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ON NILPOTENCE INDICES OF SIGN PATTERNS

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Abstract. The work in this paper was motivated by [3], where Eschenbach and Li listed four 4 by 4 sign patterns, conjectured to be nilpotent sign patterns of nilpotence index at least 3. These sign patterns with no zero entries, called full sign patterns, are shown to be potentially nilpotent of nilpotence index 3. We also generalize these sign patterns of order 4 so that we provide classes of n by n sign patterns of nilpotence indices at least 3, if they are potentially nilpotent. Furthermore it is shown that if a full sign pattern $\mathcal{A}$ of order n has nilpotence index k with $2 \leq k \leq n-1$, then sign pattern $\mathcal{A}$ has nilpotent realizations of nilpotence indices $k, k+1, \ldots, n$. Hence, the four 4 by 4 sign patterns in [3, page 91] also allow nilpotent realizations of nilpotence index 4.

1. Introduction

A sign pattern is a matrix with entries in $\{+, -, 0\}$. A sign pattern with no zero entries is said to be a *full* sign pattern. The *sign pattern class* $Q(\mathcal{A})$ of an m by n sign pattern $\mathcal{A} = [\alpha_{ij}]$ is the set of m by n real matrices with sign pattern $\mathcal{A}$, i.e.,

$$Q(\mathcal{A}) = \{A = [a_{ij}] \in \mathbb{R}^{m \times n} \mid \operatorname{sgn}(a_{ij}) = \alpha_{ij} \text{ for all } i, j\}.$$

Each matrix in $Q(\mathcal{A})$ is called a *realization* of $\mathcal{A}$. An n by n matrix A is *nilpotent* if there exists a positive integer k such that $A^k = O$, or equivalently the characteristic polynomial of A is x^n. The minimum of such a positive integer k is the *nilpotence index of A*. If an n by n sign pattern $\mathcal{A}$ has a nilpotent realization, then we say that $\mathcal{A}$ is *potentially nilpotent*. The minimum of the nilpotence index of a nilpotent realization of $\mathcal{A}$ is the *nilpotence index of $\mathcal{A}$*. It is clear that the nilpotence index of a potentially nilpotent full sign pattern is at least 2.

The nilpotence indices of realizations of a sign pattern have been studied in the literature (see, for example, [3, 4, 5]). However, the nilpotence index of a potentially nilpotent sign pattern is introduced in this paper for the first time.

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By the notation $\mathcal{N}_\ell$ in [3, 4], a nonzero sign pattern $\mathcal{A}$ of order n ($n \geq 2$) is potentially nilpotent of nilpotence index k ($2 \leq k \leq n$) if $\mathcal{A} \notin \mathcal{N}_{k-1}$ and $\mathcal{A}$ has a nilpotent realization of nilpotence index k. In order to determine if a sign pattern of order n is potentially nilpotent, one needs to solve a system of n non-linear polynomial equations in several variables, derived from setting all the coefficients of the characteristic polynomial of a realization equal to 0. Hence, such recognition problem is not an easy task, and thereby to determine the nilpotence index or to determine all nilpotence indices of nilpotent realizations that the sign pattern allows is much harder than the problem of recognizing potentially nilpotent sign patterns.

In [3], Eschenbach and Li almost finished classifying 4 by 4 sign patterns with nilpotence index 2. There are four full sign patterns of order 4 in [3, page 91] whose nilpotence indices need to be determined:

$$
\begin{aligned}
&\mathcal{A}_1 = \begin{bmatrix} + & + & + & + \\ - & - & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} + & + & + & + \\ - & - & - & - \\ - & - & + & - \\ - & - & + & + \end{bmatrix} \\
&\mathcal{A}_3 = \begin{bmatrix} + & + & + & + \\ - & - & - & - \\ - & + & - & + \\ - & + & - & + \end{bmatrix}, \quad \mathcal{A}_4 = \begin{bmatrix} + & + & + & + \\ - & - & - & - \\ + & - & - & + \\ + & - & - & + \end{bmatrix}.
\end{aligned} \tag{1}
$$

In Section 3 it is shown that these four sign patterns of order 4 are potentially nilpotent of nilpotence index 3. Furthermore, we show in Section 2 that if a full sign pattern of order n has nilpotence index k with $2 \leq k \leq n-1$, then the sign pattern allows nilpotent realizations of nilpotence indices $k, k+1, \ldots, n$. Hence, it is shown that the four 4 by 4 sign patterns in (1) also allow nilpotent realizations of nilpotence index 4.

2. Nilpotence indices allowed by a full sign pattern

We first find necessary conditions for a full sign pattern to be potentially nilpotent. These conditions are also necessary for a sign pattern (not necessarily full) to be inertially arbitrary (see, for example, [2], [5, Theorem 1.1]).

Proposition 2.1. *Let $\mathcal{A} = [\alpha_{ij}]$ be a potentially nilpotent full sign pattern of order $n \geq 2$. Then $\mathcal{A}$ has at least one $+$ entry and one $-$ entry on the main diagonal and there are indices $s \neq t$ such that nonzero entries α_{st} and α_{ts} have opposite sign.*

Proof. Since each nilpotent realization of $\mathcal{A}$ has trace 0, the existence of at least one $+$ entry and one $-$ entry on the main diagonal follows.

Next, suppose that for each $s \neq t$, α_{st} and α_{ts} have the same sign. Let $A = [a_{ij}] \in Q(\mathcal{A})$. Then the (s, s)-entry of A^2 is

$$(A^2)_{ss} = \sum_{k=1}^{n} a_{sk}a_{ks} = a_{ss}^2 + \sum_{k \neq s} a_{sk}a_{ks} > 0.$$

This implies that there are no nilpotent realizations of $\mathcal{A}$, which is a contradiction. Hence, the result follows. □

Proposition 2.1 also can obtained from [5, Theorems 1.1, 1.2]. Note that the full sign patterns in (1) satisfy the necessary conditions in Proposition 2.1. For potentially nilpotent sign patterns with zero entries, Proposition 2.1 is not necessarily true. For example, any strictly upper triangular sign pattern is potentially nilpotent, but it does not satisfy two necessary conditions in Proposition 2.1.

We now show that by perturbing the Jordan canonical form of a nilpotent realization of a full sign pattern $\mathcal{A}$, the nilpotence index k of $\mathcal{A}$, with $2 \leq k \leq n-1$, necessarily implies that $\mathcal{A}$ has a nilpotent realizations of each nilpotence index greater than k.

Let J_k denote the 0-Jordan block of order k, i.e.,

$$J_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \vdots \\ & O & & \ddots & \ddots & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}_{k \times k}.$$

We use E_{ij} to denote the square matrix of an appropriate order that has exactly one nonzero entry, equal to 1, in the (i, j)-position.

Lemma 2.2. *For positive integers s, t with $s \geq t$, let $J = J_s \oplus J_t$. Then, for $i = 1, \ldots, t$ and a nonzero real number ϵ, $J + \epsilon E_{s,s+i}$ is a nilpotent matrix of nilpotence index $s + t - (i - 1)$. In particular, $J + \epsilon E_{s,s+t}$ has nilpotence index $s + 1$ and $J + \epsilon E_{s,s+1}$ has nilpotence index $s + t$.*

Proof. Let $J_\epsilon(s + i) = J + \epsilon E_{s,s+i}$. Then, for $k = 1, \ldots, s$,

$$[J_\epsilon(s + i)]^k = J^k + \epsilon(E_{s-k+1,s+i} + E_{s-k+2,s+i+1} + \cdots + E_{\ell_1,\ell_2}),$$

where $(\ell_1, \ell_2) \in \{(s, s + i + (k - 1)), (s + t - k - i + 1, s + t)\}$. Since $J^s = O$, we have

$$[J_\epsilon(s + i)]^s = \epsilon(E_{1,s+i} + E_{2,s+i+1} + \cdots + E_{\ell_1,\ell_2}),$$

where $(\ell_1, \ell_2) \in \{(s, 2s+i-1), (t-i+1, s+t)\}$. Since, for an s by t matrix M, the product MJ_t is the matrix obtained from M by shifting columns $1, \ldots, t-1$ to their next columns and replacing the first column by zero column,

$$[J_\epsilon(s + i)]^s[J_\epsilon(s + i)]^\ell = [\epsilon(E_{1,s+i} + E_{2,s+i+1} + \cdots + E_{\ell_1,\ell_2})]J_t^\ell \neq O \tag{2}$$

for $\ell = 1, \dots, t-i$, where $(\ell_1, \ell_2) \in \{(s, 2s+i-1), (t-i+1, s+t)\}$. However, the product (2) is the zero matrix of order $s+t$ for $\ell = t-i+1$. Therefore, the nilpotence index of $J_\epsilon(s+i)$ is $(s+t)-(i-1)$ for $i = 1, \dots, t$. $\square$

Remark 2.3. Let m, s and t be positive integers with $m, t \leq s$. Then, by replacing s by m in Lemma 2.2 and then mimicking the proof of Lemma 2.2, it can be shown that the nilpotence index of $(J_s \oplus J_t) + \epsilon E_{m,s+j}$ is $\max\{i+(t-j+1), s\}$ for any nonzero real number ϵ.

Theorem 2.4. *Let $\mathcal{A}$ be an n by n potentially nilpotent full sign pattern. If $\mathcal{A}$ has a nilpotent realization of nilpotence index ℓ with $2 \leq \ell \leq n-1$, then $\mathcal{A}$ also has a nilpotent realization of nilpotence index $\ell+1$.*

Proof. Suppose that A is a nilpotent realization of $\mathcal{A}$ of nilpotence index ℓ with $2 \leq \ell \leq n-1$. Since the nilpotence index ℓ is less than n, there exists a nonsingular matrix S of order n such that

$$S^{-1}AS = J_{\ell_1} \oplus \cdots \oplus J_{\ell_p},$$

where $\ell_1 = \ell \geq \ell_2 \geq \cdots \geq \ell_p \geq 1$ and $\ell_1 + \cdots + \ell_p = n$ for some positive integer $p \geq 2$. By Lemma 2.2, it follows that

$$S^{-1}AS + \epsilon E_{\ell_1, \ell_1+\ell_2} \tag{3}$$

is a nilpotent matrix of nilpotence index $\ell_1 + 1 = \ell + 1$ for any nonzero real number ϵ. Let

$$B = A + \epsilon S E_{\ell_1, \ell_1+\ell_2} S^{-1}.$$

Since B is similar to the matrix in (3), B is a nilpotent matrix of nilpotence index $\ell+1$. Note that B is in $Q(\mathcal{A})$ for sufficiently small ϵ. Hence, the result follows. $\square$

By using Theorem 2.4 repeatedly, we get the following result.

Corollary 2.5. *Let $\mathcal{A}$ be an n by n potentially nilpotent full sign pattern of nilpotence index k. Then, for each $\ell \in \{k, k+1, \dots, n\}$, $\mathcal{A}$ allows a nilpotent realization of nilpotence index ℓ.*

The following example shows that Corollary 2.5 is not necessarily true for a sign pattern that is not full. For an n by n sign pattern $\mathcal{A}$, if $A \in Q(\mathcal{A})$ is nilpotent of nilpotence index k, then it is clear that cA and DAD^{-1} are in $Q(\mathcal{A})$ and nilpotent of nilpotence index k for some positive real number c and a diagonal matrix D with main diagonal entries all positive.

Example 2.6. Let

$$\mathcal{A} = \begin{bmatrix} + & + & 0 \\ - & - & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then it can be verified that the realization of $\mathcal{A}$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a nilpotent matrix of nilpotence index 2. Note that any nilpotent realization of $\mathcal{A}$ is a direct sum of a 2 by 2 nilpotent matrix and the 1 by 1 zero matrix. Hence, every nilpotent realization of $\mathcal{A}$ has nilpotence index 2.

We also note that there is a square sign pattern such that it is not full but having consecutive nilpotence indices.

Example 2.7. Let

$$\mathcal{A} = \begin{bmatrix} + & + & + & + \\ - & - & - & - \\ 0 & 0 & + & + \\ 0 & 0 & - & - \end{bmatrix}.$$

We consider the following realization A of $\mathcal{A}$

$$A = \begin{bmatrix} 1 & 1 & a & b \\ -1 & -1 & -c & -d \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix},$$

where $a, b, c, d > 0$. Then we have

$$A^2 = \begin{bmatrix} 0 & 0 & 2a-b-c & a-d \\ 0 & 0 & -a+d & -b-c+2d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 0 & 0 & a-b-c+d & a-b-c+d \\ 0 & 0 & -a+b+c-d & -a+b+c-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $A^4 = O$. Thus, $2a = b + c = 2d$ implies A has nilpotence index 2, and $a \neq d$ and $a + d = b + c$ imply that A has nilpotence index 3. For all other positive values of a, b, c and d, A is of nilpotence index 4.

3. Potentially nilpotent sign patterns of indices at least 3

In this section we provide classes of sign patterns $\mathcal{A}$ such that if $\mathcal{A}$ is potentially nilpotent, then its nilpotence index is at least 3.

Theorem 3.1. *For $n \geq 3$, let*

$$\mathcal{A} = \left[\begin{array}{cc|ccc} + & + & + & \cdots & + \\ - & - & - & \cdots & - \\ \hline - & - & & & \\ \vdots & \vdots & & \mathcal{B} & \\ - & - & & & \end{array}\right]_{n\times n},$$

where the main diagonal entries of $\mathcal{B}$ are in $\{+, 0\}$. Suppose that $\mathcal{A}$ is potentially nilpotent. Then the nilpotence index of $\mathcal{A}$ is at least 3.

Proof. Suppose to the contrary that there is a nilpotent $A = [a_{ij}] \in Q(\mathcal{A})$ of nilpotence index 2. Since $\mathrm{tr}(A) = 0$, $|a_{22}| \geq |a_{11}|$. Since A has a nilpotence index 2, $A^2 = O$. Note that $(A^2)_{11} = 0$ implies $a_{11}^2 > |a_{12}a_{21}|$, and $(A^2)_{22} = 0$ implies $|a_{12}a_{21}| > a_{22}^2$. This implies that $a_{11}^2 > |a_{12}a_{21}| > a_{22}^2$, which contradicts that $|a_{22}| \geq |a_{11}|$. Hence, the result follows. □

Theorem 3.2. *For $n \geq 3$, let*

$$\mathcal{A} = \left[\begin{array}{cc|ccc} + & + & + & \cdots & + \\ - & - & - & \cdots & - \\ \hline & & & & \\ & \mathcal{B} & & \mathcal{C} & \\ & & & & \end{array}\right]_{n\times n},$$

where $\mathcal{B}$ is $\begin{bmatrix} - & \cdots & - \\ + & \cdots & + \end{bmatrix}^T$ or its negation. Suppose that $\mathcal{A}$ is potentially nilpotent. Then the nilpotence index of $\mathcal{A}$ is at least 3.

Proof. We first consider sign pattern $\mathcal{A}$ with $\mathcal{B} = \begin{bmatrix} - & \cdots & - \\ + & \cdots & + \end{bmatrix}^T$. Suppose to the contrary that there is a nilpotent $A = [a_{ij}] \in Q(\mathcal{A})$ of nilpotence index 2. By using a positive scaling and then a positive diagonal similarity, we may assume that

$$A = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & \cdots & 1 \\ -a_{21} & -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \hline -a_{31} & a_{32} & * & \cdots & * \\ \vdots & \vdots & & \ddots & \\ -a_{n1} & a_{n2} & * & \cdots & * \end{array}\right],$$

where all specified $a_{ij} > 0$. Note that $(A^2)_{12} = 0$ implies $a_{22} > 1$, and $(A^2)_{21} = 0$ implies $a_{22} < 1$. Hence, the result follows.

The case when $\mathcal{B} = \begin{bmatrix} + & \cdots & + \\ - & \cdots & - \end{bmatrix}^T$ can be proved similarly by considering $(A^2)_{12} = 0$ and $(A^2)_{21} = 0$. □

Remark 3.3. The statements in Theorems 3.1 and 3.2 can be rephrased more generally by replacing $(1, 2)$- and $(2, 1)$-block of the sign patterns by blocks with some zeros, since the contradictions in the proofs involve only the entries $(A^2)_{ij}$ with $1 \le i,j \le 2$. For example, some nonzero entries in $(1,2)$- and $(2,1)$-block of $\mathcal{A}$ in Theorem 3.1 can be replaced by zero entries as long as the ith row of the $(1,2)$-block and the ith column of the $(2,1)$-block are not *combinatorially orthogonal* (see [1, page 267]) for $i = 1,2$.

We conclude this paper by showing that the full sign patterns in (1) are potentially nilpotent of nilpotence index 3. We first show that all $\mathcal{A}_i$'s in (1) are potentially nilpotent by listing nilpotent realizations $A_i \in Q(\mathcal{A}_i)$ of nilpotence index 3:

$$A_1 = \begin{bmatrix} 13 & 13 & 12 & 1 \\ -15 & -15 & -1 & -14 \\ -2 & -2 & 1 & -3 \\ -1 & -1 & -2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & \frac{-13+\sqrt{4201}}{16} & 1 & \frac{323+\sqrt{4201}}{112} \\ -1 & \frac{13-\sqrt{4201}}{16} & -1 & \frac{-323-\sqrt{4201}}{112} \\ -1 & \frac{13-\sqrt{4201}}{16} & 2 & \frac{-181+\sqrt{4201}}{56} \\ -1 & \frac{13-\sqrt{4201}}{16} & 7 & \frac{-61+\sqrt{4201}}{16} \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 1 & 1 & 21 & \frac{21}{4} \\ -1 & -1 & -21 & -\frac{21}{4} \\ -1 & 3 & -21 & \frac{63}{4} \\ -1 & 4 & -21 & 21 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & \frac{19}{3} & 19 \\ -1 & -1 & -\frac{19}{3} & -19 \\ 1 & -3 & -19 & 19 \\ 1 & -4 & -\frac{76}{3} & 19 \end{bmatrix}.$$

Note that the sign patterns $\mathcal{A}_1$ and $\mathcal{A}_2$ in (1) satisfy the conditions in Theorem 3.1, and the sign patterns $\mathcal{A}_3$ and $\mathcal{A}_4$ in (1) satisfy the conditions in Theorem 3.2. Hence, the following result is a direct consequence of the above list, Corollary 2.5 and Theorems 3.1, 3.2.

Corollary 3.4. *The* 4 *by* 4 *full sign patterns in* (1) *are potentially nilpotent of nilpotence index* 3*, which also allow nilpotent realizations of nilpotence index* 4.

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