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ON PERMUTING 3-DERIVATIONS AND COMMUTATIVITY IN PRIME NEAR-RINGS

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Abstract. In this note, we introduce a permuting 3-derivation in near-rings and investigate the conditions for a near-ring to be a commutative ring.

1. Introduction and preliminaries

A non-empty set R with two binary operations $+$ (addition) and $\cdot$ (multiplication) is called a *near-ring* if it satisfies the following axioms:

i) $(R, +)$ is a group (not necessarily abelian),
ii) $(R, \cdot)$ is a semigroup,
iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Exactly speaking, it is a *left near-ring* because it satisfies the left distributive law. We will use the word *near-ring* to mean *left near-ring* and denote xy instead of $x \cdot y$.

For a near-ring R, the set $R_0 = \{x \in R : 0x = 0\}$ is called the *zero-symmetric part* of R. A near-ring R is said to be *zero-symmetric* if $R = R_0$. Throughout this note, R will be a zero-symmetric near-ring and R is called *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. Recall that R is called *n-torsion-free*, where n is a positive integer, if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol C will represent the multiplicative center of R, that is, $C = \{x \in R : xy = yx \text{ for all } y \in R\}$. For $x \in R$, the symbol $C(x)$ will denote the centralizer of x in R. As usual, for $x, y \in R$, $[x, y]$ will denote the commutator $xy - yx$, while $\langle x, y\rangle$ will indicate the additive-group commutator $x + y - x - y$. As for terminologies concerning near-rings used here without special mention, we refer to G. Pilz [6].

An additive map $d : R \to R$ is called a *derivation* if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a *bi-derivation* we mean a bi-additive map $D : R \times R \to R$ (i.e., D is additive in both arguments) which satisfies the

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relations

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y, z \in R$. Let D be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in R$. The map $\tau : R \to R$ defined by $\tau(x) = D(x, x)$ for all $x \in R$ is called the trace of D.

A map $F : R \times R \times R \to R$ is said to be *permuting* if the equation $F(x_1, x_2, x_3) = F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ holds for all $x_1, x_2, x_3 \in R$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $f : R \to R$ defined by $f(x) = F(x, x, x)$ for all $x \in R$, where $F : R \times R \times R \to R$ is a permuting map, is called the *trace* of F. It is obvious that, in the case $F : R \times R \times R \to R$ is a permuting map which is also 3-additive (i.e., additive in each argument), the trace f of F satisfies the relation

$$f(x + y) = f(x) + 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) + f(y)$$

for all $x, y \in R$.

Since we have

$$F(0, y, z) = F(0 + 0, y, z) = F(0, y, z) + F(0, y, z)$$

for all $y, z \in R$, we obtain $F(0, y, z) = 0$ for all $y, z \in R$. Hence we get

$$0 = F(0, y, z) = F(x - x, y, z) = F(x, y, z) + F(-x, y, z)$$

and so we see that $F(-x, y, z) = -F(x, y, z)$ for all $x, y, z \in R$. This tells us that f is an odd function.

A 3-additive map $\Delta : R \times R \times R \to R$ will be called a 3-*derivation* if the relations

$$\Delta(x_1x_2, y, z) = \Delta(x_1, y, z)x_2 + x_1\Delta(x_2, y, z),$$
$$\Delta(x, y_1y_2, z) = \Delta(x, y_1, z)y_2 + y_1\Delta(x, y_2, z)$$

and

$$\Delta(x, y, z_1z_2) = \Delta(x, y, z_1)z_2 + z_1\Delta(x, y, z_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i \in R$, $i = 1, 2$.

For example, let N be a noncommutative near-ring and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, a, b \in N \right\}.$$

It is clear that R is a noncommutative near-ring under matrix addition and matrix multiplication. We define a map $\Delta : R \times R \times R \to R$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1a_2a_3 \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that Δ is a 3-derivation.

Derivations and bi-derivations in rings and near-rings have been studied by many mathematicians in several ways [1, 2, 3, 4, 5, 7, 9, 10]. Furthermore,

M. Uçkun and M. A. Öztürk [8] investigated symmetric bi-Γ-derivations and commutativity in Γ-near-rings.

In this note, we examine the conditions for a near-ring with permuting 3-derivations to be a commutative ring.

2. Lemmas

We need the following lemmas to obtain our main results in Section 3.

Lemma 2.1 ([2, Lemma 3])**.** *Let R be a prime near-ring. If $C \setminus \{0\}$ contains an element z for which $z + z \in C$, then $(R, +)$ is abelian.*

Lemma 2.2. *Let R be a $3!$-torsion free near-ring. Suppose that there exists a permuting 3-additive map $F : R \times R \times R \to R$ such that $f(x) = 0$ for all $x \in R$, where f is the trace of F. Then we have $F = 0$.*

Proof. For any $x, y \in R$,

$$f(x+y) = f(x) + 2F(x,x,y) + F(x,y,y) + F(x,x,y) + 2F(x,y,y) + f(y)$$

and so, by the hypothesis, we get

$$2F(x,x,y) + F(x,y,y) + F(x,x,y) + 2F(x,y,y) = 0 \tag{2.1}$$

for all $x, y \in R$. Putting $-x$ instead of x in (2.1), we obtain

$$2F(x,x,y) - F(x,y,y) + F(x,x,y) - 2F(x,y,y) = 0 \tag{2.2}$$

for all $x, y \in R$.

On the other hand, for any $x, y \in R$,

$$f(y+x) = f(y) + 2F(y,y,x) + F(y,x,x) + F(y,y,x) + 2F(y,x,x) + f(x)$$

and thus, by the hypothesis, we have

$$2F(x,y,y) + F(x,x,y) + F(x,y,y) + 2F(x,x,y) = 0 \tag{2.3}$$

for all $x, y \in R$ since F is permuting. Comparing (2.1) with (2.2), we get

$$2F(x,y,y) + F(x,x,y) + F(x,y,y) = F(x,x,y) - 3F(x,y,y)$$

which implies that

$$\begin{aligned} &2F(x,y,y) + F(x,x,y) + F(x,y,y) + 2F(x,x,y) \\ &= F(x,x,y) - 3F(x,y,y) + 2F(x,x,y) \end{aligned}$$

for all $x, y \in R$. Hence it follows from (2.3) that

$$F(x,x,y) - 3F(x,y,y) + 2F(x,x,y) = 0 \tag{2.4}$$

for all $x, y \in R$. The substitution $x = -x$ in (2.4) leads to

$$F(x,x,y) + 3F(x,y,y) + 2F(x,x,y) = 0 \tag{2.5}$$

for all $x, y \in R$. Combining (2.4) and (2.5), we obtain

$$F(x,y,y) = 0 \tag{2.6}$$

for all $x, y \in R$ since R is 6-torsion free. The replacement $y = y + z$ to linearize (2.6) yields

$$F(x, y, z) = 0$$

for all $x, y, z \in R$, i.e., $F = 0$ which completes the proof. □

Lemma 2.3. *Let R be a $3!$-torsion free prime near-ring and let $x \in R$. Suppose that there exists a nonzero permuting 3-derivation $\Delta : R \times R \times R \to R$ such that $x\delta(y) = 0$ for all $y \in R$, where δ is the trace of Δ. Then we have $x = 0$.*

Proof. Since we have

$$\delta(y + z) = \delta(y) + 2\Delta(y, y, z) + \Delta(y, z, z) + \Delta(y, y, z) + 2\Delta(y, z, z) + \delta(z)$$

for all $y, z \in R$, the hypothesis gives

$$2x\Delta(y, y, z) + x\Delta(y, z, z) + x\Delta(y, y, z) + 2x\Delta(y, z, z) = 0 \tag{2.7}$$

for all $y, z \in R$. Setting $y = -y$ in (2.7), it follows that

$$2x\Delta(y, y, z) - x\Delta(y, z, z) + x\Delta(y, y, z) - 2x\Delta(y, z, z) = 0 \tag{2.8}$$

for all $y, z \in R$.

On the other hand, for any $y, z \in R$,

$$\delta(z + y) = \delta(z) + 2\Delta(z, z, y) + \Delta(z, y, y) + \Delta(z, z, y) + 2\Delta(z, y, y) + \delta(y)$$

and so, by the hypothesis, we have

$$2x\Delta(y, z, z) + x\Delta(y, y, z) + x\Delta(y, z, z) + 2x\Delta(y, y, z) = 0 \tag{2.9}$$

for all $x, y, z \in R$ since Δ is permuting. Comparing (2.7) with (2.8), we get

$$2x\Delta(y, z, z) + x\Delta(y, y, z) + x\Delta(y, z, z) = x\Delta(y, y, z) - 3x\Delta(y, z, z)$$

which means that

$$\begin{aligned} &2x\Delta(y, z, z) + x\Delta(y, y, z) + x\Delta(y, z, z) + 2x\Delta(y, y, z) \\ &= x\Delta(y, y, z) - 3x\Delta(y, z, z) + 2x\Delta(y, y, z) \end{aligned}$$

for all $x, y, z \in R$. Now, from (2.9), we obtain

$$x\Delta(y, y, z) - 3x\Delta(y, z, z) + 2x\Delta(y, y, z) = 0 \tag{2.10}$$

for all $x, y, z \in R$. Taking $y = -y$ in (2.10) leads to

$$x\Delta(y, y, z) + 3x\Delta(y, z, z) + 2x\Delta(y, y, z) = 0 \tag{2.11}$$

for all $x, y, z \in R$. Combining (2.10) and (2.11), we obtain

$$x\Delta(y, z, z) = 0 \tag{2.12}$$

for all $x, y \in R$ since R is 6-torsion free. Replacing $z = z + w$ to linearize (2.12) and using the conditions show that

$$x\Delta(w, y, z) = 0 \tag{2.13}$$

for all $w, x, y, z \in R$. Substituting wv for w in (2.13), we get

$$xw\Delta(v, y, z) = 0$$

for all $v, w, x, y, z \in R$. Since R is prime and $\Delta \neq 0$, we arrive at $x = 0$. This completes the proof of the theorem. □

Lemma 2.4. *Let R be a near-ring and let $\Delta : R \times R \times R \to R$ be a permuting 3-derivation. Then we have*

$$[\Delta(x, z, w)y + x\Delta(y, z, w)]v = \Delta(x, z, w)yv + x\Delta(y, z, w)v$$

for all $v, w, x, y, z \in R$.

Proof. Since we have

$$\Delta(xy, z, w) = \Delta(x, z, w)y + x\Delta(y, z, w)$$

for all $w, x, y, z \in R$, the associative law gives

$$\begin{aligned}\Delta((xy)v, z, w) &= \Delta(xy, z, w)v + xy\Delta(v, z, w) \\ &= [\Delta(x, z, w)y + x\Delta(y, z, w)]v + xy\Delta(v, z, w)\end{aligned} \tag{2.14}$$

for all $v, w, x, y, z \in R$ and

$$\begin{aligned}\Delta(x(yv), z, w) &= \Delta(x, z, w)yv + x\Delta(yv, z, w) \\ &= \Delta(x, z, w)yv + x[\Delta(y, z, w)v + y\Delta(v, z, w)] \\ &= \Delta(x, z, w)yv + x\Delta(y, z, w)v + xy\Delta(v, z, w)\end{aligned} \tag{2.15}$$

for all $v, w, x, y, z \in R$. Comparing (2.14) and (2.15), we see that

$$[\Delta(x, z, w)y + x\Delta(y, z, w)]v = \Delta(x, z, w)yv + x\Delta(y, z, w)v$$

for all $v, w, x, y, z \in R$. The proof of the lemma is complete. □

3. Permuting 3-derivations and commutativity

Now we are ready to prove our main results in this section.

Theorem 3.1. *Let R be a 3!-torsion free prime near-ring. Suppose that there exists a nonzero permuting 3-derivation $\Delta : R \times R \times R \to R$ such that*

$$\Delta(x, y, z) \in C$$

for all $x, y, z \in R$. Then R is a commutative ring.

Proof. Assume that $\Delta(x, y, z) \in C$ for all $x, y, z \in R$. Since Δ is nonzero, there exist $x_0, y_0, z_0 \in R$ such that $\Delta(x_0, y_0, z_0) \in C \setminus \{0\}$ and

$$\Delta(x_0, y_0, z_0) + \Delta(x_0, y_0, z_0) = \Delta(x_0, y_0, z_0 + z_0) \in C.$$

So $(R, +)$ is abelian by Lemma 2.1.

Since the hypothesis implies that

$$w\Delta(x, y, z) = \Delta(x, y, z)w \tag{3.1}$$

for all $w, x, y, z \in R$, we replace x by xv in (3.1) to get

$$w[\Delta(x, y, z)v + x\Delta(v, y, z)] = [\Delta(x, y, z)v + x\Delta(v, y, z)]w$$

and thus, from Lemma 2.4 and the hypothesis, it follows that

$$\Delta(x,y,z)wv + \Delta(v,y,z)wx = \Delta(x,y,z)vw + \Delta(v,y,z)xw$$

which means that

$$\Delta(x,y,z)[w,v] = \Delta(v,y,z)[x,w] \tag{3.2}$$

for all $v,w,x,y,z \in R$. Setting $\delta(u)$ in place of v in (3.2) and using $\delta(x) \in C$ for all $x \in R$ by the hypothesis, we obtain

$$\Delta(\delta(u),y,z)[x,w] = 0 \tag{3.3}$$

for all $u,w,x,y,z \in R$. The substitution vx for x in (3.3) yields that

$$\Delta(\delta(u),y,z)v[x,w] = 0$$

for all $u,v,w,x,y,z \in R$. Since R is prime, we obtain either $\Delta(\delta(u),y,z) = 0$ or $[x,w] = 0$ for all $u,w,x,y,z \in R$.

Assume that

$$\Delta(\delta(u),y,z) = 0 \tag{3.4}$$

for all $u,y,z \in R$. Let us take $u+x$ instead of u in (3.4). Then we obtain

$$\begin{aligned} 0 &= \Delta(\delta(u+x),y,z) \\ &= \Delta(\delta(u)+\delta(x)+3\Delta(u,u,x)+3\Delta(u,x,x),y,z) \\ &= 3\Delta(\Delta(u,u,x),y,z)+3\Delta(\Delta(u,x,x),y,z), \end{aligned}$$

that is,

$$\Delta(\Delta(u,u,x),y,z)+\Delta(\Delta(u,x,x),y,z) = 0 \tag{3.5}$$

for all $v,w,x,y \in R$. Setting $u = -u$ in (3.5) and then comparing the result with (3.11), we see that

$$\Delta(\Delta(u,u,x),y,z) = 0 \tag{3.6}$$

for all $u,x,y,z \in R$. Substituting ux for x in (3.6) and employing (3.4) give the relation

$$\delta(u)\Delta(x,y,z) + \Delta(u,y,z)\Delta(u,u,x) = 0$$

and so it follows from the hypothesis that

$$\delta(u)\Delta(x,y,z) + \Delta(u,u,x)\Delta(u,y,z) = 0 \tag{3.7}$$

for all $u,x,y,z \in R$. We put $u = y = x$ in (3.7) to obtain

$$\delta(x)\Delta(x,x,w) = 0 \tag{3.8}$$

for all $w,x \in R$. Taking wx in substitute for w in (3.8) yields

$$\delta(x)w\delta(x) = 0$$

and so the primeness of R implies that $\delta(x) = 0$ for all $x \in R$. Hence, by Lemma 2.2, we have $\Delta = 0$ which is a contradiction. So R is a commutative ring. This proves the theorem. □

Theorem 3.2. *Let R be a $3!$-torsion free prime near-ring. Suppose that there exists a nonzero permuting 3-derivation $\Delta : R \times R \times R \to R$ such that*

$$\delta(x),\ \delta(x) + \delta(x) \in C(\Delta(u, v, w))$$

for all $u, v, w, x \in R$, where δ is the trace of Δ. Then R is a commutative ring.

Proof. Assume that

$$\delta(x),\ \delta(x) + \delta(x) \in C(\Delta(u, v, w)) \tag{3.9}$$

for all $u, v, w, x \in R$. From (3.9), we get

$$\begin{aligned}
&\Delta(u + t, v, w)(\delta(x) + \delta(x)) \\
&= (\delta(x) + \delta(x))\Delta(u + t, v, w) \\
&= (\delta(x) + \delta(x))[\Delta(u, v, w) + \Delta(t, v, w)] \\
&= (\delta(x) + \delta(x))\Delta(u, v, w) + (\delta(x) + \delta(x))\Delta(t, v, w) \\
&= \delta(x)\Delta(u, v, w) + \delta(x)\Delta(u, v, w) + \delta(x)\Delta(t, v, w) + \delta(x)\Delta(t, v, w) \\
&= \delta(x)[\Delta(u, v, w) + \Delta(u, v, w) + \Delta(t, v, w) + \Delta(t, v, w)] \\
&= [\Delta(u, v, w) + \Delta(u, v, w) + \Delta(t, v, w) + \Delta(t, v, w)]\delta(x)
\end{aligned} \tag{3.10}$$

for all $t, u, v, w, x \in R$ and

$$\begin{aligned}
&\Delta(u + t, v, w)(\delta(x) + \delta(x)) \\
&= \Delta(u + t, v, w)\delta(x) + \Delta(u + t, v, w)\delta(x) \\
&= [\Delta(u, v, w) + \Delta(t, v, w)]\delta(x) + [\Delta(u, v, w) + \Delta(t, v, w)]\delta(x) \\
&= [\Delta(u, v, w) + \Delta(t, v, w) + \Delta(u, v, w) + \Delta(t, v, w)]\delta(x)
\end{aligned} \tag{3.11}$$

for all $t, u, v, w, x \in R$. Comparing (3.10) and (3.11), we obtain

$$\Delta(\langle u, t\rangle, v, w)\delta(x) = 0$$

for all $t, u, v, w, x \in R$. Hence it follows from Lemma 2.3 that

$$\Delta(\langle u, t\rangle, v, w) = 0 \tag{3.12}$$

for all $t, u, v, w \in R$. We substitute uz for u and ut for t in (3.12) to get

$$\begin{aligned}
0 &= \Delta(u\langle z, t\rangle, v, w) \\
&= \Delta(u, v, w)\langle z, t\rangle + u\Delta(\langle z, t\rangle, v, w) \\
&= \Delta(u, v, w)\langle z, t\rangle.
\end{aligned}$$

That is,

$$\Delta(u, v, w)\langle z, t\rangle = 0 \tag{3.13}$$

for all $t, u, v, w, z \in R$. Letting $z = sz$ and $t = st$ in (3.13) yields

$$\Delta(u, v, w)s\langle z, t\rangle = 0 \tag{3.14}$$

for all $s, t, u, v, w, z \in R$. Since $\Delta \neq 0$, we conclude, from (3.14) and the primeness of R, that $\langle z, t\rangle = 0$ is fulfilled for all $t, z \in R$. Therefore $(R, +)$ is abelian.

By the hypothesis, we know that

$$[\delta(x), \Delta(u, v, w)] = 0 \tag{3.15}$$

for all $u, v, w, x \in R$. Hence if we let $x = x + y$ in (3.15), then we deduce from (3.15) that

$$[\Delta(x, x, y), \Delta(u, v, w)] + [\Delta(x, y, y), \Delta(u, v, w)] = 0 \tag{3.16}$$

for all $u, v, w, x, y \in R$. Setting $y = -y$ in (3.16) and comparing the result with (3.16), we obtain

$$[\Delta(x, y, y), \Delta(u, v, w)] = 0 \tag{3.17}$$

for all $u, v, w, x, y \in R$. Replacing y by $y + z$ in (3.17) and using (3.17), we have

$$[\Delta(x, y, z), \Delta(u, v, w)] = 0$$

since Δ is permuting, i.e.,

$$\Delta(x, y, z)\Delta(u, v, w) = \Delta(u, v, w)\Delta(x, y, z) \tag{3.18}$$

for all $u, v, w, x, y, z \in R$. Taking ut instead of u in (3.18), we obtain

$$\begin{aligned} &\Delta(u, v, w)t\Delta(x, y, z) - \Delta(x, y, z)\Delta(u, v, w)t \\ &\quad + u\Delta(t, v, w)\Delta(x, y, z) - \Delta(x, y, z)u\Delta(t, v, w) = 0 \end{aligned} \tag{3.19}$$

for all $t, u, v, w, x, y, z \in R$. Substituting $\delta(u)$ for u in (3.19) and then utilizing the hypothesis and (3.18), we get

$$\Delta(\delta(u), v, w)[t, \Delta(x, y, z)] = 0 \tag{3.20}$$

for all $t, u, v, w, x, y, z \in R$. Let us write in (3.20) ws instead of w. Then we have

$$\Delta(\delta(u), v, w)s[t, \Delta(x, y, z)] = 0$$

for all $s, t, u, v, w, x, y, z \in R$. Since R is prime, we arrive at either $\Delta(\delta(u), v, w) = 0$ or $[t, \Delta(x, y, z)] = 0$ for all $t, u, v, w, x, y, z \in R$.

As in the proof of Theorem 3.1, the case when $\Delta(\delta(u), v, w) = 0$ holds for all $u, v, w \in R$ leads to the contradiction.

Consequently, we arrive at

$$[t, \Delta(x, y, z)] = 0$$

for all $t, x, y, z \in R$, i.e, $\Delta(x, y, z) \in C$ for all $x, y, z \in R$. Therefore, Theorem 3.1 yields that R is a commutative ring which is complete the proof. □

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