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Antitone Galois Connections and Formal Concepts

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Abstract

In this paper, we investigate the properties of antitone Galois connection and formal concepts. Moreover, we show that order reverse generating maps induce formal, attribute oriented and object oriented concepts on a complete residuated lattice.

Key Words : Complete residuated lattices, order reverse generating maps, isotone (antitone) Galois connetion, formal (resp. attribute oriented, object oriented) concepts

1. Introduction and preliminaries

Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-4,8,10]. A fuzzy context consists of (X, Y, R) where X is a set of objects, Y is a set of attributes and R is a relation between X and Y. Bělohlávek [1-4] developed the notion of formal concepts with $R \in L^{X\times Y}$ on a complete residuated lattice L.

In this paper, we investigate the properties of antitone Galois connections. Using their properties, we define formal, attribute oriented and object oriented concepts on a complete residuated lattice. Moreover, we show that order reverse generating maps induce formal, attribute oriented and object oriented concepts on a complete residuated lattice.

Definition 1.1. [9] A triple $(L, \leq, \odot)$ is called a *complete residuated lattice* iff it satisfies the following conditions:

(L1) $L = (L, \leq, 1, 0)$ is a complete lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;

(L2) $(L, \odot, 1)$ is a commutative monoid;

(L3) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i) \odot b = \bigvee_{i\in\Gamma} (a_i \odot b).$$

Define an operation $\rightarrow$ as $a \rightarrow b = \bigvee\{c \in L \mid a \odot c \leq b\}$, for each $a, b \in L$.

Remark 1.2. [9] (1) Each frame $(L, \leq, \wedge)$ is a complete residuated lattice.

(2) The unit interval with a left-continuous t-norm t, $([0, 1], \leq, t)$, is a complete residuated lattice.

(3) Define a binary operation $\odot$ on $[0, 1]$ by $x \odot y = \max\{0, x + y - 1\}$. Then $([0, 1], \leq, \odot)$ is a complete residuated lattice.

Let $(L, \leq, \odot)$ be a complete residuated lattice. A order reversing map $^*: L \rightarrow L$ defined by $a^* = a \rightarrow 0$ is called a *strong negation* if $a^{**} = a$ for each $a \in L$.

In this paper, we assume $(L, \leq, \odot, ^*)$ is a complete residuated lattice with a strong negation *.

Lemma 1.3. [9] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $x \odot y \leq x \odot z$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(4) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(5) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(6) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(7) $x \odot y = (x \rightarrow y^*)^*$ and $x \rightarrow y = y^* \rightarrow x^*$.

2. Antitone Galois connections and formal concepts

Definition 2.1. [5] Let X and Y be two sets. Let $\omega^\rightarrow, \phi^\rightarrow, \xi^\rightarrow : L^X \rightarrow L^Y$ and $\omega^\leftarrow, \phi^\leftarrow, \xi^\leftarrow : L^Y \rightarrow L^X$ be operators.

(1) The pair $(\omega^\rightarrow, \omega^\leftarrow)$ is called *antitone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\rho \leq \omega^\rightarrow(\mu)$ iff $\mu \leq \omega^\leftarrow(\rho)$.

(2) The pair $(\phi^\rightarrow, \phi^\leftarrow)$ is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\phi^\rightarrow(\mu) \leq \rho$ iff $\mu \leq \phi^\leftarrow(\rho)$. Moreover, the pair $(\xi^\leftarrow, \xi^\rightarrow)$

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is called an *isotone Galois connection* between X and Y if for $\mu \in L^X$ and $\rho \in L^Y$, $\xi^{\leftarrow}(\rho) \leq \mu$ iff $\rho \leq \xi^{\rightarrow}(\mu)$.

Definition 2.2. Let $\omega^{\rightarrow}, \phi^{\rightarrow}, \xi^{\rightarrow} : L^X \to L^Y$ and $\omega^{\leftarrow}, \phi^{\leftarrow}, \xi^{\leftarrow} : L^Y \to L^X$ be functions. A pair $(\mu, \rho) \in L^X \times L^Y$ is called:

(1) *a formal concept* if $\rho = \omega^{\rightarrow}(\mu)$ and $\mu = \omega^{\leftarrow}(\rho)$ where $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galios connection,

(2) *an attribute oriented concept* if $\rho = \phi^{\rightarrow}(\mu)$ and $\mu = \phi^{\leftarrow}(\rho)$ where $(\phi^{\rightarrow}, \phi^{\leftarrow})$ is an isotone Galios connection,

(3) *an object oriented concept* if $\rho = \xi^{\rightarrow}(\mu)$ and $\mu = \xi^{\leftarrow}(\rho)$ where $(\xi^{\leftarrow}, \xi^{\rightarrow})$ is an isotone Galios connection.

Theorem 2.3. Let $\omega^{\rightarrow} : L^X \to L^Y$ and $\omega^{\leftarrow} : L^Y \to L^X$ be operators. Let $(\omega^{\rightarrow}, \omega^{\leftarrow})$ be an antitone Galois connection between X and Y. Then the following properties hold:

(1) For each $\mu \in L^X$ and $\mu \in L^X$, $\mu \leq \omega^{\leftarrow}(\omega^{\rightarrow}(\mu))$ and $\rho \leq \omega^{\rightarrow}(\omega^{\leftarrow}(\rho))$.

(2) If $\mu_1 \leq \mu_2$, then $\omega^{\rightarrow}(\mu_1) \geq \omega^{\rightarrow}(\mu_2)$. Moreover, if $\rho_1 \leq \rho_2$, then $\omega^{\leftarrow}(\rho_1) \geq \omega^{\leftarrow}(\rho_2)$.

(3) For each $\mu \in L^X$ and $\rho \in L^Y$, $\omega^{\leftarrow}(\omega^{\rightarrow}(\omega^{\leftarrow}(\rho))) = \omega^{\leftarrow}(\rho)$ and $\omega^{\rightarrow}(\omega^{\leftarrow}(\omega^{\rightarrow}(\mu))) = \omega^{\rightarrow}(\mu)$.

(4) For each $\mu_i \in L^X$ and $\rho_j \in L^Y$, $\omega^{\rightarrow}(\bigvee_{i\in I} \mu_i) = \bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i)$ and $\omega^{\leftarrow}(\bigvee_{j\in J} \rho_j) = \bigwedge_{j\in J} \omega^{\leftarrow}(\rho_j)$.

(5) If $\omega^{\leftarrow}(\omega^{\rightarrow}(\mu_i)) = \mu_i$, then $\omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge_{i\in I} \mu_i)) = \bigwedge_{i\in I} \mu_i$

(6) If $(\omega^{\rightarrow}(\omega^{\leftarrow}(\rho_j)) = \rho_j$, then $\omega^{\rightarrow}(\omega^{\leftarrow}(\bigwedge_{j\in J} \rho_j)) = \bigwedge_{j\in J} \rho_j$.

Proof. (1) Since $\omega^{\rightarrow}(\mu) \leq \omega^{\rightarrow}(\mu)$, we have $\mu \leq \omega^{\leftarrow}(\omega^{\rightarrow}(\mu))$. Since $\omega^{\leftarrow}(\rho) \leq \omega^{\leftarrow}(\rho)$, we have $\rho \leq \omega^{\rightarrow}(\omega^{\leftarrow}(\rho))$.

(2) Since $\mu_1 \leq \mu_2 \leq \omega^{\leftarrow}(\omega^{\rightarrow}(\mu_2))$, $\omega^{\rightarrow}(\mu_2) \leq \omega^{\rightarrow}(\mu_1)$. Since $\omega^{\rightarrow}(\omega^{\leftarrow}(\rho_2)) \geq \rho_2 \geq \rho_1$, $\omega^{\leftarrow}(\rho_1) \geq \omega^{\leftarrow}(\rho_2)$.

(3) It easily proved from (1) and (2).

(4) By (2), $\omega^{\rightarrow}(\bigvee_{i\in I} \mu_i) \leq \bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i)$. Since $\omega^{\rightarrow}(\mu_i) \geq \bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i)$ implies $\omega^{\leftarrow}(\bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i)) \geq \mu_i$, we have $\bigvee_{i\in I} \mu_i \leq \omega^{\leftarrow}(\bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i))$. Hence $\omega^{\rightarrow}(\bigvee_{i\in I} \mu_i) \geq \bigwedge_{i\in I} \omega^{\rightarrow}(\mu_i)$.

(5) By (1), $\omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge \mu_i)) \geq \bigwedge \mu_i$ and by (2),

$$\bigwedge_{i\in I} \mu_i = \bigwedge_{i\in I} \omega^{\leftarrow}(\omega^{\rightarrow}(\mu_i)) \geq \omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge_{i\in I} \mu_i)).$$

Hence, $\omega^{\leftarrow}(\omega^{\rightarrow}(\bigwedge_{i\in I} \mu_i)) = \bigwedge_{i\in I} \mu_i$.

(6) It is similarly proved as (5). □

Example 2.4. Let $X = \{a, b, c\}$, $Y = \{x, y, z, w\}$ and $L = \{0, 1\}$ be sets. Define $\omega^{\rightarrow} : P(X) \to P(Y)$ and $\omega^{\leftarrow} : P(Y) \to P(X)$ as

$$\omega^{\rightarrow}(\emptyset) = Y, \omega^{\rightarrow}(\{a\}) = \{x, y\}, \omega^{\rightarrow}(\{b\}) = \{y, w\},$$

$$\omega^{\rightarrow}(\{c\}) = \{z, w\}, \omega^{\rightarrow}(\{a, b\}) = \{y\},$$

$$\omega^{\rightarrow}(\{b, c\}) = \{w\}, \omega^{\rightarrow}(\{a, c\}) = \omega^{\rightarrow}(X) = \emptyset.$$

$$\omega^{\leftarrow}(\emptyset) = X, \omega^{\leftarrow}(\{w\}) = \{b, c\},$$

$$\omega^{\leftarrow}(\{z\}) = \{c\} = \omega^{\leftarrow}(\{z, w\}),$$

$$\omega^{\leftarrow}(\{y\}) = \{a, b\}, \omega^{\leftarrow}(\{y, w\}) = \{b\} = \omega^{\leftarrow}(\{x, y\}),$$

$$\omega^{\leftarrow}(\{x\}) = \{a\}, \omega^{\leftarrow}(\{y, z\}) = \omega^{\leftarrow}(\{y, z, w\}) = \emptyset,$$

$$\omega^{\leftarrow}(\{x, w\}) = \omega^{\leftarrow}(\{x, z\}) = \omega^{\leftarrow}(\{x, z, w\}) = \emptyset,$$

$$\omega^{\leftarrow}(\{x, y, w\}) = \omega^{\leftarrow}(\{x, y, z\}) = \omega^{\leftarrow}(Y) = \emptyset.$$

Then $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galios connection. Thus, we obtain formal concepts

$$\{(\emptyset, Y), (\{a\}, \{x, y\}), (\{b\}, \{y, w\}), (\{c\}, \{z, w\})$$

$$(\{b, c\}, \{w\}), (\{a, b\}, \{y\}), (X, \emptyset)\}$$

In general, $\{z, w\} = \omega^{\rightarrow}(\{a, c\} \cap \{b, c\}) \neq \omega^{\rightarrow}(\{a, c\}) \cup \omega^{\rightarrow}(\{b, c\}) = \{w\}$.

Definition 2.5. An operator $\phi^{\rightarrow} : L^X \to L^Y$ is called a join-generating operator, denoted by $\phi^{\rightarrow} \in J(X, Y)$, if $\phi^{\rightarrow}(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \phi^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

An operator $\psi^{\rightarrow} : L^X \to L^Y$ is called a meet-generating operator, denoted by $\psi^{\rightarrow} \in M(X, Y)$, if $\psi^{\rightarrow}(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \psi^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

An operator $\omega^{\rightarrow} : L^X \to L^Y$ is called an order reverse-generating operator, denoted by $\omega^{\rightarrow} \in K(X, Y)$, if $\omega^{\rightarrow}(\bigvee_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \omega^{\rightarrow}(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma} \subset L^X$.

Theorem 2.6. For $\omega^{\rightarrow} \in K(X, Y)$, Define functions $\phi_\omega^{\rightarrow}, \xi_\omega^{\rightarrow} : L^X \to L^Y$ and $\omega^{\leftarrow}, \phi_\omega^{\leftarrow}, \xi_\omega^{\leftarrow} : L^Y \to L^X$ as follows: for all $\lambda \in L^X, \rho \in L^Y$,

$$\omega^{\leftarrow}(\rho) = \bigvee\{\lambda \in L^X \mid \omega^{\rightarrow}(\lambda) \geq \rho\}$$

$$\phi_\omega^{\rightarrow}(\mu) = (\omega^{\rightarrow}(\mu))^*, \ \phi_\omega^{\leftarrow}(\rho) = \omega^{\leftarrow}(\rho^*)$$

$$\xi_\omega^{\leftarrow}(\rho) = \bigwedge\{\lambda \in L^X \mid \omega(\lambda^*) \geq \rho\},$$

$$\xi_\omega^{\rightarrow}(\mu) = \bigvee\{\rho \in L^Y \mid \xi_\omega^{\leftarrow}(\rho) \leq \mu\}$$

Then the following properties hold:

(1) $\omega^{\leftarrow} \in K(Y, X)$ with $\omega^{\rightarrow}(\lambda) \geq \rho \Leftrightarrow \omega^{\leftarrow}(\rho) \geq \lambda$ for all $\lambda \in L^X$ and $\rho \in L^Y$. Furthermore, $\omega^{\rightarrow}(\alpha \odot \lambda) \leq \alpha \to \omega^{\rightarrow}(\lambda)$ iff $\alpha \to \omega^{\leftarrow}(\rho) \leq \omega^{\leftarrow}(\alpha \odot \rho)$. Similarly, $\omega^{\leftarrow}(\alpha \odot \lambda) \leq \alpha \to \omega^{\leftarrow}(\lambda)$ iff $\alpha \to \omega^{\rightarrow}(\rho) \leq \omega^{\rightarrow}(\alpha \odot \rho)$.

(2) The pair $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galois connection and $(\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda)), \omega^{\rightarrow}(\lambda))$ for all $\lambda \in L^X$ are formal concepts.

(3) The pair $(\phi_\omega^{\rightarrow}, \phi_\omega^{\leftarrow})$ is an isotone Galois connection and $(\phi_\omega^{\leftarrow}(\phi_\omega^{\rightarrow}(\lambda)), \phi_\omega^{\rightarrow}(\lambda)) = (\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda)), \omega^{\rightarrow}(\lambda)^*)$ for all $\lambda \in L^X$ are attribute oriented concepts.

(4) $\omega^{\rightarrow}(\alpha \odot \mu) \leq \alpha \to \omega^{\rightarrow}(\mu)$ iff $\alpha \odot \phi_\omega^{\rightarrow}(\mu) \leq \phi_\omega^{\rightarrow}(\alpha \odot \mu)$ iff $\alpha \to \phi_\omega^{\leftarrow}(\rho) \leq \phi_\omega^{\leftarrow}(\alpha \to \rho)$.

(5) $\xi_\omega^{\leftarrow} : L^Y \to L^X$ is a join-generating function such that $\xi_\omega^{\leftarrow}(\rho) = (\omega^{\leftarrow}(\rho))^*$ and

$$\omega^{\rightarrow}(\lambda^*) \geq \rho \Leftrightarrow \omega^{\leftarrow}(\rho) \geq \lambda^* \Leftrightarrow \xi_\omega^{\leftarrow}(\rho) \leq \lambda.$$

Moreover, $\omega^{\rightarrow}(\alpha \odot \lambda) \geq \alpha \to \omega^{\rightarrow}(\lambda)$ iff $\alpha \odot \xi_\omega^{\leftarrow}(\rho) \leq \xi_\omega^{\leftarrow}(\alpha \odot \rho)$.

(6) $\xi_\omega^{\rightarrow} : L^X \to L^Y$ is a meet-generating function such that $\xi_\omega^{\rightarrow}(\lambda) = \omega^{\rightarrow}(\lambda^*)$ and

$$\omega^{\rightarrow}(\lambda^*) \geq \rho \Leftrightarrow \omega^{\leftarrow}(\rho) \geq \lambda^* \Leftrightarrow \xi_\omega^{\leftarrow}(\rho) \leq \lambda \Leftrightarrow \rho \leq \xi_\omega^{\rightarrow}(\lambda).$$

Moreover, $\omega^{\rightarrow}(\alpha \odot \lambda) \geq \alpha \to \omega^{\rightarrow}(\lambda)$ iff $\alpha \to \xi_\omega^{\rightarrow}(\lambda) \leq \xi_\omega^{\rightarrow}(\alpha \to \lambda)$.

(7) The pair $(\xi_\omega^{\leftarrow}, \xi_\omega^{\rightarrow})$ is an isotone Galois connection and $(\xi_\omega^{\leftarrow}(\rho), \xi_\omega^{\rightarrow}(\xi_\omega^{\leftarrow}(\rho))) = (\omega^{\leftarrow}(\rho)^*, \omega^{\rightarrow}(\omega^{\rightarrow}(\rho)))$ for all $\rho \in L^Y$ are object oriented concepts.

Proof. (1) Since $\omega^{\rightarrow} \in K(X,Y)$ and $\omega^{\leftarrow}(\rho) = \bigvee\{\lambda \in L^X \mid \omega^{\rightarrow}(\lambda) \geq \rho\}$, we have

$$\omega^{\rightarrow}(\lambda) \geq \rho \Leftrightarrow \omega^{\leftarrow}(\rho) \geq \lambda.$$

Moreover, $\omega^{\leftarrow} \in K(Y,X)$ from

$$\begin{aligned} \bigwedge_{i\in\Gamma} \omega^{\leftarrow}(\rho_i) \geq \mu \quad & \Leftrightarrow \omega^{\leftarrow}(\rho_i) \geq \mu, \ \ \forall i \in \Gamma \\ & \Leftrightarrow \omega^{\rightarrow}(\mu) \geq \rho_i, \ \ \forall i \in \Gamma \\ & \Leftrightarrow \omega^{\rightarrow}(\mu) \geq \bigvee_{i\in\Gamma} \rho_i, \\ & \Leftrightarrow \omega^{\leftarrow}(\bigvee_{i\in\Gamma} \rho_i) \geq \mu. \end{aligned}$$

Hence $\omega^{\leftarrow}(\bigvee_{i\in\Gamma} \rho_i) = \bigwedge_{i\in\Gamma} \omega^{\leftarrow}(\rho_i)$.

Let $\omega^{\rightarrow}(\alpha \odot \lambda) \leq \alpha \to \omega^{\rightarrow}(\lambda)$. For $\mu \leq \alpha \to \omega^{\leftarrow}(\rho)$, $\mu \odot \alpha \leq \omega^{\leftarrow}(\rho)$ iff $\omega^{\rightarrow}(\mu \odot \alpha) \geq \rho$. Since $\omega^{\rightarrow}(\alpha \odot \mu) \leq \alpha \to \omega^{\rightarrow}(\mu)$, $\alpha \to \omega^{\rightarrow}(\mu) \geq \rho$ iff $\omega^{\rightarrow}(\mu) \geq \alpha \odot \rho$ iff $\omega^{\leftarrow}(\alpha \odot \rho) \geq \mu$. Hence $\alpha \to \omega^{\leftarrow}(\rho) \leq \omega^{\leftarrow}(\alpha \odot \rho)$. Conversely, it similarly proved.

(2) By (1), since $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galois connection, $(\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda)), \omega^{\rightarrow}(\lambda))$ are formal concepts from Theorem 2.3(3).

(3) It follows from $\phi_\omega^{\rightarrow}(\lambda) \leq \rho$ iff $(\omega^{\rightarrow}(\lambda))^* \leq \rho$ iff $\omega^{\rightarrow}(\lambda) \geq \rho^*$ iff $\omega^{\leftarrow}(\rho^*) \geq \lambda$ iff $\phi_\omega^{\leftarrow}(\rho) \leq \lambda$. Moreover, $\phi_\omega^{\rightarrow}(\phi_\omega^{\leftarrow}(\phi_\omega^{\rightarrow}(\lambda))) = \phi_\omega^{\rightarrow}(\phi_\omega^{\leftarrow}((\omega^{\rightarrow}(\lambda))^*)) = \phi_\omega^{\rightarrow}(\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda))) = (\omega^{\rightarrow}(\omega^{\leftarrow}(\omega^{\rightarrow}(\lambda))))^* = (\omega^{\rightarrow}(\lambda)))^* = \phi_\omega^{\rightarrow}(\lambda)$

(4) $\omega^{\rightarrow}(\alpha \odot \mu) \leq \alpha \to \omega^{\rightarrow}(\mu)$ iff $(\omega^{\rightarrow}(\alpha \odot \mu))^* \geq (\alpha \to \omega^{\rightarrow}(\mu))^*$ iff $\alpha \odot \phi_\omega^{\rightarrow}(\mu) \leq \phi_\omega^{\rightarrow}(\alpha \odot \mu)$.

Let $\alpha \to \phi_\omega^{\leftarrow}(\rho) \leq \phi_\omega^{\leftarrow}(\alpha \to \rho)$. For $\phi_\omega^{\rightarrow}(\alpha \odot \mu) \leq \rho$, $\alpha \odot \lambda \leq \phi_\omega^{\leftarrow}(\rho)$. Then $\lambda \leq \alpha \to \phi_\omega^{\leftarrow}(\rho) \leq \phi_\omega^{\leftarrow}(\alpha \to \rho)$ implies $\phi_\omega^{\rightarrow}(\lambda) \leq \alpha \to \rho$. Hence $\alpha \odot \phi_\omega^{\rightarrow}(\lambda) \leq \rho$. Thus, $\alpha \odot \phi_\omega^{\rightarrow}(\lambda) \leq \phi_\omega^{\rightarrow}(\alpha \odot \lambda)$. Conversely, it is similarly proved.

(5) We have

$$\begin{aligned} & \xi_\omega^{\leftarrow}(\rho) = \bigwedge\{\lambda \in L^X \mid \omega^{\rightarrow}(\lambda^*) \geq \rho\} \\ & = \Big(\bigvee\{\lambda^* \in L^X \mid \omega^{\rightarrow}(\lambda^*) \geq \rho\}\Big)^* = (\omega^{\leftarrow}(\rho))^*. \end{aligned}$$

We have $\xi_\omega^{\leftarrow} \in J(Y,X)$ from:

$$\begin{aligned} \bigvee_{i\in\Gamma} \xi_\omega^{\leftarrow}(\rho_i) \leq \lambda \quad & \Leftrightarrow \xi_\omega^{\leftarrow}(\rho_i) \leq \lambda, \ \ \forall i \in \Gamma \\ & \Leftrightarrow \omega^{\rightarrow}(\lambda^*) \geq \rho_i, \ \ \forall i \in \Gamma \\ & \Leftrightarrow \omega^{\rightarrow}(\lambda^*) \geq \bigvee_{i\in\Gamma} \rho_i, \\ & \Leftrightarrow \xi_\omega^{\leftarrow}(\bigvee_{i\in\Gamma} \rho_i) \leq \lambda. \end{aligned}$$

$\omega^{\rightarrow}(\alpha \odot \lambda) \geq \alpha \to \omega^{\rightarrow}(\lambda)$ iff $\omega^{\leftarrow}(\alpha \odot \rho) \leq \alpha \to \omega^{\leftarrow}(\rho)$ iff $(\omega^{\leftarrow}(\alpha \odot \rho))^* \geq (\alpha \to \omega^{\leftarrow}(\rho))^*$ iff $\alpha \odot \xi_\omega^{\leftarrow}(\rho) \leq \xi_\omega^{\leftarrow}(\alpha \odot \rho)$.

(6) We have

$$\begin{aligned} \xi_\omega^{\rightarrow}(\lambda) \quad & = \bigvee\{\rho \in L^Y \mid \xi_\omega^{\leftarrow}(\rho) \leq \lambda\} \\ & = \bigvee\{\rho \in L^Y \mid \omega^{\rightarrow}(\lambda^*) \geq \rho\} = \omega^{\rightarrow}(\lambda^*). \end{aligned}$$

$$\begin{aligned} \xi_\omega^{\rightarrow}(\alpha \to \lambda) \quad & = \omega^{\rightarrow}((\alpha \to \lambda)^*) = \omega^{\rightarrow}(\alpha \odot \lambda^*) \\ & \geq \alpha \to \omega^{\rightarrow}(\lambda^*) = \alpha \to \xi_\omega^{\rightarrow}(\lambda). \end{aligned}$$

(7)

$$\begin{aligned} \xi_\omega^{\leftarrow}(\xi_\omega^{\rightarrow}(\xi_\omega^{\leftarrow}(\rho))) \quad & = \xi_\omega^{\leftarrow}(\xi_\omega^{\rightarrow}((\omega^{\leftarrow}(\rho))^*) \\ & = \xi_\omega^{\leftarrow}(\omega^{\rightarrow}(\omega^{\leftarrow}(\rho))) \\ & = (\omega^{\leftarrow}(\omega^{\rightarrow}(\omega^{\leftarrow}(\rho))))^* \\ & = (\omega^{\leftarrow}(\rho))^* = \xi_\omega^{\leftarrow}(\rho) \end{aligned}$$

□

Corollary 2.7. Let $P(X)$ and $P(Y)$ be families of subsets of X and Y. Let $\omega^{\rightarrow} : P(X) \to P(Y)$ be an operator with $\omega^{\rightarrow}(\bigcup A_i) = \bigcap \omega^{\rightarrow}(A_i)$ for $A_i \in P(X)$. Define functions $\phi_\omega^{\rightarrow}, \xi_\omega^{\rightarrow} : P(X) \to P(Y)$ and $\omega^{\leftarrow}, \phi_\omega^{\leftarrow}, \xi_\omega^{\leftarrow} : P(Y) \to P(X)$ as follows: for all $A \in P(X), B \in P(Y)$,

$$\omega^{\leftarrow}(B) = \bigcup\{A \in P(X) \mid \omega^{\rightarrow}(A) \supset B\}$$

$$\phi_\omega^{\rightarrow}(A) = (\omega^{\rightarrow}(A))^c, \ \phi_\omega^{\leftarrow}(B) = \omega^{\leftarrow}(B^c)$$

$$\xi_\omega^{\leftarrow}(B) = \bigcap\{A \in P(X) \mid \omega(A^c) \geq B\},$$

$$\xi_\omega^{\rightarrow}(A) = \bigcup\{B \in P(Y) \mid \xi_\omega^{\leftarrow}(B) \subset A\}$$

Then the following properties hold:

(1) $\omega^{\leftarrow}(\bigcup B_i) = \bigcap \omega^{\leftarrow}(B_i)$ for $B_i \in P(Y)$ with $\omega^{\rightarrow}(A) \geq B \Leftrightarrow \omega^{\leftarrow}(B) \geq A$ for all $A \in P(X)$ and $B \in P(Y)$.

(2) The pair $(\omega^{\rightarrow}, \omega^{\leftarrow})$ is an antitone Galois connection and $(\omega^{\leftarrow}(\omega^{\rightarrow}(A)), \omega^{\rightarrow}(A))$ for all $A \in P(X)$ are formal concepts.

(3) The pair $(\phi_\omega^{\rightarrow}, \phi_\omega^{\leftarrow})$ is an isotone Galois connection and $(\phi_\omega^{\leftarrow}(\phi_\omega^{\rightarrow}(A)), \phi_\omega^{\rightarrow}(A))$ are attribute oriented concepts.

(4) $\xi_\omega^{\leftarrow} : P(Y) \to P(X)$ is a union-preserving function such that $\xi_\omega^{\leftarrow}(B) = (\omega^{\leftarrow}(B))^c$ and

$$\omega^{\rightarrow}(A^c) \supset B \Leftrightarrow \omega^{\leftarrow}(B) \supset A^c \Leftrightarrow \xi_\omega^{\leftarrow}(B) \subset A.$$

(5) $\xi_\omega^{\rightarrow} : P(X) \to P(Y)$ is an intersection-preserving function such that $\xi_\omega^{\rightarrow}(A) = \omega^{\rightarrow}(A^c)$ and

$$\omega^{\rightarrow}(A^c) \supset B \Leftrightarrow \omega^{\leftarrow}(B) \supset A^c$$

$$\Leftrightarrow \xi_\omega^{\leftarrow}(B) \subset A \Leftrightarrow B \subset \xi_\omega^{\rightarrow}(A).$$

(6) The pair $(\xi_\omega^{\leftarrow}, \xi_\omega^{\rightarrow})$ is an isotone Galois connection and $(\xi_\omega^{\leftarrow}(B), \xi_\omega^{\rightarrow}(\xi_\omega^{\leftarrow}(B)))$ for all $B \in P(Y)$ are object oriented concepts.

Example 2.8. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $L = \{0, 1\}$ be sets. Define a function $f : X \to Y$ as follows:

$$f(x_1) = f(x_2) = y_1,\ f(x_3) = y_2.$$

Define $\omega^{\rightarrow} : P(X) \to P(Y)$ as $\omega^{\rightarrow}(A) = \{y_2\} \cup (f(A))^c$;

$$\omega^{\rightarrow}(\emptyset) = Y, \omega^{\rightarrow}(\{x_1\}) = \omega^{\rightarrow}(\{x_2\}) = \{y_2, y_3\},$$

$$\omega^{\rightarrow}(\{x_3\}) = Y, \omega^{\rightarrow}(\{x_1, x_2\}) = \{y_2, y_3\},$$

$$\omega^{\rightarrow}(\{x_2, x_3\}) = \omega^{\rightarrow}(\{x_1, x_3\}) = \omega^{\rightarrow}(X) = \{y_2, y_3\}.$$

$$\omega^{\leftarrow}(\emptyset) = \omega^{\leftarrow}(\{y_2\}) = \omega^{\leftarrow}(\{y_2, y_3\}) = \omega^{\leftarrow}(\{y_3\}) = X,$$

$$\omega^{\leftarrow}(\{y_1\}) = \omega^{\leftarrow}(\{y_1, y_3\}) = \{x_3\},$$

$$\omega^{\leftarrow}(\{y_1, y_2\}) = \omega^{\leftarrow}(Y) = \{x_3\}.$$

Thus, we obtain attribute oriented concepts $(\omega^{\leftarrow}(\omega^{\rightarrow}(\mu)), \omega^{\rightarrow}(\mu))$ as follows:

$$\{(\{x_3\}, Y), (X, \{y_2, y_3\})\}$$

(2) $\phi_\omega^{\rightarrow}(A) = \{y_1, y_3\} \cap f(A)$. Then

$$\phi_\omega^{\rightarrow}(\emptyset) = \emptyset, \phi_\omega^{\rightarrow}(\{x_1\}) = \{y_1\}, \phi_\omega^{\rightarrow}(\{x_2\}) = \{y_1\},$$

$$\phi_\omega^{\rightarrow}(\{x_3\}) = \emptyset, \phi_\omega^{\rightarrow}(\{x_1, x_2\}) = \{y_1\},$$

$$\phi_\omega^{\rightarrow}(\{x_2, x_3\}) = \phi_\omega^{\rightarrow}(\{x_1, x_3\}) = \phi_\omega^{\rightarrow}(X) = \{y_1\}.$$

$$\phi_\omega^{\leftarrow}(Y) = \phi_\omega^{\leftarrow}(\{y_1, y_3\}) = X,$$

$$\phi_\omega^{\leftarrow}(\{y_1\}) = \phi_\omega^{\leftarrow}(\{y_1, y_2\}) = X,$$

$$\phi_\omega^{\leftarrow}(\{y_2, y_3\}) = \phi_\omega^{\leftarrow}(\{y_2\}) = \phi_\omega^{\leftarrow}(\{y_3\}) = \phi_\omega^{\leftarrow}(\emptyset) = \{x_3\}.$$

Thus, we obtain attribute oriented concepts $(\phi_\omega^{\leftarrow}(\phi_\omega^{\rightarrow}(\mu)), \phi_\omega^{\rightarrow}(\mu))$ as follows:

$$\{(\{x_3\}, \emptyset\}), (X, \{y_1\})\}$$

(3) Since $\xi_\omega^{\rightarrow}(A) = \omega^{\rightarrow}(A^c), \xi_\omega^{\leftarrow}(B) = (\omega^{\rightarrow}(B))^c$,

$$\xi_\omega^{\rightarrow}(X) = Y, \xi_\omega^{\rightarrow}(\{x_2, x_3\}) = \{y_2, y_3\},$$

$$\xi_\omega^{\rightarrow}(\{x_1, x_3\}) = \{y_2, y_3\}, \xi_\omega^{\rightarrow}(\{x_1, x_2\}) = Y$$

$$\xi_\omega^{\rightarrow}(\{x_3\}) = \{y_2, y_3\},$$

$$\xi_\omega^{\rightarrow}(\{x_1\}) = \xi_\omega^{\rightarrow}(\{x_2\}) = \xi_\omega^{\rightarrow}(\emptyset) = \{y_2, y_3\}.$$

$$\xi_\omega^{\leftarrow}(\emptyset) = \xi_\omega^{\leftarrow}(\{y_2\}) = \xi_\omega^{\leftarrow}(\{y_2, y_3\}) = \xi_\omega^{\leftarrow}(\{y_3\}) = \emptyset,$$

$$\xi_\omega^{\leftarrow}(\{y_1\}) = \xi_\omega^{\leftarrow}(\{y_1, y_3\}) = \{x_1, x_2\}.$$

$$\xi_\omega^{\leftarrow}(\{y_1, y_2\}) = \xi_\omega^{\leftarrow}(Y) = \{x_1, x_2\}.$$

Thus, we obtain attribute oriented concepts $(\xi_\omega^{\rightarrow}(\mu), \xi_\omega^{\leftarrow}(\xi_\omega^{\rightarrow}(\mu)))$ as follows:

$$\{(\{x_1, x_2\}, Y), (\emptyset, \{y_2, y_3\})\}$$

Theorem 2.9. Let (X, Y, R) be a fuzzy context. Define a function $\omega_R^{\rightarrow} : L^X \to L^Y$ as follows:

$$\omega_R^{\rightarrow}(\lambda)(y) = \bigvee_{x \in X} (\lambda(x) \to R(x, y)).$$

Then we have the following properties:

(1) $\omega_R^{\rightarrow} \in K(X, Y)$ and $\omega_R^{\rightarrow}$ has a right adjoint mapping $\omega_R^{\leftarrow}$ with

$$\omega_R^{\leftarrow}(\rho)(x) = \bigwedge_{y \in Y} (\rho(y) \to R(x, y)).$$

Moreover, $\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(\lambda)) \geq \lambda$ and $\omega_R^{\rightarrow}(\omega_R^{\leftarrow}(\rho)) \geq \rho$ for all $\lambda \in L^X$ and $\rho \in L^Y$.

(2) $(\omega_R^{\rightarrow}, \omega_R^{\leftarrow})$ is an isotone Galois connections and $(\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(\lambda)), \omega_R^{\rightarrow}(\lambda))$ for all $\lambda \in L^X$ are formal concepts.

(3) $\omega_R^{\rightarrow}(\alpha \odot \lambda) = \alpha \to \omega_R^{\rightarrow}(\lambda) = \omega_{\alpha \to R}^{\rightarrow}(\lambda)$ and $\omega_R^{\leftarrow}(\alpha \odot \rho) = \alpha \to \omega_R^{\leftarrow}(\rho) = \omega_{\alpha \to R}^{\leftarrow}(\rho)$, for all $\lambda \in L^X, \rho \in L^Y$.

(4) $\phi_{\omega_R}^{\rightarrow}(\mu) = (\omega_R^{\rightarrow}(\mu))^*$ and $\phi_{\omega_R}^{\leftarrow}(\rho) = \omega_R^{\leftarrow}(\rho^*)$ where

$$\phi_{\omega_R}^{\rightarrow}(\mu)(y) = \bigvee_{y \in Y} (\mu(x) \odot R^*(x, y)),$$

$$\phi_{\omega_R}^{\leftarrow}(\rho)(x) = \bigwedge_{y \in Y} (R^*(x, y) \to \rho(y)).$$

(5) The pair $(\phi_{\omega_R}^{\rightarrow}, \phi_{\omega_R}^{\leftarrow})$ is an isotone Galois connection and $(\phi_{\omega_R}^{\leftarrow}(\phi_{\omega_R}^{\rightarrow}(\lambda)), \phi_{\omega_R}^{\rightarrow}(\lambda))$ are attribute concepts.

(6)

$$\xi_{\omega_R}^{\leftarrow}(\rho)(x) = \bigvee_{y \in Y} (\rho(y) \odot R^*(x, y)),$$

$$\xi_{\omega_R}^{\rightarrow}(\lambda)(y) = \bigwedge_{x \in X} (R^*(x, y) \to \lambda(x))$$

(7) The pair $(\xi_{\omega_R}^{\leftarrow}, \xi_{\omega_R}^{\rightarrow})$ is an isotone Galois connection and $(\xi_{\omega_R}^{\leftarrow}(\rho), \xi_{\omega_R}^{\rightarrow}(\xi_{\omega_R}^{\leftarrow}(\rho)))$ for all $\rho \in L^Y$ are object oriented concepts.

Proof. (1) Since $\omega_R^{\rightarrow}(\bigvee_{i\in\Gamma}\lambda_i)(y) = \bigwedge_{x\in X}(\bigvee_{i\in\Gamma}\lambda_i(x) \to R(x,y)) = \bigwedge_{i\in\Gamma}\Big(\bigwedge_{x\in X}(\lambda_i(x) \to R(x,y)\Big) = \bigwedge_{i\in\Gamma}\omega_R^{\rightarrow}(\lambda_i)(y)$, $\omega_R^{\rightarrow}$ has a right adjoint mapping $\omega_R^{\leftarrow}$ as follows:

$$\begin{aligned}\omega_R^{\leftarrow}(\rho)(x) &= \bigvee\{\lambda \mid \rho \le \omega_R^{\rightarrow}(\lambda)\}\\ &= \bigvee\{\lambda \mid \rho(y) \le \bigwedge(\lambda(x) \to R(x,y))\}\\ &= \bigvee\{\lambda \mid \lambda(x) \le \bigwedge_{y\in Y}(\rho(y) \to R(x,y))\}\\ &= \bigwedge_{y\in Y}(\rho(y) \to R(x,y))\end{aligned}$$

$$\begin{aligned}&\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(\lambda))(x)\\ &= \bigwedge_{y\in Y}\{\omega_R^{\rightarrow}(\lambda)(y) \to R(x,y)\}\\ &= \bigwedge_{y\in Y}\{\bigwedge_{x\in X}(\lambda(x) \to R(x,y)) \to R(x,y)\}\\ &\ge \bigwedge_{y\in Y}\{(\lambda(x) \to R(x,y)) \to R(x,y)\}\\ &\ge \lambda(x).\end{aligned}$$

$$\begin{aligned}&\omega_R^{\rightarrow}(\omega_R^{\leftarrow}(\rho))(y)\\ &= \bigwedge_{x\in X}\{\omega_R^{\leftarrow}(R)(\rho)(x) \to R(x,y)\}\\ &= \bigwedge_{x\in X}\{\bigwedge_{y\in Y}(\rho(y) \to R(x,y)) \to R(x,y)\}\\ &\ge \bigwedge_{x\in X}\{(\rho(y) \to R(x,y)) \to R(x,y)\}\\ &\ge \rho(y).\end{aligned}$$

(3) By Lemma 1.3(6), we prove:

$$\begin{aligned}\omega_R^{\rightarrow}(\alpha\odot\lambda)(y) &= \bigwedge_{x\in X}((\alpha\odot\lambda)(x) \to R(x,y))\\ &= \bigwedge_{x\in X}(\alpha \to (\lambda(x) \to R(x,y)))\\ &= \alpha \to \bigwedge_{x\in X}(\lambda(x) \to R(x,y))\\ &= \alpha \to \omega_R^{\rightarrow}(\lambda)(y)\\ &= \bigwedge_{x\in X}(\lambda(x) \to (\alpha \to R(x,y)))\\ &= \omega_{\alpha\to R}^{\rightarrow}(\lambda)(y)\end{aligned}$$

(4)

$$\begin{aligned}&\phi_{\omega_R}^{\rightarrow}(\mu)(y) = (\omega_R^{\rightarrow}(\mu))^*(y)\\ &= (\bigwedge_{x\in X}(\mu(x) \to R(x,y))^*\\ &= \bigvee_{x\in X}(\mu(x)\odot R^*(x,y))(\text{ by Lemma 1.3(7)}).\end{aligned}$$

$$\begin{aligned}&\phi_{\omega_R}^{\leftarrow}(\rho)(x) = \omega_R^{\leftarrow}(\rho^*)(x)\\ &= \bigvee_{y\in Y}(\rho^*(y) \to R(x,y))(\text{ by Lemma 1.3(7)})\\ &= \bigwedge_{y\in Y}(R^*(x,y) \to \rho(y)).\end{aligned}$$

(5) It follows from Theorem 2.6(3).

(6)

$$\begin{aligned}\xi_{\omega_R}^{\leftarrow}(\rho)(x) &= (\omega_R^{\leftarrow}(\rho)(x))^*\\ &= (\bigwedge_{y\in Y}(\rho(y) \to R(x,y)))^*\\ &= \bigvee_{y\in Y}(\rho(y)\odot R^*(x,y)).\end{aligned}$$

$$\begin{aligned}\xi_{\omega_R}^{\rightarrow}(\mu)(y) &= \omega_R^{\rightarrow}(\mu^*)(y)\\ &= \bigwedge_{x\in X}(\mu^*(x) \to R(x,y))\\ &= \bigwedge_{x\in X}(R(x,y)^* \to \mu(x)).\end{aligned}$$

(7) It follows from Theorem 2.6(7). □

Corollary 2.10. Let X and Y be sets and $R \subset X \times Y$. Define a function $\omega_R^{\rightarrow} : P(X) \to P(Y)$ as follows:

$$\omega_R^{\rightarrow}(A) = \{y \in Y \mid (\exists x \in X)(x \in A \to (x,y) \in R)\}$$

Then we have the following properties:

(1) $\omega_R^{\rightarrow} \in K(X,Y)$ and $\omega_R^{\rightarrow}$ has a right adjoint mapping $\omega_R^{\leftarrow}$ with

$$\omega_R^{\leftarrow}(B) = \{x \in X \mid (\exists y \in Y)(y \in B \to (x,y) \in R)\}.$$

Moreover, $\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(A)) \supset A$ and $\omega_R^{\rightarrow}(\omega_R^{\leftarrow}(B)) \supset B$ for all $A \in P(X)$ and $B \in P(Y)$.

(2) $(\omega_R^{\rightarrow}, \omega_R^{\leftarrow})$ is an isotone Galois connections and $(\omega_R^{\leftarrow}(\omega_R^{\rightarrow}(A)), \omega_R^{\rightarrow}(A))$ for all $A \in P(X)$ are formal concepts.

(3) $\phi_{\omega_R}^{\rightarrow}(A) = (\omega_R^{\rightarrow}(A))^c$ and $\phi_{\omega_R}^{\leftarrow}(B) = \omega_R^{\leftarrow}(B^c)$ where

$$\phi_{\omega_R}^{\rightarrow}(A) = \{y \in Y \mid (\exists x \in A)((x \in A)\wedge((x,y) \in R^c))\},$$

$$\phi_{\omega_R}^{\leftarrow}(B) = \{x \in X \mid (\forall y \in Y)((x,y) \in R^c \to y \in B)\}.$$

(5) The pair $(\phi_{\omega_R}^{\rightarrow}, \phi_{\omega_R}^{\leftarrow})$ is an isotone Galois connection and $(\phi_{\omega_R}^{\leftarrow}(\phi_{\omega_R}^{\rightarrow}(A)), \phi_{\omega_R}^{\rightarrow}(A))$ are attribute concepts.

(6)

$$\xi_{\omega_R}^{\leftarrow}(B) = \{x \in X \mid (\exists y \in B)((y \in B) \wedge ((x,y) \in R^c)\},$$

$$\xi_{\omega_R}^{\rightarrow}(A) = \{y \in Y \mid (\forall x \in X)((x,y) \in R^c \to x \in A)\}$$

(7) The pair $(\xi_{\omega_R}^{\leftarrow}, \xi_{\omega_R}^{\rightarrow})$ is an isotone Galois connection and $(\xi_{\omega_R}^{\leftarrow}(B), \xi_{\omega_R}^{\rightarrow}(\xi_{\omega_R}^{\leftarrow}(B)))$ for all $B \in P(Y)$ are object oriented concepts.

Example 2.11. Let $X = \{a,b,c\}$, $Y = \{x,y,z,w\}$ and $L = \{0,1\}$ be sets. Define a relation R as follows:

$$R = \{(a,x),(a,y),(b,y),(b,w),(c,z),(c,w)\}.$$

(1) Define $\omega_R^{\rightarrow} : P(X) \to P(Y)$ as $\omega_R^{\rightarrow}(A) = \{y \in Y \mid a \in A \to (a,y) \in R\}$;

$$\omega_R^{\rightarrow}(\emptyset) = Y, \omega_R^{\rightarrow}(\{a\}) = \{x,y\}, \omega_R^{\rightarrow}(\{b\}) = \{y,w\},$$

$$\omega_R^{\rightarrow}(\{c\}) = \{z,w\}, \omega_R^{\rightarrow}(\{a,b\}) = \{y\},$$

$$\omega_R^{\rightarrow}(\{b,c\}) = \{w\}, \omega_R^{\rightarrow}(\{a,c\}) = \omega_R^{\rightarrow}(X) = \emptyset.$$

We obtain $\omega_R^{\leftarrow}(B) = \{a \in X \mid y \in B \to (a,y) \in R\}$;

$$\omega_R^{\leftarrow}(\emptyset) = X, \omega_R^{\leftarrow}(\{w\}) = \{b,c\},$$

$$\omega_R^{\leftarrow}(\{z\}) = \{c\} = \omega_R^{\leftarrow}(\{z,w\}),$$

$$\omega_R^{\leftarrow}(\{y\}) = \{a,b\}, \omega_R^{\leftarrow}(\{y,w\}) = \{b\} = \omega_R^{\leftarrow}(\{x,y\}),$$

$$\omega_R^{\leftarrow}(\{x\}) = \{a\}, \omega_R^{\leftarrow}(\{y,z\}) = \omega_R^{\leftarrow}(\{y,z,w\}) = \emptyset,$$

$$\omega_R^{\leftarrow}(\{x,w\}) = \omega_R^{\leftarrow}(\{x,z\}) = \omega_R^{\leftarrow}(\{x,z,w\}) = \emptyset,$$

$$\omega_R^{\leftarrow}(\{x,y,w\}) = \omega_R^{\leftarrow}(\{x,y,z\}) = \omega_R^{\leftarrow}(Y) = \emptyset,$$

Thus, we obtain formal concepts

$$\{(\emptyset, Y), (\{a\}, \{x, y\}), (\{b\}, \{y, w\}), (\{c\}, \{z, w\})$$

$$(\{b, c\}, \{w\}), (\{a, b\}, \{y\})(X, \emptyset)\}$$

(2) We obtain a relation $R^* = R^c$ as follows:

$$R^c = \{(a, z), (a, w), (b, x), (b, z), (c, x), (c, y)\}.$$

We obtain $\phi^{\rightarrow}_{\omega_R} : P(X) \to P(Y)$ as

$$\phi^{\rightarrow}_{\omega_R}(A) = \{y \in Y \mid (\exists a \in A)(a \in A \wedge (a, y) \in R^c)\}$$

$$\phi^{\rightarrow}_{\omega_R}(\emptyset) = \emptyset, \phi^{\rightarrow}_{\omega_R}(\{a\}) = \{z, w\}, \phi^{\rightarrow}_{\omega_R}(\{b\}) = \{x, z\},$$

$$\phi^{\rightarrow}_{\omega_R}(\{c\}) = \{x, y\}\phi^{\rightarrow}_{\omega_R}(\{a, b\}) = \{x, z, w\},$$

$$\phi^{\rightarrow}_{\omega_R}(\{b, c\}) = \{x, y, z\}, \phi^{\rightarrow}_{\omega_R}(\{a, c\}) = \phi^{\rightarrow}_{\omega_R}(X) = Y.$$

We obtain $\phi^{\leftarrow}_{\omega_R}(B) = \{a \in X \mid y \in B \to (a, y) \in R\}$;

$$\phi^{\leftarrow}_{\omega_R}(\emptyset) = \phi^{\leftarrow}_{\omega_R}(\{x\}) = \phi^{\leftarrow}_{\omega_R}(\{y\}) = \phi^{\leftarrow}_{\omega_R}(\{z\}) = \emptyset,$$

$$\phi^{\leftarrow}_{\omega_R}(\{w\}) = \phi^{\leftarrow}_{\omega_R}(\{x, w\}) = \emptyset,$$

$$\phi^{\leftarrow}_{\omega_R}(\{y, z\}) = \phi^{\leftarrow}_{\omega_R}(\{y, w\}) = \emptyset,$$

$$\phi^{\leftarrow}_{\omega_R}(\{x, y\}) = \phi^{\leftarrow}_{\omega_R}(\{x, y, w\}) = \{c\}, \phi^{\leftarrow}_{\omega_R}(\{z, w\}) = \{b\},$$

$$\phi^{\leftarrow}_{\omega_R}(Y) = X, \phi^{\leftarrow}_{\omega_R}(\{x, z\}) = \{b\}, \phi^{\leftarrow}_{\omega_R}(\{y, z, w\}) = \{a\},$$

$$\phi^{\leftarrow}_{\omega_R}(\{x, z, w\}) = \{a, b\}, \phi^{\leftarrow}_{\omega_R}(\{x, y, z\}) = \{b, c\}.$$

Thus, we obtain attribute oriented concepts

$$\{(\emptyset, \emptyset), (\{c\}, \{x, y\}), (\{b\}, \{x, z\}), (\{a\}, \{z, w\}),$$

$$(\{b, c\}, \{x, y, z\}), (\{a, b\}, \{x, z, w\})(X, Y)\}$$

(3) We obtain $\xi^{\leftarrow}_{\omega_R}(B) = \{a \in X \mid (\exists y \in B)((y \in B) \wedge ((a, y) \in R^c))\}$;

$$\xi^{\leftarrow}_{\omega_R}(\emptyset) = \emptyset, \xi^{\leftarrow}_{\omega_R}(\{w\}) = \{a\}, \xi^{\leftarrow}_{\omega_R}(\{y\}) = \{c\},$$

$$\xi^{\leftarrow}_{\omega_R}(\{z\}) = \xi^{\leftarrow}_{\omega_R}(\{z, w\}) = \{a, b\}, \xi^{\leftarrow}_{\omega_R}(\{y, w\}) = \{a, c\},$$

$$\xi^{\leftarrow}_{\omega_R}(\{x\}) = \xi^{\leftarrow}_{\omega_R}(\{x, y\}) = \{b, c\}, \xi^{\leftarrow}_{\omega_R}(\{y, z\}) = X$$

$$\xi^{\leftarrow}_{\omega_R}(\{y, z, w\}) = \xi^{\leftarrow}_{\omega_R}(\{x, w\}) = \xi^{\leftarrow}_{\omega_R}(\{x, z\}) = X,$$

$$\xi^{\leftarrow}_{\omega_R}(\{x, z, w\}) = \xi^{\leftarrow}_{\omega_R}(\{x, y, w\}) = X,$$

$$\xi^{\leftarrow}_{\omega_R}(\{x, y, z\}) = \xi^{\leftarrow}_{\omega_R}(Y) = X.$$

$$\xi^{\rightarrow}_{\omega_R}(A) = \{y \in Y \mid ((a, y) \in R^c \to (a \in A)\};$$

$$\xi^{\rightarrow}_{\omega_R}(\emptyset) = \xi^{\rightarrow}_{\omega_R}(\{b\}) = \emptyset, \xi^{\rightarrow}_{\omega_R}(\{c\}) = \{y\},$$

$$\xi^{\rightarrow}_{\omega_R}(\{b, c\}) = \{x, y\},\ \xi^{\rightarrow}_{\omega_R}(\{a\}) = \{w\},$$

$$\xi^{\rightarrow}_{\omega_R}(\{b, c\}) = \{x, y\}, \xi^{\rightarrow}_{\omega_R}(\{a, c\}) = \{y, w\}, \xi^{\rightarrow}_{\omega_R}(X) = Y.$$

Thus, we obtain object oriented concepts

$$\{(\emptyset, \emptyset), (\{c\}, \{y\}), (\{b, c\}, \{x, y\}), (\{a\}, \{w\})$$

$$(\{a, c\}, \{y, w\}), (\{a, b\}, \{z, w\}), (X, Y)\}$$

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