Intuitionistic Vague Groups*

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1. Introduction

In 1971, Rosenfeld[12] defined the fuzzy subgroup of a group$(G,\cdot)$ as a fuzzy set in G satisfying the conditions, by using the concept of fuzzy sets introduced by Zadeh[13]. Many authors [3, 5, 9, 11] have mainly investigated various algebraic notions based on his approach. In fact, in Rosenfeld's work, only the subset are fuzzy, but the group operation is crisp. Recently, Demirci[7] introduced the concept of fuzzy equalities and fuzzy mappings. By using them, he provided a good tool for fuzzyfying the group operation on a crisp set[6].

In 1986, Atanassove[1] introduced the notion of intuitionistic fuzzy set as the generatization of fuzzy sets. In 1989, Biswas[3] introduced the intuitionistic fuzzy subgroup of a group $(G,\cdot)$ as an intuitionistic fuzzy set of G satisfying some conditions. Also many authors[2,9] have worked to present the intuitionistic fuzzy setting of various algebraic concept based on their approach.

In this paper, by taking the group operation on a crisp set as an intuitionistic fuzzy mapping in the sense of [10], we establish the group structure on a crisp set and study the validity of the classical results in this setting.

2. Preliminaries

We will list some concept and one result needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings. Throughout this paper, we will denote the unit interval [0,1] as I and X, Y, Z, etc., are nonempty crisp sets.

Definition 2.1[1,6]. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 2.2[1,9]. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset \text{IFS}(X)$. Then

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.
(3) $A^c = (\nu_A, \mu_A)$.
(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_A)$.
(4)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = (\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha}, \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha})$.
(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.
(5)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = (\bigvee_{\alpha\in\Gamma} \mu_{A_\alpha}, \bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha})$.
(6) [ ]A=$(\mu_A, 1-\mu_A)$, $<> A = (1-\nu_A, \nu_A)$.

Definition 2.3[4]. R is called an *intuitionistic fuzzy relation from* X *to* Y (or *on* $X \times Y$) if $R \in \text{IFS}(X \times Y)$. In particular, if $R \in \text{IFS}(X \times X)$ then R is called an *intuitionistic fuzzy relation on* X.

We will denote the set of all intuitionistic fuzzy relation on X as IFR(X).

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Definition 2.4[10]. Let $IE_X=(\mu_{IE_X}, \nu_{IE_X}) \in$ IFR(X). Then IE_X is called an intuitionistic fuzzy equality an X if it satisfies the following conditions:
(ie.1) $IE_X(x,y) = (1,0) \Leftrightarrow x = y, \forall x, y \in X$,
(ie.2) $IE_X(x,y)=IE_X(y,x), \forall x, y \in X$,
(ie.3) $\mu_{IE_X}(x,y) \wedge \mu_{IE_X}(y,z) \leq \mu_{IE_X}(x,z)$
and
$\nu_{IE_X}(x,y) \vee \nu_{IE_X}(y,z) \geq \nu_{IE_X}(x,z), ^\forall x, y, z \in X$.

We will denote the set of all intuitionistic fuzzy equalities on X as IE(X). Let $IE \in$IE(X) and let $a, b \in X$. Then $\mu_{IE}(a,b)$[resp. $\nu_{IE}(a,b)$] is interpreted the value as the grade of "a and b are nearly equal"[resp. the grade of "a and b are nonequal"].

Definition 2.4′[7]. Let X be a nonempty set and let E_X be a fuzzy relation on X. Then E_X is called a *fuzzy equality* on X if it satisfies the following conditions:
(e.1) $E_X(x,y) = 1 \Leftrightarrow x = y, \forall x, y \in X$,
(e.2) $E_X(x,y) = E_X(y,x), \forall x, y \in X$,
(e.3) $E_X(x,z) \geq E_X(x,y) \wedge E_X(y,z), \forall x, y, z \in X$,

Let E be a fuzzy equality on X and let $a, b \in X$. Then we interpret the value $E(a,b)$ as the grade of "a and b are nearly equal". We will denote the set of all fuzzy equalities on X as E(X).

Remark 2.4. (a) If $E_X \in$E(X), then $(E_X, E_X^c) \in$ IE(X).
(b) If $IE_X \in$ IE(X), then $[\ ]IE_X$, $<> IE_X \in$ IE(X). Moreover, $\mu_{IE_X}, \nu_{IE_X}^c \in E(X)$.

Definition 2.5[10]. Let IE_X and IE_Y be two intuitionistic fuzzy equalities on X and Y, respectively and let $f \in$ IFS$(X \times Y)$. Then f is called an *intuitionistic fuzzy mapping from* X to Y with respect to (in short, $w.r.t.$) $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y), denoted by $f : X \to Y$, if it satisfies the following condition :
(if.1) $\forall x \in X, \exists y \in Y$ such that $\mu_f(x,y) > 0$ and $\nu_f(x,y) < 1$.
(if.2) $\forall x, y \in X, \forall z, w \in Y$,
$$\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_X}(x,y) \leq \mu_{IE_Y}(z,w)$$
and
$$\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_X}(x,y) \geq \nu_{IE_Y}(z,w).$$

Definition 2.5′[7]. Let f be a fuzzy relation from X to Y, i.e., $R \in I^{X \times Y}$. Let E_X and E_Y be fuzzy equalities on X and Y, respectively. Then f is called a *fuzzy mapping from* X *to* Y *w.r.t.* E_X *and* E_Y, denoted by $f : X \to Y$, if it satisfies the following conditions :
(f.1) $\forall x \in X, \exists y \in Y$ such that $f(x,y) > 0$,
(f.2) $\forall x, y \in X \quad \forall z, w \in Y$, $f(x,z) \wedge f(y,w) \wedge E_X(x,y) \leq E_Y(z,w)$.

Result 2.A[10, Proposition 3.4]. (a) Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y). Then $(f, f^c) : X \to Y$ is an intuitionistic fuzzy mapping from X to Y w.r.t. $(E_X, E_X^c) \in$ IE(X) and $(E_Y, E_Y^c) \in$ IE(Y)
(b) Let $f = (\mu_f, \nu_f) : X \to Y$ be an intuitionistic fuzzy mapping from X to$Y$ w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y). Then $<> f$ and $[\ ]\ f$ are intuitionistic fuzzy mapping from X to Y w.r.t. intuitionistic fuzzy equalities $<> IE_X$ and $<> IE_Y$, and $[\ ]\ IE_X$ and $[\ ]\ IE_Y$, respectively.
(c) Let $f = (\mu_f, \nu_f) : X \to Y$ be an intuitionistic fuzzy mapping from X to Y w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y). Then μ_f and $\nu_f{}^c$ are fuzzzy mappings from X to Y w.r.t. fuzzy equalities μ_{IE_X} and μ_{IE_Y}, and $\nu_{IE_X}{}^c$ and $\nu_{IE_Y}{}^c$ on X and Y, respectively.

Definition 2.6[10]. For sets X and Y, let $f : Y \to Y$ be an intuitionistic fuzzy mapping from X to Y w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y). Then f is said to be :
(a) *strong* if $\forall x \in X, \exists y \in Y$ such that $f(x,y) = (1,0)$,
(b) *surjective* if $\forall y \in Y, \exists x \in X$ such that $\mu_f(x,y) > 0$ and $\nu_f(x,y) < 1$,
(c) *strong surjective* if $\forall y \in Y, \exists x \in X$ such that $f(x,y) = (1,0)$,
(d) *injective* if
$\mu_f(x,z) \wedge \mu_f(y,w) \wedge \mu_{IE_Y}(z,w) \leq \mu_{IE_X}(x,y)$
and
$\nu_f(x,z) \vee \nu_f(y,w) \vee \nu_{IE_Y}(z,w) \geq \nu_{IE_X}(x,y)\ \forall x, y \in X, \forall z, w \in Y$,
(e) *bijective* if it is surjective and injective,
(f) *strong bijective* if it is strong surjective and injective.

Definition 2.6′[7]. Let $f : X \to Y$ be a fuzzy mapping w.r.t. E_X and E_Y.
Then f is said to be :
(a) *strong* if $\forall x \in X, \exists y \in Y$ such that $f(x,y) = 1$,
(b) *surjective* if $\forall y \in Y, \exists x \in X$ such that $f(x,y) > 0$,
(c) *strong surjective* if $\forall y \in Y, \exists x \in X$ such that $f(x,y) = 1$,
(d) *injective* if $f(x,z) \wedge f(y,w) \wedge E_Y(z,w) \leq E_X(x,y), \forall x, y \in X, \forall z, w \in Y$,
(e) *bijective* if it is surjective and injective,
(f) *strong bijective* if it strong surjective and injective.

Result 2.B[10, Proposition 3.6]. (a) Let $f : X \to Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] fuzzy mapping w.r.t. fuzzy equalities E_X and E_Y on X and Y, respectively, then $(f, f^c) : X \rightarrow Y$ is a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. $(E_X, E_X^c) \in$ IE(X) and $(E_Y, E_Y{}^c) \in$IE(Y).

(b)Let $f = (\mu_f, \nu_f) : X \rightarrow Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y). Then <> f and [ ] f are a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. intuitionistic fuzzy equalities <> IE_X and <> IE_Y, and [ ]IE_X and [ ]IE_Y on X and Y, respectively.

(c) Let $f = (\mu_f, \nu_f) : X \rightarrow Y$ be a strong [surjective, strong surjective, injective, bijective, strong bijective] intuitionistic fuzzy mapping w.r.t. $IE_X \in$ IE(X) and $IE_Y \in$ IE(Y). Then μ_f and $\nu_f{}^c$ are a strong [surjective, strong surjective, injective, bijective, strong bijective] fuzzy mapping w.r.t. intuitionistic fuzzy equalities μ_{IE_X} and μ_{IE_Y}, and $\nu_{IE_X^c}$ and $\nu_{IE_Y^c}$ on X and Y, respectively.

Result 2.C[10, Proposition 3.7]. Let Δ_X be the intuitionistic fuzzy relation on a set X defined by : For each $(x, y) \in X \times X$,

$$\Delta_X(x, y) = \begin{cases} (1, 0), & \text{if } x = y, \\ (0, 1), & \text{if } x \neq y. \end{cases}$$

Then Δ_X is a strong and strong bijective intuitionistic fuzzy mapping on X w.r.t. an intuitionistic fuzzy equality IE_X on X. In fact, Δ_X is an intuitionistic fuzzy equality on X. In this case, Δ_X is called an *identity intuitionistic fuzzy mapping on* X.

3. Definition of intuitionistic vague groups and their properties.

Definition 3.1. (i) *A* strong intuitionistic fuzzy mapping $f : X \times X \rightarrow X$ w.r.t. $IE_{X\times X} \in$ IE(X × X) and $IE_X \in$ IE(X) is called an *intuitionistic vague operation* on X w.r.t. $E_{X\times X}$ and E_X.

(ii) An intuitionistic vague binary operation f on X w.r.t. $IE_{X\times X}$ and IE_X is said to be *intuitionistic transitive of first order* if

(IT.1) $\mu_f(a, b, c) \wedge \mu_{IE_X}(c, d) \leq \mu_f(a, b, d)$

and

$\nu_f(a, b, c) \vee \nu_{IE_X}(c, d) \geq \nu_f(a, b, d), \ \forall a, b, c, d \in X.$

(iii) An intuitionistic vague binary operation f on X w.r.t. $IE_{X\times X}$ and IE_X is said to be *intuitionistic transitive of second order* if

(IT.2) $\mu_f(a, b, c) \wedge \mu_{IE_X}(b, d) \leq \mu_f(a, b, d)$

and

$\nu_f(a, b, c) \vee \nu_{IE_X}(b, d) \geq \nu_f(a, d, c), \ \forall a, b, c, d \in X.$

It can easily be seen that every crisp mapping $f : X \times X \rightarrow X$ is an intuitionistic vague binary operation on X w.r.t. $\Delta_{X\times X}$ and Δ_X, and it is transitive of both first order and second order.

Definition 3.1′[6]. (i) *A* strong fuzzy mapping $f : X \times X \rightarrow X$ w.r.t. $E_{X\times X} \in$ E(X × X) and $E_X \in$ E(X) is called a *vague binary operation* on X w.r.t. $E_{X\times X}$ and E_X.

(ii) A vague binary operation f on X w.r.t. $E_{X\times X}$ and E_X is said to be *transitive of first order* if

(T.1) $f(a, b, c) \wedge E_X(c, d) \leq f(a, b, d)$.

and

(iii) An intuitionistic vague binary operation f on X w.r.t. $E_{X\times X}$ and E_X is said to be *transitive of second order* if

(T.2) $f(a, b, c) \wedge E_X(b, d) \leq f(a, d, c)$.

Remark 3.2.(a) If f is a vague binary operation on X w.r.t. $E_{X\times X} \in$ E(X × X) and $E_X \in$ E(X), then (μ_f, μ_f^c) is an intuitionistic vague binary operation on X w.r.t. $(E_{X\times X}, E_{X\times X}{}^c)\in$ IE(X × X) and $(E_X, E_X^c) \in$ IE(X).

(b) If a vague binary operation f on X w.r.t. $E_{X\times X} \in$ E(X × X) and $E(X) \in$ E(X) is transitive of first [resp. second] order, then an intuitionistic vague binary operation $(\mu_f, \mu_f{}^c)$ on X w.r.t. $(E_{X\times X}, E_{X\times X}{}^c)\in$ IE(X × X) and $(E_X, E_X^c)\in$ IE(X) is intuitionistic transitive of first[resp. second] order.

(c) If f is an intuitionistic vague binary operation on X w.r.t. $IE_{X\times X} \in$ IE(X × X) and $IE_X \in$ IE(X), then [ ]f [resp. <> f] is an intuitionistic vague binary operation on X w.r.t. [ ]$IE_{X\times X}$ $\in$IE(X × X) and [ ]$IE_X \in$ IE(X) [resp. <> $IE_{X\times X} \in$ IE(X × X) and <> $IE_X \in$ IE(X)]. Moreover, μ_f [resp. $\nu_f{}^c$] is a vague binary operation on X w.r.t. $\mu_{IE_{X\times X}} \in$ E(X × X) and $\mu_{IE_X} \in$ E(X) [resp. $\nu_{IE_{X\times X}{}^c} \in$ E(X × X) and $\nu_{IE_X^c} \in$ E(X)].

(d) If an intuitionistic vague binary operation f on X w.r.t. $IE_{X\times X} \in$ IE(X × X) and $IE_X \in$ IE(X) is intuitionitsic transitive of first [resp. second] order, then [ ]f and <> f are intuitionitsic transitive of first [resp. second] order, respectively. Moreover, μ_f and $\nu_f{}^c$ are transitive of first [resp. second] order, respectively.

Let G be a nonempty crisp set.

Definition 3.3. Let $\circ$ be an intuitionistic vague binary operation on G w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G).

(i) $(G, \circ)$ is called an *intuitionistic vague semigroup* if it satisfies the following condition :

(IVG, 1) $\forall a, b, c, d, m, q, w \in G$,

$\mu_\circ(b,c,d) \wedge \mu_\circ(a,d,m) \wedge \mu_\circ(a,b,q) \wedge \mu_\circ(q,c,w) \leq \mu_{IE_G}(m,w)$

and

$\nu_\circ(b,c,d) \vee \nu_\circ(a,d,m) \vee \nu_\circ(a,b,q) \vee \nu_\circ(q,c,w) \geq \nu_{IE_G}(m,w)$.

(ii) An intuitionistic vague semigroup $(G,\circ)$ is called an *intuitionistic vague monoid* if it satisfies the following condition :

(IVG, 2) $\exists$ an (two-sided) identity element $e \in G$ such that

$\mu_\circ(e,a,a) \wedge \mu_\circ(a,e,a) = 1$

and

$\nu_\circ(e,a,a) \vee \nu_\circ(a,e,a) = 0, \ ^\forall a \in G$.

(iii) An intuitionistic vague monoid $(G,\circ)$ is called an *intuitionistic vague group* if it satisfies the following condition :

(IVG, 3) $\forall a \in G$, $\exists$ an (two-sided) inverse element $a^{-1} \in G$ such that

$\mu_\circ(a^{-1},a,e) \wedge \mu_\circ(a,a^{-1},e) = 1$

and

$\nu_\circ(a^{-1},a,e) \vee \nu_\circ(a,a^{-1},e) = 0$.

(iv) An intuitionistic vague semigroup $(G,\circ)$ is said to be *abelian (commutative)* if it satisfies the condition :

(IVG, 4) $\forall a, b, m, w \in G$,

$\mu_\circ(a,b,m) \wedge \mu_\circ(b,a,w) \leq \mu_{IE_G}(m,w)$

and

$\nu_\circ(a,b,m) \vee \nu_\circ(b,a,w) \geq \mu_{IE_G}(m,w)$.

Definition 3.3′[6]. Let $\circ$ be a vague binary operation on G w.r.t. $E_{G\times G} \in$ E(G $\times$ G) and $E_G \in$ E(G).

(i) $(G,\circ)$ is called a *vague semigroup* if it satisfies the following condition :

(VG, 1) $\forall a, b, c, d, m, q, w \in G$,

$\circ(b,c,d)\wedge\circ(a,d,m)\wedge\circ(a,b,q)\wedge\circ(q,c,w) \leq \circ(m,w)$

(ii) A vague semigroup $(G,\circ)$ is called a *vague monoid* if it satisfies the following condition :

(VG, 2) $\exists$ an (two-sided) identity element $e \in G$ such that

$\circ(e,a,a) \wedge \circ(a,e,a) = 1, \forall a \in G$.

(iii) A vague monoid $(G,\circ)$ is called a *vague group* if it satisfies the following condition :

(VG, 3) $\forall a \in G$, $\exists$ an (two-sided) inverse element $a^{-1} \in G$ such that

$\circ(a^{-1},a,e) \wedge \circ(a,a^{-1},e) = 1$.

(iv) A vague semigroup $(G,\circ)$ is said to be *abelian (commutative)* if it satisfies the condition :

(VG, 4) $\forall a, b, m, w \in G$,

$\circ(a,b,m) \wedge \circ(b,a,w) \leq E_G(m,w)$.

Remark 3.4. (a) If $(G,\circ)$ is a vague semigroup [resp. abelian semigroup, monoid and group] w.r.t. $E_{G\times G} \in$ E(G $\times$ G) and $E_G \in$ E(G), then $(G,(\mu_\circ,{\mu_\circ}^c))$ is an intuitionistic vague semigroup [resp. abelian semigroup, monoid and group] w.r.t. $(E_{G\times G},{E_{G\times G}}^c) \in$ IE(G $\times$ G) and $(E_G,E_G^c) \in$ IE(G).

(b) Let $\circ$ be an intuitionistic vague binary operation on G w.r.t. $IE_{G\times G} \in$ IE(G $\times$ G) and $IE_G \in$ IE(G). If $(G,\circ)$ is an intuitionistic vague semigroup [resp. abelian semigroup, monoid and group], then $(G,[\ ]\circ)$ and $(G,<>\ \circ)$ are intuitionistic vague semigroup [resp. abelian semigroups, monoids and groups] w.r.t. $[\ ]IE_{G\times G} \in$ IE(G $\times$ G), $[\ ]IE_G \in$ IE(G) and $<>\ IE_{G\times G} \in$ IE(G $\times$ G), $<>\ IE_G \in$ IE(G), respectively. Moreover, $(G,\mu_\circ)$ and $(G,\nu_\circ^c)$ are vague semigroups [resp. abelian semigroups, monoids and groups] w.r.t. $\mu_{IE_{G\times G}} \in$ E(G $\times$ G), $\mu_{IE_G} \in$ E(G) and $\nu_{IE_{G\times G}}{}^c \in$ E(G $\times$ G), $\nu_{IE_G}{}^c \in$ E(G), respectively.

Let $\circ$ be an intuitionistic vague operation on G w.r.t. $\Delta_{G\times G}$ and Δ_G such that $\circ(G\times G\times G) \subset \{0,1\}\times\{0,1\}$. Then an intuitionistic vague group $(G,\circ)$ one-to-one way corresponds to a group in the classical sense. In this case, an intuitionistic vague group is simply called a *crisp group*. For a given classical group $(G,\cdot)$, an infinite number of nontrivial intuitionistic vague group can be defined on G.

Example 3.5. Let $(G,\cdot)$ be a classical group, let $\alpha,\beta,\theta,\lambda,\mu,\gamma$ be fixed real number such that $0 < \theta \leq \alpha \leq \beta < 1$ and $0 < \lambda \leq \mu \leq \gamma < 1$, where $\theta+\gamma \leq 1$, $\alpha+\mu \leq 1$ and $\beta+\lambda \leq 1$. Let IE_G and $IE_{G\times G}$ be intuitionistic fuzzy equalities on G and $G \times G$ defined as follows, respectively : $\forall x,\ y,\ z,\ \omega \in G$,

$$\mathrm{IE}_G(x,y) = \begin{cases} (1,0), & \text{if } x = y, \\ (\beta,\lambda), & \text{if } x \neq y, \end{cases}$$

and

$$\mathrm{IE}_{G\times G}((x,y),(z,\omega)) = \begin{cases} (1,0), & \text{if } (x,y) = (z,\omega), \\ (\alpha,\mu), & \text{if } (x,y) \neq (z,\omega). \end{cases}$$

We define the intuitionistic fuzzy relation $*$ on $G\times G\times G$ as follows : $\forall x,y,z \in G$

$$*(x,y,z) = \begin{cases} (1,0), & if\ z = x\cdot y, \\ (\theta,\gamma), & \text{if } z \neq x\cdot y. \end{cases}$$

Then we can easily see that $(G,*)$ is an intuitionistic vague semigroup. Furthermore, the element e of $(G,\cdot)$ and the inverse element a^{-1} of a in $(G,\cdot)$ are the identity element of $(G,*)$ and the inverse element of a in $(G,*)$, respectively, Thus $(G,*)$ is an intuitionistic vague group. If $(G,\cdot)$ is abelian, so is $(G,*)$. It should also be noticed that $*$ is neither intuitionistic transitive of first order nor intuitionistic transitive of second order for $\theta < \beta$ and $\gamma > \lambda$, and that when $(\theta,\gamma) = (\alpha,\mu) = (\beta,\lambda)$, $*$ is both intuitionistic tran-

(b) If $\circ$ is intuitionistic transitive if first order, and f is intuitionistic vague injective and surjective, then the mapping $f^{-1}: G' \to G$ is an intuitionistic vague homomorphism.

Proof. (a) The proof is the analogue of the classical case in[8].

(b) Suppose $\circ$ is intuitionistic transitive of first order, and f is an intuitionistic vague injective and surjective. Let $u, v, w \in G'$. Since f is surjective and $\circ$ is strong, $\exists a, b, c \in G$ such that $a = f^{-1}(u)$, $b = f^{-1}(v)$ and $\circ(a, b, c) = (1, 0)$. Since f is an intuitionistic vague homomorphism,

$\circ'(f(a), f(b), f(c)) = \circ'(u, v, f(c)) = (1, 0)$.

Since $\circ$ is an intuitionistic fuzzy mapping, by (if. 2),

$\mu_{\circ'}(u, v, w) = \mu_{\circ'}(u, v, w) \wedge \mu_{\circ'}(u, v, f(c)) \le \mu_{IE_{G'}}(w, f(c))$.

and (4.1)

$\nu_{\circ'}(u, v, w) = \nu_{\circ'}(u, v, w) \vee \nu_{\circ'}(u, v, f(c)) \ge \nu_{IE_{G'}}(w, f(c))$.

On the other hand,

$\mu_{IE_{G'}}(w, f(c)) = \mu_{IE_{G'}}(f(f^{-1}(w)), f(c))$[Since f is bijective]

$\le \mu_{IE_G}(f^{-1}(w), c)$ [Since f is intuitionistic vague injective]

$= \mu_{\circ}(f^{-1}(u), f^{-1}(v), c) \wedge \mu_{IE_G}(f^{-1}(w), c)$ [Since $\circ(f^{-1}(u), f^{-1}(v), c) = (1, 0)$]

$\le \mu_{\circ}(f^{-1}(u), f^{-1}(v), f^{-1}(w))$ [Since $\circ$ is intuitionistic transitive of first order]

and (4.2)

$\nu_{IE_{G'}}(w, f(c)) = \nu_{IE_{G'}}(f(f^{-1}(w)), f(c))$

$\ge \nu_{IE_G}(f^{-1}(w), c)$

$= \nu_{\circ}(f^{-1}(u), f^{-1}(v), c) \vee \nu_{IE_G}(f^{-1}(w), c)$

$\ge \nu_{\circ}(f^{-1}(u), f^{-1}(v), f^{-1}(w))$

Thus, by (4.1) and (4.2),

$\mu_{\circ'}(u, v, w) \le \mu_{\circ}(f^{-1}(u), f^{-1}(v), f^{-1}(w))$

and

$\nu_{\circ'}(u, v, w) \ge \nu_{\circ}(f^{-1}(u), f^{-1}(v), f^{-1}(w))$.

Hence $f^{-1}: G' \to G$ is an intuitionistic vague homomorphism. □

Proposition 4.10. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′) and $IE_{G'} \in$ IE(G′), respectively, and let $f: G \to G'$ be an intuitionistic vague homomorphism. Then :

(a) $\mathrm{Ker}_{\mathrm{IV}} f$ is an intuitionistic vague subgroup of G.

(b) For any intuitionistic vague subgroup H of G, $f(H)$ is an intuitionistic vague subgroup of G'.

(c) For any intuitionistic vague subgroup K of G', $f^{-1}(K)$ is an intuitionistic vague subgroup of G.

Proof. (a) For any $a, b \in \mathrm{Ker}_{\mathrm{IV}} f$ and each $c \in G$, suppose $\circ(a, b^{-1}, c) = (1, 0)$. Then $f(a) = f(b)$, i.e., $f^{-1}(a) = f^{-1}(b)$. Thus

$\circ'((f(a), f^{-1}(a), e_{G'}) = \circ'(f(a), f^{-1}(b), e_{G'}) = (1, 0)$

Since f is an intuitionistic vague homomorphism, by Proposition 4.8(b) and the hypothesis,

$\circ'((f(a), f^{-1}(b), f(c)) = \circ'(f(a), f^{-1}(b), f(c)) = (1, 0)$.

Thus

$\mu_{\circ'}((f(a), f^{-1}(b), e_{G'}) \wedge mu_{\circ'}(f(a), f^{-1}(b), f(c))$
$= 1 \le \mu_{IE_{G'}}(f(c), e_{G'})$

and

$\nu_{\circ'}((f(a), f^{-1}(b), e_{G'}) \vee nu_{\circ'}(f(a), f^{-1}(b), f(c))$
$= 0 \ge \nu_{IE_{G'}}(f(c), e_{G'})$.

So $IE_{G'}(f(c), e_{G'}) = (1, 0)$, i.e., $f(c) = e_{G'}$. Hence $c \in \mathrm{Ker}_{\mathrm{IV}} f$. Therefore, by Theorem 4.3, $\mathrm{Ker}_{\mathrm{IV}} f$ is an intuitionistic vague subgroup of G.

(b) Suppose H is an intuitionistic vague subgroup of G. For any $a, b \in f(H)$ and each $c \in G'$. Suppose $\circ'(a, b^{-1}, c) = (1, 0)$. $\circ$ is an intuitionistic fuzzy mapping, $\exists \mu, \nu \in H$ and $w \in G$ such that $f(u) = a$ $f(v) = b$ and $\circ(u, v^{-1}, w) = (1, 0)$. Since H is an intuitionistic vague subgroup of G, by Theorem 4.3, $w \in H$. Then $f(w) \in f(h)$. Since f is an intuitionistic vague homomorphism, by Proposition 4.8(b),

$\circ'(f(u), f(v^{-1}), f(w)) = \circ'(f(u), f(v)^{-1}, f(w))$
$= \circ'(a, b^{-1}, f(w)) = (1, 0)$.

Thus

$\mu_{\circ'}(a, b^{-1}, f(w)) \wedge \mu_{\circ'}(a, b^{-1}, c) = 1 \le \mu_{IE_{G'}}(f(w), c)$

and

$\nu_{\circ'}(a, b^{-1}, f(w)) \vee \nu_{\circ'}(a, b^{-1}, c) = 0 \ge \nu_{IE_{G'}}(f(w), c)$.

So $IE_{G'}(f(w), c) = (1, 0)$, i.e., $c = f(w)$. Since $w \in H$, $c \in f(H)$. Hence, by Theorem 4.3, $f(H)$ is an intuitionistic vague subgroup of G'.

(c) It can be proved in a similar manner to the proof of (b). □

For a mapping $f: X \to Y$, let $Imf = \{f(a) \in Y : a \in X\}$. Then the following is the immediate result of Proposition 4.14(b).

Corollary 4.14. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f: G \to G'$ be an intuitionistic vague homomorphism. Then Imf is an intuitionistic vague subgroup of G'.

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$IE_G \in$ IE(G), and let H be a nonempty crisp subset of G. Let $\{H_\alpha\}_{\alpha\in\Gamma}$ be the family of all intuitionistic vague subgroups of G containing H. Then $\bigcap_{\alpha\in\Gamma} H_\alpha$ is an intuitionistic vague subgroup of G. In this case, $\bigcap_{\alpha\in\Gamma} H_\alpha$ is called the *intuitionistic vague subgroup of G generated by H*, and it is denoted by $< H >$.

Definition 4.5. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups. Then a mapping(in the classical sense) $f : G \to G'$ is called an *intuitionistic vague homomorphism* if it satisfies the following conditions :
$\mu_\circ(a, b, c) \leq \mu_{\circ'}(f(a), f(b), f(c))$
and
$\nu_\circ(a, b, c) \geq \nu_{\circ'}(f(a), f(b), f(c)),\ \forall a, b, c \in G$.

Definition 4.5′[6]. Let (G, μ) and (G', μ') be two vague groups. Then a mapping $f : G \to G'$ is called a *vague homomorphism* if
$\mu(a, b, c) \leq \mu'(f(a), f(b), f(c)),\ \forall a, b, c \in G$.

Remark 4.5. (a) If $f : (G, \mu) \to (G', \mu')$ is a vague homomorphism, then $f : (G, (\mu, \mu^c)) \to (G', (\mu', \mu'^c))$ is an intuitionistic vague homomorphism.
(b) If $f : (G, \circ) \to (G', \circ')$ is an intuitionistic vague homomorphism, then $f : (G, [\,]\circ) \to (G', [\,]\circ')$[*resp.* $f : (G, <> \circ) \to (G', <> \circ')$] is an intuitionistic vague homomorphism. Moreover, $f : (G, \mu_\circ) \to (G', \mu_{\circ'})$ [resp. $f : (G, \nu_\circ^c) \to (G, \nu_{\circ'}^c)$] is a vague homomorphism.

Proposition 4.6. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G'), and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then :
(a) If e_G and $e_{G'}$ are identities of $(G, \circ)$ and $(G', \circ')$, respectively, then $f(e_G) = e_{G'}$.
(b) For each $a \in G$, $f^{-1}(a) = f(a^{-1})$.

proof. (a) Let e_G and $e_{G'}$ be identities of $(G, \circ)$ and $(G, \circ')$, respectively, and let $a \in G$. Then $\circ(a, e_G, a) = (1, 0)$. Since $f : G \to G'$ is an intuitionistic vague homomorphism,
$\circ'(f(a), f(e_G), f(a)) = (1, 0)$.
On the other hand,
$\circ'(f(a), e_{G'}, f(a)) = (1, 0)$.
Thus, by Proposition 3.8,
$\mu_{0'}(f(a), f(e_G), f(a)) \wedge \mu_{0'}(f(a), e_{G'}, f(a))$
$= 1 \leq \mu_{IE_{G'}}(f(e_G), e_{G'})$
and
$\nu_{0'}(f(a), f(e_G), f(a)) \vee \nu_{0'}(f(a), e_{G'}, f(a))$
$= 0 \geq \nu_{IE_{G'}}(f(e_G), e_{G'})$.
So $IE_{G'}(f(e_G), e_{G'}) = (1, 0)$. Hence $f(e_G) = e_{G'}$.

(b) Let $a \in G$. Then clearly $\circ(a, a^{-1}, e_G) = (1, 0)$. Since $f : G \to G'$ is an intuitionistic vague homomorphism, by (a),
$\circ'(f(a), f(a^{-1}), f(e_G)) = \circ'(f(a), f(a^{-1}), e_{G'}) = (1, 0)$.
Thus, by Proposition 3.8,
$\mu_{0'}(f(a), f(a^{-1}), e_{G'}) \wedge \mu_{0'}(f(a), f^{-1}(a), e_{G'})$
$= 1 \leq \mu_{IE_{G'}}(f(a^{-1}, f^{-1}(a)))$
and
$\nu_{0'}(f(a), f(a^{-1}), e_{G'}) \vee \nu_{0'}(f(a), f^{-1}(a), e_{G'})$
$= 0 \geq \nu_{IE_{G'}}(f(a^{-1}, f^{-1}(a)))$.
So $IE_{G'}(f(a^{-1}, f^{-1}(a))) = (1, 0)$. Hence $f(a^{-1}) = f^{-1}(a)$. □

Definition 4.7. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then crisp set $\{a \in X : f(a) = e_{G'}\}$ is called an *intuitionistic vague kernel* of f, and it is denoted by $\mathrm{Ker}_{\mathrm{IV}} f$.

Definition 4.8. Let $IE_X \in$ IE(X) and let $IE_Y \in$ IE(Y). Then a mapping $g : X \to Y$ is said to be *intuitionistic vague injective w.r.t.* IE_X *and* IE_Y if
$\mu_{IE_X}(g(a), g(b)) \leq \mu_{IE_Y}(a, b)$
and
$\nu_{IE_X}(g(a), g(b)) \geq \nu_{IE_Y}(a, b),\ \forall a, b \in X$.

It is clear that an intuitionistic vague injective mapping is injective in the classical sense.

Definition 4.8′[6]. A mapping $g : X \to Y$ is said to be *vague injective* w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y) if $E_Y(g(a), g(b)) \leq E_X(a, b),\ \forall a, b \in X$.

Remark 4.8. (a) If $g : X \to Y$ is vague injective w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y), then $g : X \to Y$ is intuitionistic vague injective w.r.t. $(E_X, E_X^c) \in$ IE(X) and $(E_Y, E_Y^c) \in$ IE(Y).
(b) If $g : X \to Y$ is intuitionistic vague injective w.r.t. $IE_X \in$ IE(X) w.r.t. $[\ ]IE_X \in$ IE(X) and $[\ ]IE_Y \in$ IE(Y) [resp. $<> IE_X \in$ IE(X) and $<> IE_Y \in$ IE(Y)]. Furthermore, g is vague injective w.r.t. $\mu_{IE_X} \in$ E(X) and $\mu_{IE_Y} \in$ E(Y) [resp. $\nu_{IE_X}^c \in$ E(X)] and $\nu_{IE_Y}^c \in$ E(Y)].

Proposition 4.9. Let $(G, \circ)$ and $(G, \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in IE(G)$ and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Let e_G be the identity of $(G, \circ)$.
(a) f is injective if and only if $\mathrm{Ker}_{\mathrm{IV}} f = \{e_G\}$.

$\nu_\circ(e_L, a_L^{-1}, a_L^{-1}) \vee \nu_\circ(a, a_L^{-1}, u) \vee \nu_\circ(a, e_L, w) \vee \nu_\circ(w, a_L^{-1}, t) = 0 \geq \nu_{IE_G}(t, u)$.
Thus $IE_G(t, u) = (1, 0)$, i.e. $t = u$. So $\circ(w, a_L^{-1}, u) = (1, 0)$.
So
$\mu_\circ(a, a_L^{-1}, u) \wedge \mu_\circ(u, u, v) \wedge \mu_\circ(u, a, w) \wedge \mu_\circ(w, a_L^{-1}, u) = 1 \leq \mu_{IE_G}(v, u)$
and
$\nu_\circ(a, a_L^{-1}, u) \vee \nu_\circ(u, u, v) \vee \nu_\circ(u, a, w) \vee \nu_\circ(w, a_L^{-1}, u) = 0 \geq \nu_{IE_G}(v, u)$.
Hence $IE_G(v, u) = (1, 0)$, i.e. $v = u$. Therefore $\circ(u, u, u) = (1, 0)$. By Proposition 3.7, $IE_G(u, e_L) = (1, 0)$, i.e., $u = e_L$.
Thus $\circ(a, a_L^{-1}, e_L) = (1, 0)$ (3.6)
Now we shall show that $\circ(a, e_L, a) = (1, 0)$, i.e., e_L is also a right identity of $(G, \circ)$, i.e., e_L is a two-sided identity of $(G, \circ)$. Let $a \in G$. Then, it is clear that
$\exists u \in G$ such that $\circ(a, e_L, u) = (1, 0)$.
Thus, by (3.6),
$\mu_\circ(a_L^{-1}, a, e_L) \wedge \mu_\circ(a, e_L, u) \wedge \mu_\circ(a, a_L^{-1}, e_L) \wedge \mu_\circ(e_L, a, a) = 1 \leq \mu_{IE_G}(u, a)$
and
$\nu_\circ(a_L^{-1}, a, e_L) \vee \nu_\circ(a, e_L, u) \vee \nu_\circ(a, a_L^{-1}, e_L) \vee \nu_\circ(e_L, a, a) = 0 \geq \nu_{IE_G}(u, a)$.
So $IE_G(u, a) = (1, 0)$, i.e., $u = a$. Therefore $\circ(a, e_L, a) = (1, 0)$.
Since e_L is a two-sided identity of $(G, \circ)$, by using the hypothesis and (3.6), we can immediately see that $(IVG.3)$ is satisfied. Hence $(G, \circ)$ is an intuitionistic vague group. □

Theorem 3.11 Let $(G, \circ)$ be an intuitionistic vague semigroup w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G). Then $(G, \circ)$ is an intuitionistic vague group if and only if
(i) $\forall a, b \in G, \exists x \in G$ such that $\circ(a, x, b) = (1, 0)$,
(ii) $\forall a, b \in G, \exists y \in G$ such that $\circ(y, a, b) = (1, 0)$.
Proof. $(\Rightarrow)$: Suppose $(G, \circ)$ is an intuitionistic vague group and let $a, b \in G$. Then $\exists x, u \in G$ such that
$\circ(a^{-1}, b, x) = \circ(a, x, u) = (1, 0)$.
Thus
$\mu_\circ(a^{-1}, b, x) \wedge \mu_\circ(a, x, u) \wedge \mu_\circ(a, a^{-1}, e) \wedge \mu_\circ(e, b, b) = 1 \leq \mu_{IE_G}(u, b)$
and
$\nu_\circ(a^{-1}, b, x) \vee \nu_\circ(a, x, u) \vee \nu_\circ(a, a^{-1}, e) \vee \nu_\circ(e, b, b) = 0 \geq \nu_{IE_G}(u, b)$.
So $IE_G(u, b) = (1, 0)$, i.e. $u = b$. Hence $\circ(a, x, b) = (1, 0)$.
On the other hand, for $a, b \in G$, $\exists y, v \in G$ such that
$\circ(b, a^{-1}, y) = \circ(y, a, v) = (1, 0)$.
Then
$\mu_\circ(a^{-1}, a, e) \wedge \mu_\circ(b, e, b) \wedge \mu_\circ(b, a^{-1}, y) \wedge \mu_\circ(y, a, v) = 1 \leq \mu_{IE_G}(b, v)$
and
$\nu_\circ(a^{-1}, a, e) \vee \nu_\circ(b, e, b) \vee \nu_\circ(b, a^{-1}, y) \vee \nu_\circ(y, a, v) = 1 \geq \nu_{IE_G}(b, v)$.
Thus $IE_G(b, v) = (1, 0)$, i.e., $b = v$. So $\circ(y, a, b) = (1, 0)$.
$(\Leftarrow)$ Suppose the necessary conditions hold. Let $m \in G$ be fixed and let $a \in G$. Then $\exists e^*, x \in G$ such that
$\circ(e^*, m, m) = \circ(m, x, a) = (1, 0)$.
Since $\circ$ is an intuitionistic fuzzy mapping, for $a \in G$, $\exists u \in G$ such that $\circ(e^*, a, u) = (1, 0)$. Thus
$\mu_\circ(m, x, a) \wedge \mu_\circ(e^*, a, u) \wedge \mu_\circ(e^*, m, m) \wedge \mu_\circ(m, x, a) = 1 \leq \mu_{IE_G}(u, a)$
and
$\nu_\circ(m, x, a) \vee \nu_\circ(e^*, a, u) \vee \nu_\circ(e^*, m, m) \vee \nu_\circ(m, x, a) = 0 \geq \nu_{IE_G}(u, a)$.
So $IE_G(u, a) = (1, 0)$, i.e., $u = a$. Hence $\circ(e^*, a, a) = (1, 0)$, i.e., e^* is a left identity of $(G, \circ)$.
On the other hand, by the hypothesis, for each $a \in G$, $\exists w \in G$ such that $\circ(w, a, e) = (1, 0)$. Thus w is a left inverse of a. So, the required result is immediately obtained from Theorem 3.10. This completes the proof. □

4. Intuitionistic vague subgroups and intuitionistic vague homomorphisms

For a given intuitionistic fuzzy equality IE_X on X and for a crisp subset H of X, the restriction of the complex mapping IE_X on X and for a crisp subset H of X, the restriction of the complex mapping IE_X on $H \times H$, denoted by IE_X^H, is clearly an intuitionistic fuzzy equality on H. For a given intuitionistic vague binary operation f on X, we say that a crisp subset B of X is *intuitionistic vague closed under* f if it satisfies the following condition :
(IVGC) $f(a, b, c) = (1, 0) \Rightarrow c \in B, ^\forall a, b, c \in X$.
For given intuitionistic vague operation f on X w.r.t. $IE_{X\times X} \in$ IE(X × X) and $IE_X \in$ IE(X), if a crisp subset H of X is intuitionistic vague closed under f, then it is easily seen that $f|_{H\times H\times H}$ is an intuitionistic vague operation on H and $f|_{H\times H\times H}$ preserves the transitive properties of f.

Definition 4.1. Let $(G, \circ)$ be an intuitionistic vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let H be a nonempty crisp subset of G that is intuitionistic vague closed under $\circ$. Then H is called an *intuitionistic vague subgroup* of G if $(H, \circ|_{H\times H\times H})$ is itself an intuitionistic vague group.

For a given fuzzy equality E_X on X and for a crisp subset H of X, the restriction of the complex mapping E_X on $H \times H$, denoted by E_X^H, is clearly a fuzzy equality on H. For a given vague binary operation f on X, we say that a crisp subset B of X is *vague closed under* f if it satisfies the following condition :

(**VGC**) $f(a,b,c) = (1,0) \Rightarrow c \in B, \forall a,b \in B, \forall c \in X$.

For given vague operation f on X w.r.t. $E_{X\times X} \in$ E(X × X) and $E_X \in$ E(X), if a crisp subset H of X is vague closed under f, then it is easily seen that $f|_{H\times H\times H}$ is a vague operation on H and $f|_{H\times H\times H}$ preserves the transitive properties of f.

Definition 4.1′[6]. Let $(G,\circ)$ be a vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let H be a nonempty crisp subset of G that is vague closed under $\circ$. Then H is called a *vague subgroup* of G if $(H, \circ|_{H\times H\times H})$ is itself a vague group.

Remark 4.2. (a) Let $(G,\circ)$ be a vague group w.r.t. $E_{G\times G} \in$ E(G × G) and $E_G \in$ E(G), and let H be a nonempty crisp subset of G. If $(H, \circ|_{H\times H\times H})$ is a vague subgroup of G, then $(H, (\mu_\circ, \mu_\circ^c)|_{H\times H\times H})$ is an intuitionistic vague subgroup of the intuitionistic vague group $(G, (\mu_\circ, \mu_\circ^c))$ w.r.t. $(\mu_{E_{G\times G}}, \mu^c_{E_{G\times G}}) \in$ IE(G × G) and $(\mu_{E_G}, \mu^c_{E_G}) \in$ IE(G).

(b) Let $(G,\circ)$ be an intuitionistic vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let H be a nonempty crisp subset of G. If $(H, \circ|_{H\times H\times H})$ is an intuitionistic vague subgroup of $(G,\circ)$, then $(H, [\ ]\circ|_{H\times H\times H})$ [resp. $(H, <>\circ|_{H\times H\times H})$] is an intuitionistic vague subgroup of the intuitionistic vague group $(G, [\ ]\circ)$ [resp. $(G, <>\circ)$] w.r.t. $[\ ]IE_{G\times G} \in$ IE(G × G) and $[\ ]IE_G \in$ IE(G) [resp. $<> IE_{G\times G} \in IE(G\times G)$ and $<> IE_G \in$ IE(G)]. Moreover, $(H, \mu_\circ|_{H\times H\times H})$[resp. $(H, \nu_\circ^c|_{H\times H\times H})$] is a vague subgroup of the vague group $(G, \mu_\circ)$[resp. $(G, \nu_\circ^c)$] w.r.t. $\mu_{IE_{G\times G}} \in$ E(G × G) and $\mu_{IE_G} \in$ E(G)[resp. $\nu_{IE^c_{G\times G}} \in$ E(G × G) and $\nu_{IE^c_G} \in$ E(G)].

Theorem 4.3. Let $(G,\circ)$ be an intuitionistic vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let H be a nonempty crisp subset of G. Then H is an intuitionistic vague subgroup of G if and only if $(^\forall a,b \in H)(^\forall c \in G)(\circ(a,b^{-1},c) = (1,0) \Rightarrow c \in H)$.

Proof $(\Rightarrow)$: Suppose H be an intuitionistic vague subgroup of G. Let e_G[resp. e_H] be an identity of $(G,\circ)$[resp. $(H, \circ|_{H\times H\times H})$]. Then, for $a \in H$, it is clear that

$\circ|_{H\times H\times H}(a, e_H, a) = \circ(a, e_H, a) = \circ(a, e_G, a) = (1,0)$

By Proposition 3.8,

$\mu_0(a, e_H, a) \wedge \mu_0(a, e_G, a) = 1 \le \mu_{IE_G}(e_H, e_G)$

and

$\nu_0(a, e_H, a) \vee \nu_0(a, e_G, a) = 0 \ge \nu_{IE_G}(e_H, e_G)$.

Thus $IE_G(e_H, e_G) = (1,0)$, i.e., $e_H = e_G$. For $b \in H$, let b_H^{-1} be the inverse of b in $(H, \circ|_{H\times H\times H})$. Then, it is clear that

$\circ|_{H\times H\times H}(b, b_H^{-1}, e_H) = \circ(b, b_H^{-1}, e_H) = \circ(b, b^{-1}, e_H) = (1,0)$.

Thus

$\mu_0(b, b_H^{-1}, e_H) \wedge \mu_0(b, b^{-1}, e_H) = 1 \le \mu_{IE_G}(b_H^{-1}, b^{-1})$

and

$\nu_0(b, b_H^{-1}, e_H) \vee \nu_0(b, b^{-1}, e_H) = 0 \ge \nu_{IE_G}(b_H^{-1}, b^{-1})$.

So $IE_G(b_H^{-1}, b^{-1}) = (1,0)$, i.e., $b_H^{-1} = b^{-1}$.

Now, for $a, b^{-1} \in H$ and $c \in G$, suppose $\circ(a, b^{-1}, c) = (1,0)$. Since $a, b^{-1} \in H$ and H is intuitionistic vague closed under $\circ$, by $(IVGC)$, $c \in H$.

$(\Leftarrow)$: Suppose the necessary condition holds. Since $H \neq \varnothing$, $\exists u \in H$. Then $\circ(u, u^{-1}, e_G) = (1,0)$. Thus, by the hypothesis, $e_G \in H$. Let $a \in H$. Then clearly $\circ(e_G, a^{-1}, a^{-1}) = (1,0)$. Thus, by the hypothesis, $a^{-1} \in H$.

For any $a, b \in H$ and each $c \in G$, suppose $\circ(a,b,c) = (1,0)$. Then $\circ(a,b,c) = \circ(a, (b^{-1})^{-1}, c) = 1$. Since $b^{-1} \in H$, by the hypothesis, $c \in H$. Thus H is intuitionistic vague closed under $\circ$. Since $(G,\circ)$ is an intuitionictic vague group, it can easily be be seen that $(H, \circ|_{H\times H\times H})$ satisfies the condition (IVG. 1) w.r.t. $IE_{G\times G}H \times H \in$ IE(H × H) and $IE_G{}^H \in$ IE(H). Hence H is an intuitionistic vague subgroup of G. □

Theorem 4.4. Let $(G,\circ)$ be an intuitionistic vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let H be a nonempty crisp subset of G. Then H is an intuitionistic vague subgroup of G if and only if it satisfies the following conditions:

(i) H is intuitionistic vague closed under $\circ$

(ii) For each $a \in H$, $a^{-1} \in H$.

Proof The proof can be obtained in a similar manner to that of the classical case in [8]. Thus it is omitted. □

The following is the immediate result of Theorem 4.4.

corollary 4.4-1. Let $(G,\circ)$ be an intuitionistic vague w.r.t. $IE_{G\times G} \in$ IE(G × G) and $IE_G \in$ IE(G), and let $\{H_\alpha\}_{\alpha\in\Gamma}$ be a nonempty family of intuitionistic vague subgroup of G such that $\bigcap_{\alpha\in\Gamma} H_\alpha \neq \varnothing$. Then $\bigcap_{\alpha\in\Gamma} H_\alpha$ is an intuitionistic vague subgroup of G.

The following is the immediate result of corollary 4.4-1.

corollary 4.4-2. Let $(G,\circ)$ be an intuitionistic vague group w.r.t. $IE_{G\times G} \in$ IE(G × G) and

$IE_G \in$ IE(G), and let H be a nonempty crisp subset of G. Let $\{H_\alpha\}_{\alpha\in\Gamma}$ be the family of all intuitionistic vague subgroups of G containing H. Then $\bigcap_{\alpha\in\Gamma} H_\alpha$ is an intuitionistic vague subgroup of G. In this case, $\bigcap_{\alpha\in\Gamma} H_\alpha$ is called the *intuitionistic vague subgroup of G generated by H*, and it is denoted by $< H >$.

Definition 4.5. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups. Then a mapping(in the classical sense) $f : G \to G'$ is called an *intuitionistic vague homomorphism* if it satisfies the following conditions :

$\mu_\circ(a, b, c) \leq \mu_{\circ'}(f(a), f(b), f(c))$

and

$\nu_\circ(a, b, c) \geq \nu_{\circ'}(f(a), f(b), f(c)),\ \forall a, b, c \in G.$

Definition 4.5′[6]. Let (G, μ) and (G', μ') be two vague groups. Then a mapping $f : G \to G'$ is called a *vague homomorphism* if

$\mu(a, b, c) \leq \mu'(f(a), f(b), f(c)),\ \forall a, b, c \in G.$

Remark 4.5. (a) If $f : (G, \mu) \to (G', \mu')$ is a vague homomorphism, then $f : (G, (\mu, \mu^c)) \to (G', (\mu', \mu'^c))$ is an intuitionistic vague homomorphism.

(b) If $f : (G, \circ) \to (G', \circ')$ is an intuitionistic vague homomorphism, then $f : (G, [\,]\circ) \to (G', [\,]\circ')$[*resp.* $f : (G, <> \circ) \to (G', <> \circ')$] is an intuitionistic vague homomorphism. Moreover, $f : (G, \mu_\circ) \to (G', \mu_{\circ'})$ [resp. $f : (G, \nu^c_\circ) \to (G, \nu^c_{\circ'})$] is a vague homomorphism.

Proposition 4.6. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then :

(a) If e_G and $e_{G'}$ are identities of $(G, \circ)$ and $(G', \circ')$, respectively, then $f(e_G) = e_{G'}$.

(b) For each $a \in G$, $f^{-1}(a) = f(a^{-1})$.

proof. (a) Let e_G and $e_{G'}$ be identities of $(G, \circ)$ and $(G, \circ')$, respectively, and let $a \in G$. Then $\circ(a, e_G, a) = (1, 0)$. Since $f : G \to G'$ is an intuitionistic vague homomorphism,

$\circ'(f(a), f(e_G), f(a)) = (1, 0).$

On the other hand,

$\circ'(f(a), e_{G'}, f(a)) = (1, 0).$

Thus, by Proposition 3.8,

$\mu_{0'}(f(a), f(e_G), f(a)) \wedge \mu_{0'}(f(a), e_{G'}, f(a))$

$= 1 \leq \mu_{IE_{G'}}(f(e_G), e_{G'})$

and

$\nu_{0'}(f(a), f(e_G), f(a)) \vee \nu_{0'}(f(a), e_{G'}, f(a))$

$= 0 \geq \nu_{IE_{G'}}(f(e_G), e_{G'}).$

So $IE_{G'}(f(e_G), e_{G'}) = (1, 0)$. Hence $f(e_G) = e_{G'}$.

(b) Let $a \in G$. Then clearly $\circ(a, a^{-1}, e_G) = (1, 0)$. Since $f : G \to G'$ is an intuitionistic vague homomorphism, by (a),

$\circ'(f(a), f(a^{-1}), f(e_G)) = \circ'(f(a), f(a^{-1}), e_{G'}) = (1, 0).$

Thus, by Proposition 3.8,

$\mu_{0'}(f(a), f(a^{-1}), e_{G'}) \wedge \mu_{0'}(f(a), f^{-1}(a), e_{G'})$

$= 1 \leq \mu_{IE_{G'}}(f(a^{-1}, f^{-1}(a)))$

and

$\nu_{0'}(f(a), f(a^{-1}), e_{G'}) \vee \nu_{0'}(f(a), f^{-1}(a), e_{G'})$

$= 0 \geq \nu_{IE_{G'}}(f(a^{-1}, f^{-1}(a))).$

So $IE_{G'}(f(a^{-1}, f^{-1}(a))) = (1, 0)$. Hence $f(a^{-1}) = f^{-1}(a)$. □

Definition 4.7. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then crisp set $\{a \in X : f(a) = e_{G'}\}$ is called an *intuitionistic vague kernel* of f, and it is denoted by $\mathrm{Ker}_{\mathrm{IV}} f$.

Definition 4.8. Let $IE_X \in$ IE(X) and let $IE_Y \in$ IE(Y). Then a mapping $g : X \to Y$ is said to be *intuitionistic vague injective w.r.t.* IE_X *and* IE_Y if

$\mu_{IE_X}(g(a), g(b)) \leq \mu_{IE_Y}(a, b)$

and

$\nu_{IE_X}(g(a), g(b)) \geq \nu_{IE_Y}(a, b),\ \forall a, b \in X.$

It is clear that an intuitionistic vague injective mapping is injective in the classical sense.

Definition 4.8′[6]. A mapping $g : X \to Y$ is said to be *vague injective* w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y) if $E_Y(g(a), g(b)) \leq E_X(a, b),\ \forall a, b \in X$.

Remark 4.8. (a) If $g : X \to Y$ is vague injective w.r.t. $E_X \in$ E(X) and $E_Y \in$ E(Y), then $g : X \to Y$ is intuitionistic vague injective w.r.t. $(E_X, E^c_X) \in$ IE(X) and $(E_Y, E^c_Y) \in$ IE(Y).

(b) If $g : X \to Y$ is intuitionistic vague injective w.r.t. $IE_X \in$ IE(X) w.r.t. $[\ ]IE_X \in$ IE(X) and $[\ ]IE_Y \in$ IE(Y) [resp. $<> IE_X \in$ IE(X) and $<> IE_Y \in$ IE(Y)]. Furthermore, g is vague injective w.r.t. $\mu_{IE_X} \in$ E(X) and $\mu_{IE_Y} \in$ E(Y) [resp. $\nu^c_{IE_X} \in$ E(X)] and $\nu^c_{IE_Y} \in$ E(Y)].

Proposition 4.9. Let $(G, \circ)$ and $(G, \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in IE(G)$ and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Let e_G be the identity of $(G, \circ)$.

(a) f is injective if and only if $\mathrm{Ker}_{\mathrm{IV}} f = \{e_G\}$.

(b) If $\circ$ is intuitionistic transitive if first order, and f is intuitionistic vague injective and surjective, then the mapping $f^{-1} : G' \to G$ is an intuitionistic vague homomorphism.

Proof. (a) The proof is the analogue of the classical case in[8].

(b) Suppose $\circ$ is intuitionistic transitive of first order, and f is an intuitionistic vague injective and surjective. Let $u, v, w \in G'$. Since f is surjective and $\circ$ is strong, $\exists a, b, c \in G$ such that $a = f^{-1}(u)$, $b = f^{-1}(v)$ and $\circ(a, b, c) = (1, 0)$. Since f is an intuitionistic vague homomorphism,

$\circ'(f(a), f(b), f(c)) = \circ'(u, v, f(c)) = (1, 0)$.

Since $\circ$ is an intuitionistic fuzzy mapping, by (if. 2),

$\mu_{\circ'}(u, v, w) = \mu_{\circ'}(u, v, w) \wedge \mu_{\circ'}(u, v, f(c)) \leq \mu_{IE_{G'}}(w, f(c))$.

and (4.1)

$\nu_{\circ'}(u, v, w) = \nu_{\circ'}(u, v, w) \vee \nu_{\circ'}(u, v, f(c)) \geq \nu_{IE_{G'}}(w, f(c))$.

On the other hand,

$\mu_{IE_{G'}}(w, f(c)) = \mu_{IE_{G'}}(f(f^{-1}(w)), f(c))$[Since f is bijective]

$\leq \mu_{IE_G}(f^{-1}(w), c)$ [Since f is intuitionistic vague injective]

$= \mu_\circ(f^{-1}(u), f^{-1}(v), c) \wedge \mu_{IE_G}(f^{-1}(w), c)$ [Since $\circ(f^{-1}(u), f^{-1}(v), c) = (1, 0)$]

$\leq \mu_\circ(f^{-1}(u), f^{-1}(v), f^{-1}(w))$ [Since $\circ$ is intuitionistic transitive of first order]

and (4.2)

$\nu_{IE_{G'}}(w, f(c)) = \nu_{IE_{G'}}(f(f^{-1}(w)), f(c))$
$\geq \nu_{IE_G}(f^{-1}(w), c)$
$= \nu_\circ(f^{-1}(u), f^{-1}(v), c) \vee \nu_{IE_G}(f^{-1}(w), c)$
$\geq \nu_\circ(f^{-1}(u), f^{-1}(v), f^{-1}(w))$

Thus, by (4.1) and (4.2),

$\mu_{\circ'}(u, v, w) \leq \mu_\circ(f^{-1}(u), f^{-1}(v), f^{-1}(w))$

and

$\nu_{\circ'}(u, v, w) \geq \nu_\circ(f^{-1}(u), f^{-1}(v), f^{-1}(w))$.

Hence $f^{-1} : G' \to G$ is an intuitionistic vague homomorphism. □

Proposition 4.10. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′) and $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then :

(a) $\mathrm{Ker}_{\mathrm{IV}} f$ is an intuitionistic vague subgroup of G.

(b) For any intuitionistic vague subgroup H of G, $f(H)$ is an intuitionistic vague subgroup of G'.

(c) For any intuitionistic vague subgroup K of G', $f^{-1}(K)$ is an intuitionistic vague subgroup of G.

Proof. (a) For any $a, b \in \mathrm{Ker}_{\mathrm{IV}} f$ and each $c \in G$, suppose $\circ(a, b^{-1}, c) = (1, 0)$. Then $f(a) = f(b)$, i.e., $f^{-1}(a) = f^{-1}(b)$. Thus

$\circ'((f(a), f^{-1}(a), e_{G'}) = \circ'(f(a), f^{-1}(b), e_{G'}) = (1, 0)$

Since f is an intuitionistic vague homomorphism, by Proposition 4.8(b) and the hypothesis,

$\circ'((f(a), f^{-1}(b), f(c)) = \circ'(f(a), f^{-1}(b), f(c)) = (1, 0)$.

Thus

$\mu_{\circ'}((f(a), f^{-1}(b), e_{G'}) \wedge mu_{\circ'}(f(a), f^{-1}(b), f(c))$
$= 1 \leq \mu_{IE_{G'}}(f(c), e_{G'})$

and

$\nu_{\circ'}((f(a), f^{-1}(b), e_{G'}) \vee nu_{\circ'}(f(a), f^{-1}(b), f(c))$
$= 0 \geq \nu_{IE_{G'}}(f(c), e_{G'})$.

So $IE_{G'}(f(c), e_{G'}) = (1, 0)$, i.e., $f(c) = e_{G'}$. Hence $c \in \mathrm{Ker}_{\mathrm{IV}} f$. Therefore, by Theorem 4.3, $\mathrm{Ker}_{\mathrm{IV}} f$ is an intuitionistic vague subgroup of G.

(b) Suppose H is an intuitionistic vague subgroup of G. For any $a, b \in f(H)$ and each $c \in G'$. Suppose $\circ'(a, b^{-1}, c) = (1, 0)$. $\circ$ is an intuitionistic fuzzy mapping, $\exists \mu, \nu \in H$ and $w \in G$ such that $f(u) = a$ $f(v) = b$ and $\circ(u, v^{-1}, w) = (1, 0)$. Since H is an intuitionistic vague subgroup of G, by Theorem 4.3, $w \in H$. Then $f(w) \in f(h)$. Since f is an intuitionistic vague homomorphism, by Proposition 4.8(b),

$\circ'(f(u), f(v^{-1}), f(w)) = \circ'(f(u), f(v)^{-1}, f(w))$
$= \circ'(a, b^{-1}, f(w)) = (1, 0)$.

Thus

$\mu_{\circ'}(a, b^{-1}, f(w)) \wedge \mu_{\circ'}(a, b^{-1}, c) = 1 \leq \mu_{IE_{G'}}(f(w), c)$

and

$\nu_{\circ'}(a, b^{-1}, f(w)) \vee \nu_{\circ'}(a, b^{-1}, c) = 0 \geq \nu_{IE_{G'}}(f(w), c)$.

So $IE_{G'}(f(w), c) = (1, 0)$, i.e., $c = f(w)$. Since $w \in H$, $c \in f(H)$. Hence, by Theorem 4.3, $f(H)$ is an intuitionistic vague subgroup of G'.

(c) It can be proved in a similar manner to the proof of (b). □

For a mapping $f : X \to Y$, let $Imf = \{f(a) \in Y : a \in X\}$. Then the following is the immediate result of Proposition 4.14(b).

Corollary 4.14. Let $(G, \circ)$ and $(G', \circ')$ be two intuitionistic vague groups w.r.t. $IE_{G\times G} \in$ IE(G × G), $IE_G \in$ IE(G) and $IE_{G'\times G'} \in$ IE(G′ × G′), $IE_{G'} \in$ IE(G′), respectively, and let $f : G \to G'$ be an intuitionistic vague homomorphism. Then Imf is an intuitionistic vague subgroup of G'.

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