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# Vague Continuous Mappings

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**Abstract**

Up to now, in the area of fuzzy topology, almost all the researchers have investigated fuzzy continuities by using ordinary mappings. However, in this paper, we study continuities by using fuzzy mappings introduced by Demirci [2].

**Keywords and phrases** : fuzzy point, fuzzy equality, fuzzy mapping, vague continuous[resp. open and closed] mapping.

## 1. Introduction

In 1965, Zadeh [7] introduced the concept of fuzzy sets as the generalization of ordinary subsets. Thereafter many investigations have been carried out, in the general theoretical field based on that. First, Chang [1] introduced the notion of fuzzy topology and after that, Srivastava et. al[5], Pu and Liu [3,4] and Yalvac [6], etc., studied fuzzy topological spaces.

Throughout this paper, $X, Y, Z$, etc., denote crisp sets, I [resp. $I_{0,1}$ and $I_0$] denotes the closed unit interval $[0, 1]$(resp. open interval $(0, 1)$ and open-closed interval $(0, 1]$) and $I^X$ denotes the set of all the fuzzy sets in $X$.

## 2. Preliminaries

In this section, we list some concept introduced and some results obtained by Demirci [2].

**Definition 2.1 [2].** A mapping $E_X : X \times X \to I$ is called a *fuzzy equality on* $X$ if it satisfies the following conditions :

(e.1) $E_X(x, y) = 1 \Leftrightarrow x = y, \forall x,\, y \in X$,

(e.2) $E_X(x, y) = E_X(y, x), \forall x,\, y \in X$,

(e.3) $E_X(x, y) \wedge E_X(y, z) \leq E_X(x, z), \forall x,\, y,\, z \in X$.

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The real number $E_X(x, y)$ represents the degree of the equality of $x$ and $y$ for $x, y \in X$. We will denote the set of all the fuzzy equalities on as $E(X)$.

**Definition 2.2 [6].**

(1) $R$ is called a *fuzzy relation from* $X$ *to* $Y$ (or *on* $X \times Y$) if $R \in I^{X \times Y}$.

(2) Let $R$ and $S$ be fuzzy relations on $X \times Y$ and $Y \times Z$ respectively. Then the (*sup - min*) *composition* of $R$ and $S$, denoted by $S \circ R$, is a fuzzy relation on $X \times Z$ defined by

$(S \circ R)(x, z) = \bigvee_{y \in Y}[R(x, y) \wedge S(y, z)], \forall x \in X, \forall z \in Z$.

(3) Let $R$ be a fuzzy relation on $X \times Y$. Then the *inverse* of $R$, denoted by $R^{-1}$, is the fuzzy relation on $Y \times X$ defined by $R^{-1}(y, x) = R(x, y) \forall x \in X, \forall y \in Y$.

**Definition 2.3 [2].** A fuzzy relation $f$ on $X \times Y$ is called a fuzzy mapping with respect to (for short w.r.t.) $E_X \in E(X)$ and $E_Y \in E(Y)$, denoted by $f : X \to Y$ , it satisfies the following conditions:

($f$.1 ) $\forall x \in X, \exists y \in Y$ such that $f(x, y) > 0$,

($f$.2 ) $\forall x, \forall y \in X, \forall z, \forall w \in Y, f(x, z) \wedge f(y, w) \wedge E_X(x, y) \leq E_Y(z, w)$.

In particular, if $f(x, y) = 1$, then we will write $y = f(x)$.

**Definition 2.4 [2].** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f$ is said to be :

(1) *strong* if $\forall x \in X$, $\exists y \in Y$ such that $f(x, y) = 1$,

(2) *surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x, y) > 0$,

(3) *strong surjective* if $\forall y \in Y$, $\exists x \in X$ such that $f(x, y) = 1$,

(4) *injective* if $f(x, z) \wedge f(y, w) \wedge E_Y(z, w) \leq E_X(x, y), \forall x, y \in X, \forall z, w \in Y$,

(5) *bijective* if it is surjective and injective,

(6) *strong bijective* if it is strong surjective and injective.

**Definition 2.5 [2].** The *fuzzy identity mapping* $I_X : X \to X$ on $X$ is the fuzzy relation on $X \times X$ defined by

$$I_X(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \ \forall x, \forall y \in X \end{cases}$$

It is clear that $I_X : X \to X$ is a strong fuzzy mapping w.r.t. a fuzzy equality $E_X$ on $X$. Moreover, it is strong bijective w.r.t. a fuzzy equality $E_X$ on $X$.

**Result 2.A [2, Proposition 2.1].** Let$f : X \to Y$ and $g : Y \to Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$, $E_Y \in E(Y)$ and $E_Z \in E(Z)$. Then $g \circ f : X \to Z$ is a fuzzy mapping w.r.t. $E_X$ and $E_Z$.

**Result 2.B [2, Proposition 2.2].** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f^{-1} : Y \to Y$ is a fuzzy mapping w.r.t. $E_Y$ and $E_X$ if and only if $f$ is bijective.

**Result 2.C [2, Proposition 2.3].** Let $f : X \to Y$ be strong and injective w.r.t. $E_X = I_X \in E(X)$ and $E_Y \in E(Y)$. Then $f^{-1} \circ f = I_X$.

**Result 2.D [2, Proposition 2.4].** Let $f : X \to Y$ and $g : Y \to Z$ be bijective w.r.t. $E_X \in E(X), E_Y \in E(Y)$ and $E_Z \in E(Z)$. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ and $(g \circ f)^{-1} : Z \to X$ is a fuzzy mapping w.r.t. $E_Z$ and $E_X$.

**Definition 2.6 [2].** Let $f : X \to Y$ be a fuzzy mapping, let $A \in I^X$ and let $B \in I^Y$. Then :

(1) The *image of $A$ under $f$*, denoted by $f(A)$, is a fuzzy set in $Y$ defined by

$$f(A)(y) = \bigvee_{x \in X}[A(x) \wedge f(x, y)], \ \forall y \in Y.$$

(2) The *preimage of $B$ under $f$*, denoted by $F^{-1}(B)$ is a fuzzy set in $X$ defined by

$$f^{-1}(B)(x) = \bigvee_{y \in Y}[B(y) \wedge f(x, y)], \ \forall x \in X.$$

The following are the immediate results of Result 2.A and Definition 2.6.

**Proposition 2.7.** Let $f : X \to Y$ and $g : Y \to Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$. $E_Y \in E(Y)$ and $E_Z \in E(Z)$, and let $A \in I^X$. Then $(g \circ f)(A) = g(f(A))$.

**Proposition 2.8.** (1) ${I_X}^{-1}(A) = A, \ \forall A \in I^X$.

(2) Let $f : X \to Y$ and $g : Y \to Z$ be fuzzy mappings w.r.t. $E_X \in E(X), E_Y \in E(Y)$ and $E_Z \in E(Z)$. Then $(g \circ f)^{-1}(A) = (f^{-1} \circ g^{-1})(A) = f^{-1}(g^{-1}(A)), \forall A \in I^Z$.

**Result 2.E [2, Proposition 2.5].** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$, let $A \in I^X$ and let $B \in I^Y$.

(1) If $f$ is strong, then $A \subset f^{-1}(f(A))$.

(2) If $E_X = I_X$ and $f$ is injective, then $f^{-1}(f(A)) \subset A$.

(3) If $f$ is strong surjective, then $B \subset f(f^{-1}(B))$.

(4) If $E_Y = I_Y$, then $f(f^{-1}(B)) \subset B$.

**Result 2.F [2, Proposition 2.6].** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$, let $A \in I^X$ and let $B \in I^Y$.

(1) If $E_X = I_X$ and $f$ is injective, then $f(A^c) \subset [f(A)]^c$.

(2) If $f$ is strong surjective, then $[f(A)]^c \subset f(A^c)$.

(3) If $f$ is strong, then $[f^{-1}(B)]^c \subset f^{-1}(B^c)$.

(4) If $E_Y = Y_Y$, then $f^{-1}(B^c) \subset [f^{-1}(B)]^c$.

**Result 2.G [2, Proposition 2.7].** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$, let $\{A_\alpha : \alpha \in \Gamma\} \subset I^X$ and let $\{B_\alpha : \alpha \in \Gamma\} \subset I^Y$.

(1) $f(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} f(A_\alpha)$.

(2) $f^{-1}(\bigcup_{\alpha \in \Gamma} B_\alpha) = \bigcup_{\alpha \in \Gamma} f^{-1}(B_\alpha)$.

(3) $f(\bigcap_{\alpha \in \Gamma} A_\alpha) \subset \bigcap_{\alpha \in \Gamma} f(A_\alpha)$.

(4) $f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha) \subset \bigcap_{\alpha \in \Gamma} f^{-1}(B_\alpha)$.

(5) If $A_\alpha \subset A_\beta$ for $\alpha, \beta \in \Gamma$, then $f(A_\alpha) \subset f(A_\beta)$.

(6) If $B_\alpha \subset B_\beta$ for $\alpha, \beta \in \Gamma$, then $f^{-1}(B_\alpha) \subset f^{-1}(B_\beta)$.

(7) If $E_X = I_X$ and $f$ is injective, then $\bigcap_{\alpha \in \Gamma} f(A_\alpha) \subset f(\bigcap_{\alpha \in \Gamma} A_\alpha)$.

(8) If $E_Y = I_Y$, then $\bigcap_{\alpha \in \Gamma} f^{-1}(\beta_\alpha) \subset f^{-1}(\bigcap_{\alpha \in \Gamma} B_\alpha)$.

## 3. Vague continuous, open and closed mappings

*The fuzzy point in $X$ with support $x \in X$ and value* $\lambda \in I_0$ [*resp.* $\lambda \in I_{0,1}$ ], denoted by $x_\lambda$, is a fuzzy set in $X$ defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{if } y \neq x, \quad \forall y \in X. \end{cases}$$

We will denote the set of all the fuzzy points in $X$ as $F_p(X)$. Let $x_\lambda \in F_p(X)$ and let $A \in I^X$. Then $x_\lambda$ is said to *belong to* $A$, denoted by $x_\lambda \in_1 A$ [5] [resp. $x_\lambda \in_2 A$ [3] if $\lambda < A(x)$ [resp. $\lambda \leq A(x)$]. In this paper $x_\lambda \in A$ will stand for either $x_\lambda \in_1 A$ or $x_\lambda \in_2 A$.

**Definition 3.1 [3].** Let $x_\lambda \in F_p(X)$ and let $A, B \in I^X$. Then :

(1) $x_\lambda$ is said to be *quasi-coincident with* $A$, denoted by $x_\lambda q A$, if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$.

(2) $A$ is said to be *quasi-coincident with* $B$, denoted by $AqB$, if $\exists x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$. In this case, we say that $A$ *and* $B$ *are quasi-coincident(with each other) at* $x$.

(3) $A$ and $B$ are said to be *intersecting* if $\exists x \in X$ such that $(A \cap B)(x) \neq 0$. In this case, we say that $A$ *and* $B$ *intersect at* $x$.

It is clear that if $A$ and $B$ are quasi-coincident at $x$, then $A$ and $B$ intersect at $x$.

**Result 3.A [3, Proposition 2.1].** Let $A, B \in I^X$ and let $x_\lambda \in F_p(x)$. Then $A \subset B$ if and only if $A\bar{q}B^c$, i.e., $A$ and $B^c$ are not quasi-coincident. In particular, $x_\lambda \in_2 A$ if and only if $x_\lambda \bar{q} A^c$.

**Result 3.B [3, Proposition 2.3].** Let $\{A_\alpha : \alpha \in \Gamma\} \subset I^X$ and let $x_\lambda \in F_p(X)$. Then $x_\lambda q(\bigcup_{\alpha\in\Gamma} A_\alpha)$ if and only if $\exists \alpha \in \Gamma$ such that $x_\lambda q A_\alpha$.

**Proposition 3.2.** Let $f : X \to Y$ be a strong fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$, let $A \in I^X$ and let $B \in I^Y$.

(1) For each $x_\lambda \in F_p(X)$, $f(x_\lambda) \in F_p(Y)$ and $f(x_\lambda) = (f(x))_\lambda$.

(2) If $f(x_\lambda)qB$, $\forall x_\lambda \in F_p(X)$, then $x_\lambda q f^{-1}(B)$.

(3) If $x_\lambda q A$, $\forall x_\lambda \in F_p(X)$, then $f(x_\lambda) q f(A)$.

**Proof.** (1) By Definition 2.6, for each $y \in Y$,

$$\begin{aligned} f(x_\lambda)(y) &= \textstyle\bigvee_{z\in X}[x_\lambda(z) \wedge f(z, y)] \\ &= \lambda \wedge f(x, y). \end{aligned}$$

Since $f$ is strong, $\exists y_0 \in Y$ such that $f(x, y_0) = 1$. Thus $f(x_\lambda)(y_0) = \lambda$ and $y_0 = f(x)$. Hence $y_{0\lambda} = (f(x))_\lambda = f(x_\lambda)$.

(2) Suppose $f(x_\lambda)qB$. Since $f(x_\lambda) = y_{0\lambda}$ for some $y_0 \in Y$ by (1), $\lambda_0 + B(y_0) > 1$ and $f(x, y_0) = 1$. Thus

$$\begin{aligned} \lambda + f^{-1}(B)(x) &= \lambda + \textstyle\bigvee_{u\in Y}[B(y) \wedge f(x, y)] \\ &= \lambda + B(y_0) > 1. \end{aligned}$$

Hence $x_\lambda q f^{-1}(B)$.

(3) Suppose $x_\lambda q A$. Then $\lambda + A(x) > 1$. By (1), $\exists y_0 \in Y$ such that $f(x_\lambda) = y_{0\lambda}$ and $f(x, y_0) = 1$. Thus

$$\begin{aligned} \lambda + f(A)(y_0) &= \lambda + \textstyle\bigvee_{z\in X}[A(z) \wedge f(z, y_0)] \\ &= \lambda + A(x) \wedge f(x, y_0) \\ &= \lambda + A(x) > 1. \end{aligned}$$

Hence $f(x_\lambda) q f(A)$. □

In 1968, Chang [1] defines a *fuzzy topology* on $X$ as a subset $\mathcal{T} \subset I^X$ such that

(i) $\emptyset, X \in \mathcal{T}$,
(ii) $\forall A, B \in \mathcal{T}, A \cap B \in \mathcal{T}$,
(iii) $\forall \{A_\alpha : \alpha \in \Gamma\} \subset \mathcal{T}, \bigcup_{\alpha\in\Gamma} A_\alpha \in \mathcal{T}$.

Each member of $\mathcal{T}$ is called a *fuzzy open set* in $X$. A fuzzy set $A \in I^X$ is called *closed* in $X$ if $A^c$ is open in $X$, i.e., $A^c \in \mathcal{T}$. The pair $(X, \mathcal{T})$ is called a *fuzzy topological space*(in short, *fts*).

For a fts $X$, we will denote the set of all fuzzy open sets [resp. closed sets] in $X$ as $FO(X)$[resp. $FC(X)$].

**Definition 3.3 [3].** Let $(X, \mathcal{T})$ be a fts, let $A \in I^X$ and let $x_\lambda \in F_p(X)$.

(1) $A$ is called a *neighborhood*(for short, *nbd*) of $x_\lambda$ if $\exists B \in \mathcal{T}$ such that $x_\lambda \in B \subset A$.

(2) $A$ is called a *Q-neighborhood*(for short, *Q-nbd)* of $x_\lambda$ if $\exists B \in \mathcal{T}$ such that $x_\lambda q B \subset A$.

The set of all the nbds[resp. Q-nbds] of $x_\lambda$ is called *the system of nbds*[resp. *Q-nbds* ] of $x_\lambda$ and will be denoted by $\mathcal{N}_Q(x_\lambda)$[resp. $\mathcal{N}_Q(x_\lambda)$].

**Result 3.C [3, Proposition 2.4].** Let$(X, \mathcal{T})$ be a fts. Then $\mathfrak{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ if and only if $\forall x_\lambda \in F_p(X)$, $\forall$ open $Q$-nbd $U$ of $x_\lambda$, $\exists B \in \mathfrak{B}$ such that $x_\lambda q B \subset U$.

**Result 3.D [6, Theorem 3.2].** Let $(X, \mathcal{T})$ be a fts and let $A \in I^X$. Then $A \in \mathcal{T}$ if and only if $\forall x_\lambda \in F_p(X)$ with $x_\lambda q A$, $A \in \mathcal{N}_Q(x_\lambda)$.

**Definition 3.4.** Let $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f$ is said to be :

(1) *vague continuous* if $f^{-1}(B) \in \mathcal{T}$, $\forall B \in \mathcal{U}$,

(2) *vague open* if $f(A) \in \mathcal{U}$, $\forall A \in \Im$,

(3) *vague closed* if $f(A^c) \in \mathcal{U}$, $\forall A^c \in \mathcal{T}$,

(4) a *vague homeomorphism* if it is bijective, continuous and open.

The following is the immediate result of Definition 3.4.

**Proposition 3.5.** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$.

(1) If $X$ is a fuzzy discrete space, then $f$ is vague continuous.

(2) If $Y$ is a fuzzy indiscrete space, then $f$ is vague continuous.

(3) If $X$ and $Y$ are fuzzy discrete spaces, then $f$ is vague continuous and open.

The following is the immediate result of Definition 3.4 and Propositions 2.7 and 2.8.

**Proposition 3.6.** (1) The identity fuzzy mapping $I_X : X \to X$ is vague continuous.

(2) Let $f : X \to Y$ and $g : Y \to Z$ be fuzzy mappings w.r.t. $E_X \in E(X)$, $E_Y \in E(Y)$ and $E_Z \in E(Z)$. If $f$ and $g$ are vague continuous [resp. open and closed], then $g \circ f$ is vague continuous [resp. open and closed].

The following is the immediate result of Definition 3.4 and Result 2. F.

**Proposition 3.7.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then $f$ is vague continuous if and only if $f^{-1}(A) \in FC(X)$, $\forall A \in FC(Y)$.

**Theorem 3.8.** Let $f : (X,\mathcal{T}) \to (Y,\mathcal{U})$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then $f$ is continuous if and only if $\forall x_\lambda \in F_p(X)$, $\forall B \in \mathcal{N}(f(x_\lambda))$, $\exists A \in \mathcal{N}(x_\lambda)$ such that $x_\lambda \in A$ and $f(A) \subset B$.

**Proof.** $(\Rightarrow)$: Suppose $f$ is vague continuous. Let $x_\lambda \in F_p(X)$ and let $B$ be $B \in \mathcal{N}(f(x_\lambda))$ . Then $\exists C \in \mathcal{U}$ such that $f(x_\lambda) \in C \subset B$. Since $f$ is vague continuous and strong, $f^{-1}(C) \in \Im$ and

$x_\lambda \in f^{-1}(f(x_\lambda)) \subset f^{-1}(C) \subset f^{-1}(B)$.( by Result 2.E(1) and 2.G(6))

Let $A = f^{-1}(C)$. Then clearly $A \in \mathcal{N}(x_\lambda)$ and $x_\lambda \in A \subset f^{-1}(B)$. Since $E_Y = I_Y$, by Results 2. E(4) and 2. G(5), $f(A) \subset f(f^{-1}(B)) \subset B$.

$(\Leftarrow)$: Suppose the necessary condition holds and let $B \in \mathcal{U}$. If $f^{-1}(B) = \emptyset$, then it is obvious. Suppose $x_\lambda \in f^{-1}(B)$. Then it is easily seen that $B \in \mathcal{N}(f(x_\lambda))$. By the hypothesis,

$\exists A_{x_\lambda} \in \mathcal{N}(x_\lambda)$ such that $x_\lambda \in A_{x_\lambda}$ and $f(A_{x_\lambda}) \subset B$.

Since $A_{x_\lambda} \in \mathcal{N}(x_\lambda)$ and $f$ is strong, $\exists C_{x_\lambda} \in \mathcal{T}$ such that $x_\lambda \in C_{x_\lambda} \subset A_{x_\lambda} \subset f^{-1}(f(A_{x_\lambda})) \subset f^{-1}(B)$.

Thus

$$\begin{aligned} f^{-1}(B) &= \bigcup\{x_\lambda : x_\lambda \in f^{-1}(B)\} \\ &\subset \bigcup\{C_{x_\lambda} \in \mathcal{T}\} \\ &\subset f^{-1}(B). \end{aligned}$$

So $f^{-1}(B) = \bigcup\{C_{x_\lambda} \in \mathcal{T} : x_\lambda \in f^{-1}(B)\} \in \mathcal{T}$. Hence $f$ is vague continuous. □

**Definition 3.9 [3].** Let $(X,\mathcal{T})$ be a fts.

(1) A subfamily $\mathcal{B}$ of $\mathcal{T}$ is called a *base* for $\mathcal{T}$ if $\forall A \in \mathcal{T}$, $\exists$ $\mathcal{B}'_A \subset \mathcal{B}$ such that $A = \bigcup \mathcal{B}_A$.

(2) A subfamily $\mathcal{S}$ of $\mathcal{T}$ is called a *subbase* for $\mathcal{T}$ if the family $\mathcal{B} = \{\bigcap \mathcal{F}$ is a finite subset of $\mathcal{S}\}$ is a base for $\mathcal{T}$.

(3) $(X,\mathcal{T})$ is said to *satisfies the second axiom of countability* or is said to be *a $C_{II}$ space* if $\mathcal{T}$ has a countable base.

The following is the immediate result of Definition 3.9 and Result 2.F.

**Theorem 3.10.** Let $f : (X,\mathcal{T}) \to (Y,\mathcal{T})$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then $f$ is vague continuous if and only if $f^{-1}(V) \in \mathcal{T}$ for each member $V$ of a subbase $\mathcal{S}$ for $\mathcal{U}$.

**Definition 3.11 [3].** Let $D$ be a nonempty set and let $\leq$ be a semi-order on $D$.

(1) The pair $(D,\leq)$ is called a *directed set, directed by* $\leq$, if $\forall m,n \in D$, $\exists p \in D$ such that $m \leq p$ and $n \leq p$.

(2) $S$ is called a *fuzzy net in* $X$ if $S : D \to F_p(X)$ is an(ordinary) mappings, where $D$ is a directed set by $\leq$. In this case, for $n \in D$, $S(n)$ is often denoted by $S_n$ and hence a net $S$ is usually denoted by $\{S_n : n \in D\}$.

**Definition 3.12 [3].** Let $S = \{S_n : n \in D\}$ be a fuzzy net in $X$ and let $A \in I^X$. Then $S$ is said to be :

(1) *quasi-coincident with* $A$ if $\forall n \in D$, $S_n q A$.

(2) *eventually quasi-coincident with* $A$ if $\exists m \in D$ such that $\forall n \in D$ with $n \geq m$, $S_n q A$.

(3) *frequently quasi-coincident with* $A$ if $\forall m \in D$, $\exists n \in D$ such that $n \geq m$ and $S_n q A$.

(4) *in* $A$ if $\forall n \in D$, $S_n \in A$.

**Definition 3.13 [3].** Let $S$ be a fuzzy net in a fts $(X,\mathcal{T})$ and let $x_\lambda \in F_p(X)$. Then $S$ is said to converge to $x_\lambda$ in $X$ relative to $\mathcal{T}$ if $S$ is eventually quasi-coincident with each $Q$-nbd of $x_\lambda$.

**Proposition 3.14.** Let $f : X \to Y$ be strong and strong surjective w.r.t. $E_X \in E(X)$ and $I_Y \in E(Y)$. Consider the following statements :

(1) $f$ is vague continuous.

(2) For each $x_\lambda \in F_p(X)$ and each $V \in \mathcal{N}_Q(f(x_\lambda)), \exists U \in \mathcal{N}_Q(x_\lambda)$ such that $f(U) \subset V$.

(3) For each fuzzy net $S$ in $X$, if $S$ converges to $x_\lambda \in F_p(X)$, then $T = \{f(S_n) : n \in D\}$ is a fuzzy net in $Y$ and converges to $f(x_\lambda)$. Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

**Proof.** (1) $\Rightarrow$ (2): Suppose $f$ is continuous. Let $x_\lambda \in F_p(X)$ and let $V \in \mathcal{N}_Q(f(x_\lambda))$. Since $f$ is strong, by Proposition 3.2(1), $\exists y \in Y$ such that $f(x,y) = 1$ and $f(x_\lambda) = y_\lambda \in F_p(Y)$. Since $V \in \mathcal{N}_Q(f(x_\lambda))$, by Definition 3.3, $\exists B \in FO(Y)$ such that $f(x_\lambda) q B \subset V$. By proposition 3.2(2) and Result

2.G, $x_\lambda q f^{-1}(B) \subset f^{-1}(V)$. Since $f$ is vague continuous, $f^{1-}(B) \in FO(X)$. Let $f^{-1}(V) = U$. Since $x_\lambda q f^{-1}(B) \subset f^{-1}(V)$,

$$\begin{aligned} \lambda + U(x) &= \lambda + f^{-1}(V)(x) \\ &= \lambda + \textstyle\bigvee_{z\in Y}[V(z) \wedge f(x,z)] \\ &= \lambda + V(y)\,[\text{Since} f(x,y) = 1] \\ &> 1. \end{aligned}$$

Thus $U \in \mathcal{N}_Q(x_\lambda)$. Since $E_Y = I_Y$, by Result 2.E(4), $f(U) = f(f^{-1}(V)) \subset V$. This completes the proof.

(2) ⇒ (3) : Suppose the condition (2) holds and for each fuzzy net $S$ in $X$, suppose $S$ converges to $x_\lambda \in F_p(X)$. Then, by Proposition 3.2(1), $f(S_n) \in F_p(Y)$. Thus $T$ is a fuzzy net in $Y$. Let $V \in \mathcal{N}_Q(f(x_\lambda))$. Then, by the condition (2),

$$\exists U \in \mathcal{N}_Q(x_\lambda) \text{ such that} f(U) \subset V.$$

Thus, by the hypothesis and Definition 3.13,

$$\exists m \in D \text{ `such that} \forall n \in D \,\text{with}\, n \geq m,\ S_n q U.$$

By Proposition 3.2(3), $f(S_n) q f(U) \subset V$. So $f(S_n) q V$. Hence $T$ is converges to $f(\lambda)$. □

**Definition 3.15 [3].** Let $(X, \mathcal{T})$ be a fts and let $A \in I^X$. Then :

(1) The union of all the $\mathcal{T}$-open sets contained in $A$ is called the *interior* of $A$, denoted by Å or $Int_{\mathcal{T}} A$.

(2) The intersection of all the $\mathcal{T}$-closed sets containing $A$ is called the *closure* of $A$, denoted by $\bar{A}$ or $cl_{\mathcal{T}} A$.

It is clear that $A^\circ$[resp. $\bar{A}$] is the largest open set contained in $A$ [resp. the smallest closed set containing $A$] and $(A^\circ)^\circ = A^\circ$ [resp. $(\bar{\bar{A}}) = \bar{A}$ ].

**Result 3.E [3, Propositions 4.1 and 4.1' ].**

(1) $x_\lambda \in A^\circ$ if and only if $x_\lambda$ has a nbd contained in $A$.

(2) $x_\lambda \in \bar{A}$ if and only if $VqA$, $\forall V \in \mathcal{N}_Q(x_\lambda)$.

**Result 3.G [3, Theorem 4.2].** $A^\circ = (\bar{A^c})^c$, $\bar{A} = ((A^c)^\circ)^c$, $(\bar{A})^c = (A^c)^\circ$, $\bar{A^c} = (A^\circ)^c$.

**Result 3.F [6, Theorem 3.1].** Let $X$ be a fts, let $x_\lambda \in F_p(X)$ and let $A \in I^X$. If $AqM, M \in \mathcal{N}_Q(x_\lambda)$ then $x_\lambda \in A$.

**Theorem 3.16.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$, let $A \in I^X$ and let $B \in I^Y$. Then the following are equivalent :

(1) $f$ is vague continuous.

(2) $f(\bar{A}) \subset \overline{f(A)}$.

(3) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

(4) $f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$.

**Proof.**(1) ⇒ (2) : Suppose $f$ is vague continuous. Since $\overline{f(A)}$ is closed in $Y$, by Proposition 3.7, $f^{-1}(\overline{f(A)})$ is closed in $X$. Then, by Result 2.E(1),

$$\bar{A} \subset \overline{f^{-1}(f(A))} \subset \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)}).$$

Since $E_Y = I_Y$, by Result 2.E(4),

$$f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}.$$

(2) ⇒ (3) : Suppose the condition (2) holds. Since $f$ is strong surjective and $E_Y = I^Y$, by Result 2.E, and the condition (2),

$$f(\overline{f}^{-1}(B)) \subset \overline{f}(f^{-1}(B)) = \bar{B}.$$

Since $f$ is strong, by Result 2.E (1),

$$\overline{f}^{-1}(B) \subset f^{-1}(f(\overline{f}^{-1}(B))) \subset f^{-1}(\bar{B}).$$

(3) ⇒ (4) : Suppose the condition (3) holds. Then, by the condition (3), $\overline{f^{-1}(B^c)} \subset f^{-1}(\overline{B^c})$. By Results 3.G and 2.F,

$$\overline{f^{-1}(B^c)} = \overline{[f^{-1}(B)]^c} = [(f^{-1}(B))^\circ]^c$$

and

$$f^{-1}(\overline{B^c}) = f^{-1}((B^\circ)^c) = [f^{-1}(B^\circ)]^c.$$

Thus $[(f^{-1}(B))^\circ]^c \subset [f^{-1}(B^\circ)]^c$. So $f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$.

(4) ⇒ (1) : Suppose the condition (4) holds. Let $U \in FO(Y)$. Then, by the condition (4), $f^{-1}(U^\circ) \subset (f^{-1}(U))^\circ$. Since $U \in FO(Y)$, $U = U^\circ$. Thus $f^{-1}(U) \subset (f^{-1}(U))^\circ$. Since $(f^{-1}(U))^\circ$ is the largest open fuzzy set in $X$ contained in $f^{-1}(U)$, $(f^{-1}(U))^\circ \subset f^{-1}(U)$. So $f^{-1}(U) = (f^{-1}(U))^\circ$, i.e., $f^{-1}(U) \in \mathcal{T}$. Hence $f$ is vague continuous. This completes the proof. □

The conditions in Theorem 3.16 are not equivalent to the condition $(f(A))^\circ \subset f(A^\circ)$ for each $A \in I^X$, in general.

**Example 3.16.** (1) Let $X = \{ㄱ, ㄴ\}$ and let $A, B \in I^X$ be defined as

$A(ㄱ) = 0.3$, $A(ㄴ) = 0.3$, $B(ㄱ) = 0.3$, $B(ㄴ) = 0$.

Let $\mathcal{T}_1 = \{\emptyset, X, A\}$ and let $\mathcal{T}_2 = \{\emptyset, X, B\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Let $E_1 : X \times X \to I$ and $E_2 : X \times X \to I$ be mappings defined by

| $E_1$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0.4 |
| ㄴ | 0.4 | 1 , |

| $E_2$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0 |
| ㄴ | 0 | 1 . |

Then clearly $E_1, E_2 \in E(X)$. Consider the fuzzy relation $f$ on $X \times X$ defined as follows :

| $f$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0 |
| ㄴ | 1 | 0 . |

Then $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is strong w.r.t. $E_1 \in E(X)$ and $E_2 = I_X \in E(X)$. Thus $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(X) = X$

and $f^{-1}(B) = A$ are open in $(X, \mathcal{T}_1)$. So $f$ is vague continuous. On the other hand, $(f(B))^\circ = B^\circ = B$ in $(X, \mathcal{T}_2)$ and $f(B^\circ) = f(\emptyset) = \emptyset$ in $(X, \mathcal{T}_1)$. Thus $(f(B))^\circ \not\subset f(B^\circ)$.

(2) Let $X = \{$ㄱ, ㄴ$\}$ and let $A \in I^X$ be defined by $A($ㄱ$) = 0.4$ and $A($ㄴ$) = 1$. Let $\mathcal{T}_1 = \{\emptyset, X\}$ and let $\mathcal{T}_2 = \{\emptyset, X, A\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Let $E_1$ and $E_2$ be fuzzy equalities defined in (1). Also let $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be fuzzy mapping defined in (1). Then $f(B)($ㄴ$) = 0$. Thus $(f(B))^\circ = \emptyset \subset f(B^\circ)$. But $f^{-1}(A)$ is not open in $(X, \mathcal{T}_1)$. So $f$ is not vague continuous. □

However, if a fuzzy mapping $f$ has some conditions, then we have the following result.

**Theorem 3.17.** Let $f : X \to Y$ be strong and strong bijective w.r.t. $E_X = I_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then the following are equivalent :

(1) $f$ is vague continuous.

(2) $f(\bar{A}) \subset \overline{f(A)}$, $\forall A \in I^X$.

(3) $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$, $\forall B \in I^Y$.

(4) $f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$, $\forall B \in I^Y$.

(5) $(f(A)^\circ \subset f(A^\circ)$, $\forall A \in I^X$.

**Proof.** By Theorem 3.16, it suffices to show that (4) $\Leftrightarrow$ (5).

(4) $\Rightarrow$ (5) : Suppose the condition (4) holds and let $A \in I^X$. Then $f(A) \in I^Y$. Thus, by the condition (4),

$$f^{-1}((f(A))^\circ) \subset (f^{-1}(f(A)))^\circ.$$

Since $f$ is strong, injective and $E_X = I_X$, by Result 2.E,

$$f^{-1}(f(A))^\circ) \subset (f^{-1}(f(A)))^\circ = A^\circ.$$

Thus

$$f(f^{-1}(f(A))^\circ) \subset f(A^\circ).$$

Since $f$ is strong surjective and $E_Y = I_Y$, by Result 2.E,

$$(f(A))^\circ = f(f^{-1}(f(A))^\circ) \subset f(A^\circ).$$

(5) $\Rightarrow$ (4) : Suppose the condition (5) holds and let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. Then, by the condition (5),

$$(f(f^{-1}(B)))^\circ \subset f(f^{-1}(B))^\circ.$$

Since $f$ is strong surjective and $E_Y = I_Y$, by Result 2.E,

$$B^\circ = (f(f^{-1}(B)))^\circ \subset f(f^{-1}(B))^\circ.$$

Since $f$ is strong, injective and $E_X = I_X$, by Result 2.E,

$$f^{-1}(B^\circ) \subset f^{-1}(f(f^{-1}(B))^\circ) = (f^{-1}(B))^\circ.$$

This completes the proof. □

**Proposition 3.18.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. If $f$ is vague open, then $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$, $\forall B \in I^Y$.

**Proof.** Suppose $f$ is vague open. Let $B \in I^Y$ and let $x_\lambda \in f^{-1}(\bar{B})$. Since $f$ is strong, $\exists y \in Y$ such that $f(x, y) = 1$ and $y_\lambda = f(x_\lambda)$. Let $V \in \mathcal{N}_Q(x_\lambda)$. Then $\exists U \in FO(X)$ such that $x_\lambda q U \subset V$. By Proposition 3.2 (3), $y_\lambda = f(x_\lambda) q f(U) \subset f(V)$. Since $f$ is vague open, $f(U) \in FO(Y)$. Thus $f(V) \in \mathcal{N}_Q(f(x_\lambda))$. Since $x_\lambda \in f^{-1}(\bar{B})$ and $E_Y = I_Y$, by Result 2.E(4),

$$y_\lambda = f(x_\lambda) \in f(f^{-1}(\bar{B})) \subset \bar{B}.$$

By Result 3.E(2), $f(V) q B$. Thus $\exists y_0 \in Y$ such that $f(V)(y_0) + B(y_0) > 1$. Let $\varepsilon > 0$ such that $f(V)(y_0) + B(y_0) - \varepsilon > 1$. Since

$$f(V)(y_0) = \bigvee_{x \in X}[V(x) \wedge f(x, y_0)],$$

$\exists x_0 \in X$ such that $f(V)(y_0) - \varepsilon < V(x_0) \wedge f(x_0, y_0) < V(x_0)$. For this $x_0 \in X$,

$$f^{-1}(B)(x_0) = \bigvee_{z \in Y}[B(z) \wedge f(x_0, z)] \geq B(y_0) \wedge f(x_0, y_0).$$

Then

$$V(x_0) + f^{-1}(B)(x_0) > f(V)(y_0) - \varepsilon + B(y_0) > 1.$$

Thus $V q f^{-1}(B)$. So, by Result 3.E(2), $x_\lambda \in \overline{f^{-1}(B)}$. Hence $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$ . □

The following is the immediate result of Theorem 3.16 and Proposition 3.18.

**Corollary 3.18.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. If $f$ is vague open and continuous then $f^{-1}(\bar{B}) = \overline{f^{-1}(B)}, \forall B \in I^Y$.

**Theorem 3.19.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then the following are equivalent :

(1) $f$ is vague open.

(2) $f(A^\circ) \subset (f(A))^\circ$, $\forall A \in I^X$.

(3) $(f^{-1}(B))^\circ \subset f^{-1}(B^\circ)$, $\forall B \in I^Y$.

**Proof.** (1) $\Rightarrow$ (2) : Suppose $f$ is vague open and let $A \in I^X$. Then clearly $A^\circ \in FO(X)$. Since $f$ is vague open, $f(A^\circ) \in FO(Y)$. Thus $f(A^\circ) = (f(A^\circ))^\circ \subset (f(A))^\circ$.

(2) $\Rightarrow$ (3) : Suppose the condition (2) holds and let $B \in I^Y$. Then $f^{-1}(B) \in I^X$. Thus, by condition (2),

$$f(f^{-1}(B))^\circ) \subset (f(f^{-1}(B)))^\circ.$$

Since $E_Y = I_Y$, by Result 2.E(4), $f(f^{-1}(B)) \subset B$. So $f(f^{-1}(B))^\circ) \subset B^\circ$. Since $f$ is strong, by Result 2.E(1),

$$(f^{-1}(B))^\circ \subset f^{-1}(f(f^{-1}(B))^\circ) \subset f^{-1}(B^\circ).$$

(3) $\Rightarrow$ (1) : Suppose the condition (3) holda and let $U \in FO(X)$. Then $U^\circ = U$ and $f(U) \in I^Y$. Thus, by condition (3), $(f^{-1}(f(U)))^\circ \subset f^{-1}(f(U))^\circ$.

Since $f$ is strong, $U = U^\circ \subset (f^{-1}(f(U)))^\circ \subset f^{-1}(f(U))^\circ$.

Since $E_Y = I_Y$, $f(U) \subset f(f^{-1}(f(U))^\circ) \subset (f(U))^\circ \subset f(U)$.

Thus $f(U) = (f(U))^\circ$. So $f(U) \in FO(Y)$. Hence $f$ is vague open. □

**Theorem 3.20.** Let $f : X \to Y$ be a fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y \in E(Y)$. Then $f$ is vague closed if and only if $\overline{f(A)} \subset f(\bar{A})$, $\forall A \in I^X$.

**Proof.** $(\Rightarrow)$ : Suppose $f$ is vague closed and let $A \in I^X$. Then clearly $\bar{A} \in FC(X)$. Since $f$ is vague closed, $f(\bar{A}) \in FC(Y)$. Thus $\overline{f}(A) \subset \overline{f(\bar{A})} = f(\bar{A})$.

$(\Leftarrow)$ : Suppose the necessary condition holds and let $A \in FC(X)$. Then $\bar{A} = A$. By the hypothesis,

$$\overline{f(A)} \subset f(\bar{A}) = f(A) \subset \overline{f(A)}.$$

Thus $f(A) = \overline{f}(A)$. So $f(A) \in FC(X)$. Hence $f$ is vague closed. □

The following is the immediate result of Theorems 3.16 and 3.20. The conditions in Theorem 3.20 are not equivalent to the condition $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$, $\forall B \in I^Y$, in general.

**Corollary 3.20.** Let $f : X \to Y$ be strong w.r.t. $E_X \in E(X)$ and $I_Y \in E(Y)$. Then $f$ is vague continuous and closed iff $f(\bar{A}) = \overline{f(A)}$, $\forall A \in I^X$.

**Example 3.20.** (1) Let $X = \{\text{ㄱ}, \text{ㄴ}\}$ and let $A, B$ and $C$ be fuzzy sets in $X$ defined as follows :

$$A(\text{ㄱ}) = 0.3, \quad A(\text{ㄴ}) = 0.3,$$
$$B(\text{ㄱ}) = 0.3, \quad B(\text{ㄴ}) = 0,$$
$$C(\text{ㄱ}) = 1, \quad C(\text{ㄴ}) = 0.$$

Let $\mathcal{T}_1 = \{\emptyset, X, A^c\}$ and let $\mathcal{T}_2 = \{\emptyset, X, B^c, C^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Let $E_1 : X \times X \to I$ and $E_2 : X \times X \to I$ be mappings defined as follows :

| $E_1$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0.4 |
| ㄴ | 0.4 | 1 , |

| $E_2$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0.6 |
| ㄴ | 0.6 | 1 . |

Then clearly $E_1, E_2, \in E(X)$. Consider the fuzzy relation $f$ on $X \times X$ defined as follows :

| $f$ | ㄱ | ㄴ |
|---|---|---|
| ㄱ | 1 | 0 |
| ㄴ | 1 | 0 . |

Then it is easily seen that $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a fuzzy mapping w.r.t. $E_1, E_2 \in E(X)$. Then $f(\emptyset) = \emptyset$, $f(X) = C$ and $f(A) = B$ are closed in $(X, \mathcal{T}_2)$. Thus $f$ is vague closed. On the other hand, $f^{-1}(\bar{A}) = f^{-1}(X) = X$ in $(X, \mathcal{T}_2)$ and $\overline{f^{-1}(A)} = \bar{A}$ in $(X, \mathcal{T}_1)$. So $f^{-1}(\bar{A}) \not\subset \overline{f^{-1}(A)}$.

(2) Let $X = \{\text{ㄱ}, \text{ㄴ}\}$ and let $A \in I^X$ be defined by $A(\text{ㄱ}) = 0$, $A(\text{ㄴ}) = 1$.

Let $\mathcal{T}_1 = \{\emptyset, X\}$ and let $\mathcal{T}_2 = \{\emptyset, X, A^c\}$. Then clearly $\mathcal{T}_1$ and $\mathcal{T}_2$ are fuzzy topologies on $X$. Let $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be a fuzzy mapping w.r.t. $E_1$ and $E_2$ defined in (1), respectively. Let $B$ be any fuzzy set in $(X, \mathcal{T}_2)$.

Suppose $B(\text{ㄱ}) = 0$. Then

$$f^{-1}(\bar{B}) = f^{-1}(A) = \emptyset \subset \overline{f^{-1}(B)}.$$

Suppose $B(\text{ㄱ}) \neq 0$. Then $f^{-1}(B) \neq \emptyset$. Thus $f^{-1}(\bar{B}) \subset X = \overline{f^{-1}(B)}$.

So, in either case, $f^{-1}(B) \subset \overline{f^{-1}(B)}$, $\forall B \in I^X$. On the other hand, $f(X) = A^c$ is not closed in $(X, \mathcal{T}_2)$. Thus $f$ is not vague closed. □

However, if $f$ has a fuzzy mapping with some conditions, then obtain the following result.

**Theorem 3.21.** Let $f : X \to Y$ be strong and strong bijective w.r.t. $E_X = I_X \in E(X)$ and $E_Y = I_Y \in E(Y)$. Then the following are equivalent :

(1) $f$ is vague closed.
(2) $\overline{f(A)} \subset f(\bar{A})$, $\forall \in I^X$.
(3) $f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$, $\forall B \in I^Y$.

**Proof.** By Theorem 3.17, it is sufficient to show that (2) ⇔ (3).

(2) ⇒ (3) : Suppose the condition (2) holds and let $B \in I^Y$. Then clearly $f^{-1}(B) \in I^X$. Thus, by the condition (2),

$$\overline{f(f^{-1}(B))} \subset f(\overline{f^{-1}(B)}).$$

Since $f$ is strong surjective and $E_Y = I_Y$, by Result 2.E,

$$\bar{B} = \overline{f(f^{-1}(B))} \subset f(\overline{f^{-1}(B)}).$$

Since $f$ is strong, injective and $E_X = I_X$, by Result 2.E,

$$f^{-1}(\bar{B}) \subset f^{-1}(f(\overline{f^{-1}(B)})) = \overline{f^{-1}(B)}.$$

(3) ⇒ (2) : Suppose the condition (3) holds and let $A \in I^X$. Then clearly $f(A) \in I^Y$. Thus, by the condition (3),

$$f^{-1}(\overline{f(A)}) \subset \overline{f}^{-1}(f(A)).$$

Since $f$ is strong, injective and $E_X = I_X$, by Result 2.E,

$$f^{-1}(\overline{f(A)}) \subset \overline{f^{-1}(f(A))} = \bar{A}.$$

So $f(f^{-1}(\overline{f(A)})) \subset f(\bar{A})$. Since $f$ is strong surjective and $E_Y = I_Y$, by Result 2.E,

$$\overline{f(A)} = f(f^{-1}(\overline{f(A)})) \subset f(\bar{A}).$$

This completes the proof. □

The following is the immediate result of Theorems 3.17, 3.19 and 3.21.

**Theorem 3.22.** Let $f : X \to Y$ be strong and strong bijective w.r.t. $I_X \in E(X)$ and $I_Y \in E(Y)$. Then the following are equivalent :

(1) $f$ is a vague homeomorphism.

(2) $f$ is vague continuous and open.
(3) $\overline{f(\bar{A})} = \overline{f}(A)$, $\forall A \in I^X$.
(4) $\overline{f^{-1}(B)} = f^{-1}(\bar{B})$, $B \in I^Y$.
(5) $f^{-1}(B^\circ) = (f^{-1}(B))^\circ$, $\forall B \in I^Y$.
(6) $(f(A))^\circ = f(A^\circ)$, $\forall A \in I^X$.

**Proposition 3.23.** Let $f : X \to Y$ be strong and strong bijective w.r.t. $E_X = I_X \in E(X)$ and $E_Y \in E(Y)$. If $f$ is vague continuous, then $(f(A))^\circ \subset f(A^\circ), \forall A \in I^X$.
**Proof.** Let $A \in I^X$. Then clearly $(f(A))^\circ \in FO(Y)$. Since $f$ is vague continuous $f^{-1}((f(A))^\circ) \in FO(X)$. By Results 2.E and 2.G(6), $f^{-1}((f(A))^\circ) \subset f^{-1}(f(A)) = A$. Since $A^\circ$ is the largest open set in $X$ contained in $A$, $f^{-1}((f(A))^\circ) \subset A^\circ$. So, by Results 2.E and 2.G(5), $(f(A))^\circ \subset f(A^\circ)$.

The following is the immediate result of Proposition 3.23 and Theorem 3.19.

**Corollary 3.23.** Let $f : X \to Y$ be strong and strong bijective w.r.t. $I_X \in E(X)$ and $E_Y \in E(Y)$. If $f$ is vague continuous and open, then $f(A^\circ) = (f(A))^\circ$, $\forall A \in I^X$.

**Proposition 3.24.** Let $X$ be a fts, let $f : X \to Y$ be a strong fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$, and let $\mathcal{U} = \{U \in I^Y : f^{-1}(U) \in FO(X)\}$.
(1) $\mathcal{U}$ is a fuzzy topology on $Y$.
(2) $f$ is vague continuous.
(3) If $\mathcal{T}$ is a fuzzy topology on $Y$ such that $f : X \to (Y, \mathcal{T})$ is vague continuous, then $\mathcal{T} \subset \mathcal{U}$.

**Proof.** (1) Clearly $f^{-1}(\emptyset) = \emptyset \in FO(X)$ and $f^{-1}(Y) = X \in FO(X)$. Then $\emptyset, Y \in \mathcal{U}$. Let $U, V \in \mathcal{U}$. Then $f^{-1}(U), f^{-1}(V) \in FO(X)$. Since $E_Y = I_Y$, Result 2.G, $f^{-1}(u) \wedge f^{-1}(V) = f^{-1}(U \cap V) \in FO(X)$. Thus $U \cap V \in \mathcal{U}$. Now let $\{U_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{U}$. Then clearly $f^{-1}(U_\alpha) \in FO(X)$, $\forall \alpha \in \Gamma$ and $f^{-1}(\bigcup_{\alpha\in\Gamma} U_\alpha)$. Since $\bigcup_{\alpha\in\Gamma} f^{-1}(U_\alpha) \in FO(X)$, $f^{-1}(\bigcup_{\alpha\in\Gamma} U_\alpha) \in FO(X)$. So $\bigcup_{\alpha\in\Gamma} U_\alpha \in \mathcal{U}$. Hence $\mathcal{U}$ is a fuzzy topology on $Y$.
(2) It is obvious from the definition of $\mathcal{U}$.
(3) Let $U \in \mathcal{T}$. Since $f : X \to (Y, \mathcal{T})$ is vague continuous, $f^{-1}(U) \in FO(X)$. Thus, by the definition of $\mathcal{U}$, $U \in \mathcal{U}$. So $\mathcal{T} \subset \mathcal{U}$. □

**Proposition 3.25.** Let $Y$ be a fts, let $f : X \to Y$ be a strong fuzzy mapping w.r.t. $E_X \in E(X)$ and $E_Y = I_Y \in E(Y)$, and let $\mathcal{T} = \{U \in I^X : \exists V \in FO(Y) \text{such that} U = f^{-1}(V)\}$.
(1) $\mathcal{T}$ is a fuzzy topology on $X$.
(2) $f : (X, \mathcal{T}) \to Y$ is vague continuous.
(3) If $\mathcal{U}$ is a fuzzy topology on $X$ such that $f : (X, \mathcal{U}) \to Y$ is vague continuous, then $\mathcal{T} \subset \mathcal{U}$.

**Proof.** (1) Clearly $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$. Since $\emptyset, Y \in FO(Y)$, $\emptyset, X \in \mathcal{T}$. Let $U, V \in \mathcal{T}$. Then, $\exists A, B \in FO(Y)$ such that $f^{-1}(A) = U$ and $f^{-1}(B) = V$. Thus $A \cap B \in FO(Y)$ and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) = U \cap V$, by Result 2.G. So $U \cap V \in \mathcal{T}$. Now let $\{U_\alpha\}_{\alpha\in\Gamma} \subset \mathcal{T}$ Then $U_\alpha \in \mathcal{T}$, $\forall \alpha \in \Gamma$. Thus, $\forall \alpha \in \Gamma$, $\exists A_\alpha \in FO(Y)$ such that $U_\alpha = f^{-1}(A_\alpha)$. Thus $f^{-1}(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(A_\alpha) = \bigcup_{\alpha\in\Gamma} U_\alpha$ and $\bigcup_{\alpha\in\Gamma} A_\alpha \in FO(Y)$. So $\bigcup_{\alpha\in\Gamma} U_\alpha \in \mathcal{T}$. Hence $\mathcal{T}$ is a fuzzy topology on $X$.
(2) It is clear from the definition of $\mathcal{T}$.
(3) Let $U \in \mathcal{T}$. Then $\exists V \in FO(Y)$ such that $U = f^{-1}(V)$. Since $f : (X, \mathcal{U}) \to Y$ is vague continuous, $f^{-1}(V) \in \mathcal{U}$. Thus $U \in \mathcal{U}$. Hence $\mathcal{T} \subset \mathcal{U}$. □

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