d),$$

and with the *Parseval-Steklov equalities*

$$\int_{\mathbb{Q}_p^d} g(\mathbf{x})\overline{h(\mathbf{x})}\, d_H\mathbf{x} = \int_{\mathbb{Q}_p^d} \widetilde{g}(\boldsymbol{\xi})\overline{\widetilde{h}(\boldsymbol{\xi})}\, d_H\boldsymbol{\xi},$$
$$\int_{\mathbb{Q}_p^d} g(\mathbf{x})\widetilde{h}(\mathbf{x})\, d_H\mathbf{x} = \int_{\mathbb{Q}_p^d} \widetilde{g}(\boldsymbol{\xi}) h(\boldsymbol{\xi})\, d_H\boldsymbol{\xi},\ g, h \in \mathcal{D}(\mathbb{Q}_p^d).$$

Moreover, $\mathfrak{F}$ is a unitary isomorphism from $L^2(\mathbb{Q}_p^d)$ to $L^2(\mathbb{Q}_p^d)$ with the inversion formula

$$g(\mathbf{x}) = \lim_{\gamma\to\infty} \int_{B_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle) \widetilde{g}(\boldsymbol{\xi})\, d_H\boldsymbol{\xi} \ \text{ in } L^2(\mathbb{Q}_p^d),\ \ g \in \mathcal{D}(\mathbb{Q}_p^d),$$

and with the Parseval-Steklov equalities on $L^2(\mathbb{Q}_p^d)$, because $\mathcal{D}(\mathbb{Q}_p^d)$ is a dense subset of $L^2(\mathbb{Q}_p^d)$ (see [8]).

Let $\mathcal{M}(\mathbb{Q}_p^d)$ denote the set of all complex-valued measurable functions on $\mathbb{Q}_p^d$. For $f, g \in \mathcal{M}(\mathbb{Q}_p^d)$, we define the convolution $f * g$ of f and g by

$$f * g(\mathbf{x}) = \int_{\mathbb{Q}_p^d} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, d_H\mathbf{y},\ \mathbf{x} \in \mathbb{Q}_p^d.$$

For a complex number z with $\mathrm{Re}(z) > 0$, the p-adic *Riesz kernel* $\mathfrak{R}_z$ and *Bessel kernel* $\mathfrak{B}_z$ of order z are defined by

$$\mathfrak{R}_z(\mathbf{x}) = \frac{1 - p^{-z}}{1 - p^{z-d}}\, |\mathbf{x}|_p^{-d+z},\ \mathfrak{B}_z(\mathbf{x}) = \mathfrak{F}^{-1}[(1 + |\boldsymbol{\xi}|_p^2)^{-z/2}](\mathbf{x}),\ \mathbf{x} \in \mathbb{Q}_p^d,$$

respectively; here, $\mathfrak{R}_z(\mathbf{x})$ and $\mathfrak{B}_z(\mathbf{x})$ are in fact multi-valued functions, and so we take their values only in the principle branch to guarantee the single-valuedness. In what follows, we shall always assume that the values of every complex exponents are taken in the principle branch. Then we say that the operators $\mathcal{I}_z$ and $\mathcal{J}_z$ given by

$$\mathcal{I}_z(f)(\mathbf{x}) = \mathfrak{R}_z * f(\mathbf{x}),\ \mathcal{J}_z(f)(\mathbf{x}) = \mathfrak{B}_z * f(\mathbf{x}),\ f \in \mathcal{M}(\mathbb{Q}_p^d),$$

are the p-adic *Riesz potential* and *Bessel potential* of order z, respectively.

In what follows, we shall use notations; given two quantities A and B, we write $A \lesssim B$ or $B \gtrsim A$ if there is a positive constant c (possibly depending on the dimension d and a prime number p to be given) such that $A \leq cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We denote by $\mathcal{C}_F$ the characteristic function of a measurable subset F of $\mathbb{Q}_p^d$. For a complex number z, we write $z = s + it$ where $s, t \in \mathbb{R}$.

Originally, Theorem 1.1 and Theorem 1.4 were obtained in [6]. In contrast to the method used in [6], we give a simple proof of them in this paper by using the p-adic version of the Calderón-Zygmund decomposition technique.

Theorem 1.1. *Let z be a complex number with $0 < \mathrm{Re}(z) < d$ and let $1 \leq q < r < \infty$ satisfy $1/r = 1/q - \mathrm{Re}(z)/d$. If $q > 1$, then there exists a constant $C_0 = C(p, q, r, d) > 0$ such that*

$$\|\mathcal{I}_z(f)\|_{L^r(\mathbb{Q}_p^d)} \leq C_0 \, \|f\|_{L^q(\mathbb{Q}_p^d)}$$

for any $f \in L^q(\mathbb{Q}_p^d)$. Moreover, $\mathcal{I}_z$ is of weak type $(1, r)$; that is to say, there is a constant $C_1 = C_1(p, r, d) > 0$ such that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{I}_z(f)(\mathbf{x})| > \lambda\}|_H \leq \frac{C_1}{\lambda^r} \, \|f\|_{L^1(\mathbb{Q}_p^d)}^r, \ \lambda > 0$$

for any $f \in L^1(\mathbb{Q}_p^d)$.

Corollary 1.2. *Let z be a complex number with $0 < \mathrm{Re}(z) < d$ and let $1 \leq q < r < \infty$ satisfy $1/r = 1/q - \mathrm{Re}(z)/d$. If $q > 1$, then there exists a constant $C_2 = C_2(p, q, r, d) > 0$ such that*

$$\|\mathcal{J}_z(f)\|_{L^r(\mathbb{Q}_p^d)} \leq C_2 \, \|f\|_{L^q(\mathbb{Q}_p^d)}$$

for any $f \in L^q(\mathbb{Q}_p^d)$. Moreover, $\mathcal{J}_z$ is of weak type $(1, r)$; that is to say, there is a constant $C_3 = C_3(p, r, d) > 0$ such that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{J}_z(f)(\mathbf{x})| > \lambda\}|_H \leq \frac{C_3}{\lambda^r} \, \|f\|_{L^1(\mathbb{Q}_p^d)}^r, \ \lambda > 0$$

for any $f \in L^1(\mathbb{Q}_p^d)$.

For $0 \leq s < d$ and $f \in L^1_{loc}(\mathbb{Q}_p^d)$, we define the *fractional maximal function* $\mathcal{M}_s(f)$ by

$$\mathcal{M}_s(f)(\mathbf{x}) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{p^{\gamma(d-s)}} \int_{B_\gamma(\mathbf{x})} |f(\mathbf{y})| \, d_H\mathbf{y}, \ \mathbf{x} \in \mathbb{Q}_p^d.$$

Corollary 1.3. *Let $0 < s < d$ and let $1 \leq q < r < \infty$ satisfy $1/r = 1/q - s/d$. If $q > 1$, then there exists a constant $c_0 = c_0(p, q, r, d) > 0$ such that*

$$\|\mathcal{M}_s(f)\|_{L^r(\mathbb{Q}_p^d)} \leq c_0 \, \|f\|_{L^q(\mathbb{Q}_p^d)}$$

for any $f \in L^q(\mathbb{Q}_p^d)$. Moreover, $\mathcal{M}_s$ is of weak type $(1, r)$; that is to say, there is a constant $c_1 = c_1(p, r, d) > 0$ such that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{M}_s(f)(\mathbf{x})| > \lambda\}|_H \leq \frac{c_1}{\lambda^r} \, \|f\|_{L^1(\mathbb{Q}_p^d)}^r, \ \lambda > 0$$

for any $f \in L^1(\mathbb{Q}_p^d)$.

For $s \geq 0$ and $1 < q < \infty$, we denote by $L^q_s(\mathbb{Q}_p^d)$ the space of all generalized functions $u \in \mathcal{D}'(\mathbb{Q}_p^d)$ such that $\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{s/2}\,\widetilde{u}\,] \in L^q(\mathbb{Q}_p^d)$ and we call it the *p-adic Sobolev space* which has the norm $\|u\|_{L^q_s(\mathbb{Q}_p^d)} = \|\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{s/2}\,\widetilde{u}\,]\|_{L^q(\mathbb{Q}_p^d)}$. Then we obtain the p-adic analogue of the Sobolev embedding theorem.

Theorem 1.4. (a) *Let $1 < q \leq r < \infty$. If $s \geq 0$ is a real number satisfying $1/q - 1/r = s/d$, then the space $L^q_s(\mathbb{Q}_p^d)$ is continuously embedded into $L^r(\mathbb{Q}_p^d)$.*

(b) *If $s > d/q$ and $q > 1$, then the space $L^q_s(\mathbb{Q}_p^d)$ is continuously embedded into $L^\infty(\mathbb{Q}_p^d)$ and any element $f \in L^q_s(\mathbb{Q}_p^d)$ can be modified on a set $E \subset \mathbb{Q}_p^d$ with $|E|_H = 0$ so that the resulting function is uniformly continuous.*

2. Preliminary estimates and the proof of main theorems

In this section, we furnish several useful propositions and lemma and prove the main theorems.

Proposition 2.1. *If $\mathfrak{m}$ is a complex-valued function on $\mathbb{R}_+$ with*

$$\sum_{\gamma=0}^{\infty} |\mathfrak{m}(p^{-\gamma})| p^{-\gamma d} < \infty,$$

then we have that for any $\mathbf{x} \in \mathbb{Q}_p^d \setminus \{\mathbf{0}\}$,

$$\int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\mathfrak{m}(|\boldsymbol{\xi}|_p)\, d_H\boldsymbol{\xi} = \frac{1-p^{-d}}{|\mathbf{x}|_p^d} \sum_{\gamma=0}^{\infty} p^{-\gamma d}\mathfrak{m}(p^{-\gamma}|\mathbf{x}|_p^{-1}) - \frac{1}{|\mathbf{x}|_p^d}\mathfrak{m}(p\,|\mathbf{x}|_p^{-1}).$$

Proof. It easily follows from (1.4) and the change of variable that

$$\int_{B_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\, d_H\boldsymbol{\xi} = p^{\gamma d}\, \mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x})$$

for any $\gamma \in \mathbb{Z}$. Thus for $\gamma \in \mathbb{Z}$ we have that

(2.1)
$$\begin{aligned}\int_{S_\gamma(\mathbf{0})} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\, d_H\boldsymbol{\xi} &= p^{\gamma d}\, \mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x}) - p^{(\gamma-1)d}\, \mathcal{C}_{B_{-\gamma+1}(\mathbf{0})}(\mathbf{x}) \\ &= p^{\gamma d}(1-p^{-d})\mathcal{C}_{B_{-\gamma}(\mathbf{0})}(\mathbf{x}) - p^{(\gamma-1)d}\, \mathcal{C}_{S_{-\gamma+1}(\mathbf{0})}(\mathbf{x}).\end{aligned}$$

Hence by (2.1) and simple calculation we obtain that

$$\begin{aligned}\int_{\mathbb{Q}_p^d} \chi_p(-\langle \mathbf{x}, \boldsymbol{\xi}\rangle)\mathfrak{m}(|\boldsymbol{\xi}|_p)\, d_H\boldsymbol{\xi} &= \lim_{N\to\infty} \sum_{\gamma=-\infty}^{N} \mathfrak{m}(p^\gamma) \int_{S_\gamma(\mathbf{0})} \chi_p(-\langle \boldsymbol{\xi}, \mathbf{x}\rangle)\, d_H\mathbf{x} \\ &= \frac{1-p^{-d}}{|\mathbf{x}|_p^d} \sum_{\gamma=0}^{\infty} p^{-\gamma d}\mathfrak{m}(p^{-\gamma}|\mathbf{x}|_p^{-1}) - \frac{1}{|\mathbf{x}|_p^d}\mathfrak{m}(p\,|\mathbf{x}|_p^{-1}).\end{aligned}$$

Therefore we complete the proof. □

Lemma 2.2. *If z is a complex number with $0 < \mathrm{Re}(z) < d$, then we have;*

(a) $|\mathcal{B}_z(\mathbf{x})| \leq \frac{|1-p^{z-d}|}{|1-p^{-z}|}\,\frac{p^{\mathrm{Re}(z)} - 2p^{\mathrm{Re}(z)-d} + 1}{p^{\mathrm{Re}(z)}(1-p^{\mathrm{Re}(z)-d})}\,|\mathcal{R}_z(\mathbf{x})|,\ \mathbf{x} \in \mathbb{Q}_p^d \setminus \{\mathbf{0}\}.$

(b) $|\mathcal{B}_z(\mathbf{x})| \leq \frac{p^{\mathrm{Re}(z)} - 2p^{\mathrm{Re}(z)-d} + 1}{p^{\mathrm{Re}(z)}(1-p^{\mathrm{Re}(z)-d})}\,\frac{1}{|\mathbf{x}|_p^{d-\mathrm{Re}(z)}}\,\mathcal{C}_{B_0(\mathbf{0})}(\mathbf{x}) + \frac{2}{|\mathbf{x}|_p^d}\,\mathcal{C}_{\mathbb{Q}_p^d \setminus B_0(\mathbf{0})}(\mathbf{x}).$

Proof. (a) It easily follows from Proposition 2.1 that

$$\begin{aligned}
|\mathcal{B}_z(\mathbf{x})| &\leq \frac{1-p^{-d}}{|\mathbf{x}|_p^d} \sum_{\gamma=0}^{\infty} \frac{p^{-\gamma d}}{(1+p^{-2\gamma}|\mathbf{x}|_p^{-2})^{\mathrm{Re}(z)/2}} + \frac{1}{|\mathbf{x}|_p^d(1+p^2|\mathbf{x}|_p^{-2})^{\mathrm{Re}(z)/2}} \\
&\leq (1-p^{-d}) \sum_{\gamma=\log_p(|\mathbf{x}|_p)}^{\infty} p^{-\gamma(d-\mathrm{Re}(z))} + \frac{1}{p^{\mathrm{Re}(z)}}\,\frac{1}{|\mathbf{x}|_p^{d-\mathrm{Re}(z)}} \\
&= \frac{1-p^{-d}}{1-p^{\mathrm{Re}(z)-d}}\,\frac{1}{|\mathbf{x}|_p^{d-\mathrm{Re}(z)}} + \frac{1}{p^{\mathrm{Re}(z)}}\,\frac{1}{|\mathbf{x}|_p^{d-\mathrm{Re}(z)}} \\
&= \frac{|1-p^{z-d}|}{|1-p^{-z}|}\,\frac{p^{\mathrm{Re}(z)} - 2p^{\mathrm{Re}(z)-d} + 1}{p^{\mathrm{Re}(z)}(1-p^{\mathrm{Re}(z)-d})}\,|\mathcal{R}_z(\mathbf{x})|.
\end{aligned}$$

(b) The second part follows from calculation similar to (a) in two cases $|\mathbf{x}|_p \leq 1$ or $|\mathbf{x}|_p > 1$. Therefore we complete the proof. □

Proposition 2.3. *Let $\mathfrak{b} \in L^1(\mathbb{Q}_p^d)$ be supported in a p-adic ball B and satisfy*

$$\int_{\mathbb{Q}_p^d} \mathfrak{b}(\mathbf{x})\, d_H\mathbf{x} = 0.$$

If q is a real number with $q > 0$, then we have that

$$\int_{\mathbb{Q}_p^d \setminus B} |\mathcal{I}_z(\mathfrak{b})(\mathbf{x})|^q\, d_H\mathbf{x} = 0.$$

Proof. Since $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^d$ and $\mathfrak{b}$ is supported in B, $|\mathbf{x}-\mathbf{y}|_p = |\mathbf{x}|_p$ for $\mathbf{x} \in \mathbb{Q}_p^d \setminus B$ and $\mathbf{y} \in B$. By the cancellation property of $\mathfrak{b}$ we easily obtain that

$$|\mathcal{I}_z(\mathfrak{b})(\mathbf{x})| \leq \frac{|1-p^{-z}|}{|1-p^{z-d}|} \int_{\mathbb{Q}_p^d} \big| |\mathbf{x}-\mathbf{y}|_p^{-d+z} - |\mathbf{x}|_p^{-d+z} \big|\, |\mathfrak{b}(\mathbf{y})|\, d_H\mathbf{y} = 0,$$

provided that $\mathbf{x} \in \mathbb{Q}_p^d \setminus B$. This implies the required result. □

Lemma 2.4. *Let z be a complex number satisfying $0 < \mathrm{Re}(z) < d$. If $1 \leq q < d/\mathrm{Re}(z)$, then there exists a constant $C_4 = C_4(d,p,q,z) > 0$ such that*

$$\|\mathcal{I}_z(f)\|_{L^\infty(\mathbb{Q}_p^d)} \leq C_4\, \|f\|_{L^q(\mathbb{Q}_p^d)}^{q\,\mathrm{Re}(z)/d}\, \|f\|_{L^\infty(\mathbb{Q}_p^d)}^{1-q\,\mathrm{Re}(z)/d}$$

for any $f \in L^q(\mathbb{Q}_p^d) \cap L^\infty(\mathbb{Q}_p^d)$.

Proof. Take any $f \in L^q(\mathbb{Q}_p^d) \cap L^\infty(\mathbb{Q}_p^d)$. Then we observe that for any $\gamma_0 \in \mathbb{Z}$,

$$
\begin{aligned}
|\mathcal{I}_z(f)(\mathbf{x})| &\le \|f\|_{L^\infty(\mathbb{Q}_p^d)} \int_{B_{\gamma_0}(\mathbf{0})} |\mathbf{y}|_p^{\mathrm{Re}(z)-d}\, d_H\mathbf{y} \\
&\quad + \|f\|_{L^q(\mathbb{Q}_p^d)} \left(\int_{\mathbb{Q}_p^d \setminus \{B_{\gamma_0}(\mathbf{0})\}} |\mathbf{y}|_p^{(\mathrm{Re}(z)-d)q'}\, d_H\mathbf{y} \right)^{1/q'} \\
&= \|f\|_{L^\infty(\mathbb{Q}_p^d)} (1-p^{-d}) \sum_{-\infty<\gamma\le\gamma_0} p^{\gamma(\mathrm{Re}(z)-d)} p^{\gamma d} \\
&\quad + \|f\|_{L^q(\mathbb{Q}_p^d)} \left((1-p^{-d}) \sum_{\gamma=\gamma_0+1}^{\infty} p^{\gamma(\mathrm{Re}(z)-d)q'} p^{\gamma d} \right)^{1/q'} \\
&= \|f\|_{L^\infty(\mathbb{Q}_p^d)} \frac{1-p^{-d}}{1-p^{-\mathrm{Re}(z)}}\, p^{\mathrm{Re}(z)\gamma_0} \\
&\quad + \|f\|_{L^q(\mathbb{Q}_p^d)} \frac{(1-p^{-d})p^{d-(d-\mathrm{Re}(z))q'}}{1-p^{d-(d-\mathrm{Re}(z))q'}}\, p^{[d-(d-\mathrm{Re}(z))q']\gamma_0}.
\end{aligned}
\tag{2.2}
$$

We now choose some $\gamma_0 \in \mathbb{Z}$ in (2.2) so that

$$
p^{\gamma_0} = \left(\frac{\|f\|_{L^q(\mathbb{Q}_p^d)}}{\|f\|_{L^\infty(\mathbb{Q}_p^d)}} \right)^{q/d} (1-p^{-d})^{-1/d} (1-p^{-\mathrm{Re}(z)})^{q/d} \left(\frac{p^{d-(d-\mathrm{Re}(z))q'}}{1-p^{d-(d-\mathrm{Re}(z))q'}} \right)^{(q-1)/d}.
$$

Then it follows from (2.2) and simple computation that

$$
\|\mathcal{I}_z(f)\|_{L^\infty(\mathbb{Q}_p^d)} \le C_4 \|f\|_{L^q(\mathbb{Q}_p^d)}^{q\,\mathrm{Re}(z)/d} \|f\|_{L^\infty(\mathbb{Q}_p^d)}^{1-q\,\mathrm{Re}(z)/d},
$$

where the constant C_4 is explicitly given by

$$
C_4 = 2(1-p^{-d})^{1-\mathrm{Re}(z)/d} (1-p^{-\mathrm{Re}(z)})^{-1+q\,\mathrm{Re}(z)/d} \left(\frac{p^{d-(d-\mathrm{Re}(z))q'}}{1-p^{d-(d-\mathrm{Re}(z))q'}} \right)^{\frac{(q-1)\mathrm{Re}(z)}{d}}.
$$

Hence we complete the proof. □

Proof of Theorem 1.1. First, we prove that the Riesz potential $\mathcal{I}_z$ is of weak type $(1, r)$. We observe that if $q = 1$, then $r = d/(d - \mathrm{Re}(z))$. Take any $f \in L^1(\mathbb{Q}_p^d)$. For this estimate, we employ the p-adic version [4] of the Calderón-Zygmund decomposition of f with aperture $\mu > 0$ as follows;

$$
f = \mathfrak{g} + \mathfrak{b} \fallingdotseq \mathfrak{g} + \sum_{k=1}^{\infty} \mathfrak{b}_k,
$$

where $\{B_k : k \in \mathbb{N}\}$ is a countable family of pairwise disjoint p-adic balls so that

(a) $|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathfrak{g}(\mathbf{x})| > p^d\mu\}|_H = 0$,

(b) $\mathfrak{b}_k(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbb{Q}_p^d \setminus B_k$ and $\displaystyle\int_{\mathbb{Q}_p^d} \mathfrak{b}_k(\mathbf{x})\, d_H\mathbf{x} = 0$,

(c) $\sum_{k=1}^{\infty} |B_k|_H \le \dfrac{1}{\mu} \|f\|_{L^1(\mathbb{Q}_p^d)}$,

(d) $\|\mathfrak{g}\|_{L^1(\mathbb{Q}_p^d)} + \sum_{k=1}^{\infty} \|\mathfrak{b}_k\|_{L^1(\mathbb{Q}_p^d)} \le 3\,\|f\|_{L^1(\mathbb{Q}_p^d)}$.

We may assume that $\|f\|_{L^1(\mathbb{Q}_p^d)} = 1$ by normalization. Applying Lemma 2.4 with $q = 1$, we obtain that

$$\begin{aligned}\|\mathcal{I}_z(\mathfrak{g})\|_{L^\infty(\mathbb{Q}_p^d)} &\le C_4\,\|f\|_{L^1(\mathbb{Q}_p^d)}^{\mathrm{Re}(z)/d}\,\|\mathfrak{g}\|_{L^\infty(\mathbb{Q}_p^d)}^{1-\mathrm{Re}(z)/d} \\ &\le C_4\,3^{\mathrm{Re}(z)/d} p^{d-\mathrm{Re}(z)} \mu^{1-\mathrm{Re}(z)/d} \\ &= C_4\,3^{\mathrm{Re}(z)/d} p^{d-\mathrm{Re}(z)} \mu^{1/r}.\end{aligned}$$

If we set $\lambda = 2C_4\,3^{\mathrm{Re}(z)/d} p^{d-\mathrm{Re}(z)} \mu^{1/r} \eqqcolon C_1^{1/r}\mu^{1/r}$, then we have that

$$|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{I}_z(f)(\mathbf{x})| > \lambda\}|_H \le |\{\mathbf{x} \in \mathbb{Q}_p^d : \sum_{k\in\mathbb{N}} |\mathcal{I}_z(\mathfrak{b}_k)(\mathbf{x})| > \lambda/2\}|_H.$$

If we set $\Omega = \cup_{k\in\mathbb{N}} B_k$, then by (c) we get that $|\Omega|_H \le 1/\mu = C_1/\lambda^r$. It also follows from Proposition 2.3 that

$$\begin{aligned}&\big|\{\mathbf{x} \in \mathbb{Q}_p^d \setminus \Omega : \sum_{k\in\mathbb{N}} |\mathcal{I}_z(\mathfrak{b}_k)(\mathbf{x})| > \lambda/2\}\big|_H^{1/r} \\ &\le \frac{2}{\lambda} \sum_{k\in\mathbb{N}} \left(\int_{\mathbb{Q}_p^d \setminus B_k} |\mathcal{I}_z(\mathfrak{b}_k)(\mathbf{x})|^r \, d_H\mathbf{x} \right)^{1/r} = 0.\end{aligned}$$

Therefore we conclude that $|\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{I}_z(f)(\mathbf{x})| > \lambda\}|_H \le C_1/\lambda^r$.

For $q > 1$, we may assume that $\|f\|_{L^q(\mathbb{Q}_p^d)} = 1$ by normalization. Then we have that

$$\|\mathcal{I}_z(f)\|_{L^r(\mathbb{Q}_p^d)}^r = r \int_0^\infty \lambda^{r-1}\,\omega(\lambda)\,d\lambda, \tag{2.3}$$

where $\omega(\lambda) = |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{I}_z(f)(\mathbf{x})| > \lambda\}|_H$ for $\lambda > 0$. In order to estimate $\omega(\lambda)$, we split f into $f = g + h$ where $g = f \cdot \mathcal{C}_{\nu(\mu)}$ for $\nu(\mu) = \{\mathbf{x} \in \mathbb{Q}_p^d : |f(\mathbf{x})| > \mu\}$ and $\mu > 0$. By Lemma 2.4, we obtain that

$$\|\mathcal{I}_z(h)\|_{L^\infty(\mathbb{Q}_p^d)} \le C_4\,\|h\|_{L^q(\mathbb{Q}_p^d)}^{q\,\mathrm{Re}(z)/d}\,\|h\|_{L^\infty(\mathbb{Q}_p^d)}^{1-q\,\mathrm{Re}(z)/d} \le C_4\,\mu^{1-q\,\mathrm{Re}(z)/d} = C_4\,\mu^{q/r}.$$

We now choose $\mu > 0$ so that $\lambda/2 = C_4\,\mu^{q/r}$. From the weak type $(1, r_0)$-estimate of $\mathcal{I}_z$ with $r_0 = d/(d - \mathrm{Re}(z))$ in the above, we see that

$$\omega(\lambda) \le |\{\mathbf{x} \in \mathbb{Q}_p^d : |\mathcal{I}_z(g)(\mathbf{x})| > \lambda/2\}|_H \le 2^{r_0} C_1\,\frac{\|g\|_{L^1(\mathbb{Q}_p^d)}^{r_0}}{\lambda^{r_0}}. \tag{2.4}$$

Thus it follows from (2.3), (2,4), and the p-adic versions of integral Minkowski's inequality and changing the order of integration that

$$\begin{aligned}\|\mathcal{I}_z(f)\|_{L^r(\mathbb{Q}_p^d)}^r &\le r2^{r_0} C_1 \int_0^\infty \lambda^{r-1-r_0} \left(\int_{\nu(\mu)} |f(\mathbf{x})|\,d_H\mathbf{x} \right)^{r_0} d\lambda \\ &= q2^r C_1 C_4^{r-r_0} \int_0^\infty \mu^{q-1-qr_0/r} \left(\int_{\nu(\mu)} |f(\mathbf{x})|\,d_H\mathbf{x} \right)^{r_0} d\mu\end{aligned}$$

$$\leq q2^r C_1 C_4^{r-r_0} \left(\int_{\mathbb{Q}_p^d} \left(\int_0^{|f(\mathbf{x})|} \mu^{q-1-qr_0/r} \, d\mu \right)^{1/r_0} |f(\mathbf{x})| \, d_H\mathbf{x} \right)^{r_0}$$

$$\leq q2^r C_1 C_4^{r-r_0} \left(\int_{\mathbb{Q}_p^d} |f(\mathbf{x})|^q \, d_H\mathbf{x} \right)^{r_0} = q2^r C_1 C_4^{r-r_0}.$$

Therefore we complete the proof. □

Proof of Corollary 1.2 and Corollary 1.3. It easily follows from Theorem 1.1, Lemma 2.2, and the relation $\mathcal{M}_s(f) \lesssim \mathcal{I}_s(|f|)$ to be obtained by simple computation. □

Proof of Theorem 1.4. (a) Let $f \in L_s^q(\mathbb{Q}_p^d)$ be given. For the proof, it suffices to prove that

$$\|f\|_{L^r(\mathbb{Q}_p^d)} \leq C \, \|\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{s/2} \, \widetilde{f}\,]\|_{L^q(\mathbb{Q}_p^d)}. \tag{2.5}$$

If $s = 0$, then we have $q = r$, and so the result is obvious. Thus we may assume that $s > 0$. If we set $g(\mathbf{x}) = \mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{s/2} \, \widetilde{f}\,](\mathbf{x})$, then proving (2.5) is equivalent to showing the estimate

$$\|\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{-s/2} \, \widetilde{g}\,]\|_{L^r(\mathbb{Q}_p^d)} \leq C \, \|g\|_{L^q(\mathbb{Q}_p^d)}. \tag{2.6}$$

We observe in (2.6) that $\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{-s/2} \, \widetilde{g}\,](\mathbf{x}) = \mathfrak{B}_s * g(\mathbf{x})$. Since $1 < q < r < \infty$ and $1/q - 1/r = s/d$ implies that $0 < s < d$, the estimate (2.6) can immediately be obtained from Corollary 1.2.

(b) If $s > d/q$ and $q > 1$, then we have $d - s < d(1 - 1/q)$, and thus $(d-s)q' < d$ where q' is the dual exponent of q. Then by (b) of Lemma 2.2 we have that $\mathfrak{B}_s \in L^{q'}(\mathbb{Q}_p^d)$; indeed,

$$\begin{aligned} &\int_{\mathbb{Q}_p^d} |\mathfrak{B}_s(\mathbf{x})|^{q'} \, d_H\mathbf{x} \\ &\leq \frac{p^s - 2p^{s-d} + 1}{p^s(1-p^{s-d})} \int_{B_0(\mathbf{0})} \frac{1}{|\mathbf{x}|_p^{(d-s)q'}} \, d_H\mathbf{x} + 2 \int_{\mathbb{Q}_p^d \setminus B_0(\mathbf{0})} \frac{1}{|\mathbf{x}|_p^{dq'}} \, d_H\mathbf{x} \\ &= \frac{p^s - 2p^{s-d} + 1}{p^s(1-p^{s-d})} (1-p^{-d}) \sum_{\gamma=0}^{\infty} p^{-\gamma[d-(d-s)q']} + 2(1-p^{-d}) \sum_{\gamma=1}^{\infty} p^{-\gamma d(q'-1)} \\ &= \frac{p^s - 2p^{s-d} + 1}{p^s(1-p^{s-d})} \, \frac{1-p^{-d}}{1-p^{(d-s)q'-d}} + \frac{2(1-p^{-d})p^{d(1-q')}}{1-p^{d(1-q')}} < \infty. \end{aligned}$$

Let $f \in L_s^q(\mathbb{Q}_p^d)$ be given. For our proof, as in the above we have only to prove that

$$\|\mathfrak{B}_s * g(\mathbf{x})\|_{L^\infty(\mathbb{Q}_p^d)} \leq C \, \|g\|_{L^q(\mathbb{Q}_p^d)}, \tag{2.7}$$

where $g(\mathbf{x}) = \mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{s/2} \, \widetilde{f}\,](\mathbf{x})$. Since $\mathfrak{B}_s \in L^{q'}(\mathbb{Q}_p^d)$, the estimate (2.7) can be obtained by applying the p-adic version of Hölder's inequality.

If $\tau_{\mathbf{y}}$ denotes the translation operator defined by $\tau_{\mathbf{y}} f(\mathbf{x}) = f(\mathbf{x}+\mathbf{y})$ for $\mathbf{y} \in \mathbb{Q}_p^d$, then it easily follows from (2.7) that

$$\|\tau_{\mathbf{y}} f - f\|_{L^\infty(\mathbb{Q}_p^d)} = \|\mathfrak{F}^{-1}[(1+|\boldsymbol{\xi}|_p^2)^{-s/2}(\widetilde{\tau_{\mathbf{y}} g} - \widetilde{g})]\|_{L^\infty(\mathbb{Q}_p^d)} \leq C\,\|\tau_{\mathbf{y}} g - g\|_{L^q(\mathbb{Q}_p^d)}.$$

Thus we conclude that $\lim_{|\mathbf{y}|_p \to 0} \|\tau_{\mathbf{y}} f - f\|_{L^\infty(\mathbb{Q}_p^d)} = 0$, because $g \in L^q(\mathbb{Q}_p^d)$. Hence this implies that f can be modified on a set $E \subset \mathbb{Q}_p^d$ with $|E|_H = 0$ so that the resulting function is continuous. □

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