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Estimating exponentiated parameter and distribution of quotient and ratio in an exponentiated Pareto

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Abstract

We shall consider estimations of an exponetiated parameter of the exponentiated Pareto distribution with known scale and threshold parameters. A quotient distribution of two independent exponentiated Pareto random variables is obtained. We also derive the distribution of the ratio of two independent exponentiated Pareto random variables.

Keywords: Approximate maximum likelihood estimator, exponentiated Pareto distribution, generalized hypergeometric function, quotient, ratio.

1. Introduction

Let $F(x)$ be the density function of a continuous random variable X. Then $G(x) \equiv F^{\alpha}(x)$ is also a density function of a continuous random variable where α is a positive real number. Hence, the distribution $G(x)$ is called an exponentiated distribution of a given density function $F(x)$ (Gupta, 2001). Gupta (2001) considered an exponentiated exponential family. Ali *et al.* (2006) considered the exponentiated Weibull. Lee and Won (2006) considered inferences on the reliability in an exponentiated uniform distribution. Ali *et al.* (2007) introduced some exponentiated distributions. Ali and Woo (2010) considered estimations of the tail probability and the reliability in exponent.iated Pareto case. Ali *et al.* (2005) studied the ratio $X/(X+Y)$ for the power function distribution. Woo (2008) considered estimations for the reliability and the distribution of ratio in two independent different variates. Moon *et al.* (2009) considered the reliability and the ratio in two exponentiated complementary power function distributions. Lee and Lee (2010) studied the reliability and the ratio in a right truncated Rayleigh distribution

In this paper, we consider estimations of the exponentiated parameter in the exponentiated Pareto distribution with known scale and shape parameters. We then consider distributions of the quotient X/Y and the ratio $X/(X+Y)$ in two independent exponentiated Pareto random variables X and Y.

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2. Estimation of the exponentiated parameter

From the density function of the Pareto distribution an exponentiated Pareto distribution is as given in Ali *et al*. (2007, 2010):

$$G(x) = [1 - (\beta/x)^c]^a, \quad x \geq \beta, \quad \alpha > 0, \; c > 0, \tag{2.1}$$

which the Pareto distribution is as given in Johnson *et al*. (1994), where parameters α and β are called the exponentiated parameter and the threshold parameter, respectively.

When α and c in the density function (2.1) are known, estimation problem of threshold parameter β was considered by Ali and Woo (2010). Here, we consider estimations of an exponentiated parameter in the density function (2.1) with known parameters c and β.

Let us first consider a power function random variable with the density (2.2) in Ali *et al*. (2005) given by:

$$p(x) = \alpha x^\alpha - 1, \quad 0 < x < 1, \quad \alpha \geq 1. \tag{2.2}$$

In considering estimation of the parameter α, we introduce the following well-known Lemma:

Lemma 1. Let $Z_1, Z_2, ..., Z_n$ be a random sample from the density (2.2). Then

(a) The maximum likelihood estimation (MLE) $\widehat{\alpha}$ of α is $\widehat{\alpha} = n/(-\sum_{i=1}^n lnZ_i)$, whose the mean and the variance are $n\alpha/(n-1)$ and $n^2\alpha^2/((n-1)^2(n-2))$, $n > 2$, respectively.

(b) An unbiased estimator $\tilde{\alpha}$ of α is $\tilde{\alpha} = (n-1)/(-\sum_{i=1}^n lnZ_i)$, whose the variance is $\alpha^2/(n-1)$, $n > 1$.

(c) The unbiased estimator performs better in the sense of mean squared errors (MSE) than the MLE.

From the method of obtaining the approximate maximum likelihood estimator (AML) as in Balakrishnan and Cohen (1991), we obtain AML of α in the density (2.2) as follows:

$$\widehat{\alpha}(d) = 2d - d^2(-\sum_{i=1}^n lnZ_i/n), \tag{2.3}$$

where d is any real number.

Especially from Lemma 1 (a) and (2.3), it's clear that $\widehat{\alpha}(d) = \widehat{\alpha}$, when d is the MLE $\widehat{\alpha}$ of α. If d is the unbiased estimator $\widehat{\alpha}$ in Lemma 1 (b), then from (2.3) we introduce an AML of α as follows:

$$\widehat{\alpha}(\widetilde{\alpha}) = (n-1)(n^2+1)/(-n\sum_{i=1}^n lnZ_i), \tag{2.4}$$

whose the mean and the variance are given by:

$$E(\widehat{\alpha}(\widetilde{\alpha})) = (n+1)\alpha/n \text{ and } Var(\widehat{\alpha}(\widetilde{\alpha})) = (n+1)^2\alpha^2/(n^2(n-2)), n > 2. \tag{2.5}$$

From Lemma 1 (a) and (b) and (2.5), we obtain the following:

Fact 1. The AML $\widehat{\alpha}(\widetilde{\alpha})$ performs better in a sense of MSE than the MLE $\widehat{\alpha}$, but the AML $\widehat{\alpha}(\widetilde{\alpha})$ performs worse than the unbiased estimator $\tilde{\alpha}$.

Assume $X_1, X_2, ..., X_n$ be a random sample from the density function (2.1) with known c and β.

Then using a transformation, we obtain the following:

Fact 2. $Z_1 \equiv 1 - (\beta/X_1)^c$ follows a power function distribution as in (2.2).

From Lemma 1 (b) and Facts 1 and 2, among three introduced estimators above, the first favorable estimator of an exponentiated parameter α in the density function (2.1) is given as:

$$\widetilde{\alpha} = \frac{(n-1)}{[-\sum_{i=1}^{n} \ln(1-(\beta/X_i)^c)]},$$

whose the mean and the variance are as given in Lemma 1 (b).

From (2.4), the second favorable estimator of α is given by

$$\widehat{\alpha}(\widetilde{\alpha}) = \frac{(n-1)(n^2+1)}{[-n\sum_{i=1}^{n} \ln(1-(\beta/X_i)^c)]},$$

whose the mean and the variance are as given in (2.5).

3. Distribution of the quotient X/Y

Let X and Y be independent exponentiated Pareto random variables having the density function (2.1) each with parameters (α_1, β_1, c) and (α_2, β_2, c), respectively.

Then using formula 3.197(3) in Gradshteyn and Ryzhik (1965), the distribution of the quotient $Q = X/Y$ is obtained as follows:

$$F_Q(z) = F(-\alpha_2, 1; \alpha_1+1; (\frac{\beta_2}{\beta_1}\cdot\frac{1}{z})^c), \quad \text{if} \quad z \geq \beta_2/\beta_1, \tag{3.1}$$

where $F(a, b; c; z)$ is the hypergeometric function.

From (3.1) and formula 15.2.1 in Abramowitz and Stegun (1970), we obtain the density of the quotient Q as follows:

$$f_Q(z) = \frac{c\alpha_2}{1+\alpha_1}(\frac{\beta_2}{\beta_1})^c \cdot z^{-c-1} \cdot F(-\alpha_2, 1; \alpha_1+1; (\frac{\beta_2}{\beta_1}\cdot\frac{1}{z})^c), \quad \text{if} \quad z \geq \beta_2/\beta_1. \tag{3.2}$$

From the density (3.2) and 7.512(5) in Gradshteyn and Ryzhik (1965), the k-th moment of the quotient Q is obtained as follows:

$$E(Q^k) = (\frac{\alpha_2}{1+\alpha_1})(\frac{\beta_2}{\beta_1})^k(1-\frac{k}{c})^{-1}F(1-\alpha_2, 2, 1-\frac{k}{c}; 2+\alpha_1, 2-\frac{k}{c}; 1), \quad \text{if } k < c, \tag{3.3}$$

where $F(a, b, c; p, q; 1)$ is the generalized hypergeometric function in Gradshteyn and Ryzhik (1965).

From (3.3) and formula 9.14(1) in Gradshteyn and Ryzhik (1965), Table 1 provides the asymptotic means, variances, and coefficients of skewness of the density (3.2) for $c = 5$. From Table 1 in the Appendix, we observe the following:

Fact 3. For the density (3.2) with $c = 5$,
(a) the density is skewed to the right when (α_1, α_2) are as given in Table 1.
(b) the density with $\alpha_1 > \alpha_2$ has smaller mean and variance than the density with $\alpha_1 < \alpha_2$ has.

Let $z = 1$ in the density function $F_Q(z)$ in (3.1). Then we obtain the following reliability:

$$R = P(Y < X) = \begin{cases} F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1})^c), \text{ if } \beta_1 \geq \beta_2 \\ 1 - F(-\alpha_1, 1; \alpha_2 + 1; (\frac{\beta_1}{\beta_2})^c), \text{if } \beta_2 > \beta_1, \end{cases}$$

which is an extension of reliability in Ali and Woo (2010).

4. Distribution of the Ratio $X/(X + Y)$

In this section, we consider the distribution of the ratio $R = X/(X + Y)$, where X and Y are independent exponentiated Pareto random variables having the density function (2.1) each with parameters (α_1, β_1, c) and (α_2, β_2, c), respectively.

In a similar manner as in Section 3, using formula 3.197(3) in Gradshteyn and Ryzhik (1965) we obtain the cdf of the ratio R as follows:

$$F_R(r) = 1 - F(-\alpha_2, 1; \alpha_1 + 1; (\frac{\beta_2}{\beta_1} \cdot \frac{t}{1} - t)^c), \quad 0 < t < \frac{\beta_1}{\beta_1} + \beta_2. \tag{4.1}$$

From (4.1) and formula 15.2.1 in Abramowitz and Stegun (1970), we obtain the density of the ratio R as follows:

$$f_R(t) = (\frac{c\alpha_2}{\alpha_1 + 1})(\frac{\beta_2}{\beta_1})^c \frac{t^{c-1}}{(1 - t)^{c+1}}.$$

$$F(1 - \alpha_2, 2; \alpha_{1+2}; (\frac{\beta_2}{\beta_1} \cdot \frac{t}{1 - t})^c), \text{ if } 0 < t < \frac{\beta_1}{\beta_1 + \beta_2}. \tag{4.2}$$

From (4.2), formulas 1.110 and 7.512(5) in Gradshteyn and Ryzhik (1965), the k-th moment of the ratio R is obtained as follows:

$$E(R^k) = \frac{c\alpha_2}{\alpha_1 + 1} \sum_{i=0}^{\infty} \frac{{}_{(-k)}P_i}{i!(k + i + c)} (\frac{\beta_1}{\beta_2})^{k+i}.$$

$$F(1 - \alpha_2, 2, (k + c + i)/c; 2 + \alpha_1, (k + 2c + i)/c; 1), \quad \text{if } \beta_2 > \beta_1, \tag{4.3}$$

where ${}_aP_i = a(a-1)(a-2)\cdots(a-i+1)$ and ${}_aP_0 \equiv 1$.

From (4.3) and formula 9.14(1) in Gradshteyn and Ryzhik (1965), Table 2 in the Appendix provides the asymptotic means, variances, and coefficients of skewness of the density (4.2) for $c = 5$ and $\beta_1 = 1$ and $\beta_2 = 2$. From Table 2, we observe the following:

Fact 4. For the density (4.2) with $c = 5$ and $\beta_1 = 1$ and $\beta_2 = 2$,

(a) the density (4.2) is skewed to the right when $\alpha_1 > \alpha_2$, otherwise it's skewed to the left.

(b) its mean and variance are slightly increasing as α_1 increases when $\alpha_1 > \alpha_2$, but are slightly decreasing as α_1 increases when $\alpha_1 < \alpha_2$.

Appendix

Appendix 1.1 Asymptotic mean, variance, and coefficient of skewness of the density (3.2) with $c = 5$(units of mean and variance areβ_2/β_1 and, $(\beta_2/\beta_1)^2$ respectively)

(α_1, α_2)	mean	variance	skewness
(1.5, 3.5)	1.20386	0.18101	2.73908
(3.5, 1.5)	0.91806	0.10180	2.66768
(3.5, 5.5)	1.14940	0.18179	2.59641
(5.5, 3.5)	0.97778	0.13092	2.58397
(5,5, 7.5)	1.12560	0.18122	2.55614
(7.5, 5.5)	1.00342	0.14379	2.55186
(7.5, 9.5)	1.11269	0.18079	2.53732
(9.5, 7.5)	1.01776	0.15115	2.53535

Appendix 1.2 Asymptotic mean, variance, and coefficient of skewness of the density (4.2) with $c = 5$and $\beta_1 = 1$ and$\beta_2 = 2$.

(α_1, α_2)	mean	variance	skewness
(1.5, 3.5)	0.19267	0.01719	-0.66526
(3.5, 1.5)	0.08694	0.01809	0.94283
(3.5, 5.5)	0.16872	0.01926	-0.30083
(5.5, 3.5)	0.11085	0.01996	0.54016
(5,5, 7.5)	0.15947	0.01973	-0.16961
(7.5, 5.5)	0.11968	0.02026	0.40410
(7.5, 9.5)	0.15456	0.01991	-0.10117
(9.5, 7.5)	0.12425	0.02033	0.33500

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