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Estimation for ordered means in normal distributions

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Abstract

In this paper, we obtain the restricted maximum likelihood estimators (RMLE's) for means in normal distributions with the ordered mean constraints. The biases and mean squared errors (MSE's) of these RMLE's are approximated by Mote Carlo methods. In every case a substantial savings in MSE is obtained at the expense of a small loss in bias when using RMLE's instead of the unrestricted MLE's.

Keywords: MLE, ordered means, restricted maximum likelihood estimator (RMLE).

1. Introduction

When we know explicit information about any parameters we want to find the estimators under these informations. Hence it may be expected that the efficiency of our estimation is increased by taking such information into account.

Many papers have been written on this topic (Barlow *et al.*, 1972; Cho and Cho, 2006; Cho *et al.*, 2005; Cho and Kim, 2003; Croydon, 2010; Jeong *et al.*, 2009; Kallenberg, 2002; Rao, 1965).

Maximum likelihood estimation of increasing mean of the normal distributions was studied in Bartholomew (1959).

Maximum likelihood estimators for Poisson parameters consisting of $\lambda,\ \lambda_i,\ \lambda_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i$ subject to (a) $\widehat{\lambda} \geq \sum_{i=1}^{n} \widehat{\lambda}_i$, and subject to (a) and (b) $\widehat{\lambda}_i \geq \sum_{j=1}^{n_i} \widehat{\lambda}_{ij},\ j = 1,\ 2,\ \cdots,\ n_i$ are considered by Richard and Richard (1976).

In this paper, we obtain the restricted maximum likelihood estimators (RMLEs) for means in the normal distributions with constraints as follows. Let $x,\ x_i,\ x_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i$, *etc.*, be independently normal random variables with means $\mu,\ \mu_i,\ \mu_{ij},\ \cdots$, and common variance σ^2. We can consider the parameter space with $\mu \geq \sum_{i=1}^{n} \mu_i,\ \mu_i \geq \sum_{j=1}^{n_i} \mu_{ij},\ i = 1,\ 2,\ \cdots,\ n,$ *etc.* However, in the restricted maximum likelihood estimation, the sample estimates may turn out in reverse order. Thus we find the RMLE's with restrictions (1) $\widehat{\mu} \geq \sum_{i=1}^{n} \widehat{\mu}_i$ and (2) $\widehat{\mu}_i \geq \sum_{j=1}^{n_i} \widehat{\mu}_{ij},\ i = 1,\ 2,\ \cdots,\ n,$ *etc.*, to remove this objectionable characteristics. Also, we compare the proposed RMLE's with the unrestricted maximum likelihood estimators in the sense of the biases and mean squared errors of these estimators through the Mote Carlo methods.

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2. Restricted maximum likelihood estimators

Let $x,\ x_i,\ x_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i$, *etc.*, be independently normal random variables with means $\mu,\ \mu_i,\ \mu_{ij},\ \cdots$, and common variance σ^2.

Consider the random samples of size p from each population, say,

$$
\begin{aligned}
&x_{(1)},\ x_{(2)},\ \cdots,\ x_{(p)},\\
&x_{i(1)},\ x_{i(2)},\ \cdots,\ x_{i(p)},\ i = 1,\ 2,\ \cdots,\ n\\
&x_{ij(1)},\ x_{ij(2)},\ \cdots,\ x_{ij(p)},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots\ n_i,
\end{aligned}
$$

etc.

Then the unrestricted maximum likelihood estimators for these means are the sample means, that is,

$$
\begin{aligned}
&\sum_{k=1}^{p} x_{(k)}/p,\ \text{denote}\ \bar{x},\ \text{for}\ \mu\\
&\sum_{k=1}^{p} x_{i(k)}/p,\ \text{denote}\ \bar{x}_i,\ \text{for}\ \mu_i,\ i = 1,\ 2,\ \cdots,\ n,\\
&\sum_{k=1}^{p} x_{ij(k)}/p,\ \text{denote}\ \bar{x}_{ij},\ \text{for}\ \mu_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i,
\end{aligned}
$$

etc.

We wish to find the RMLE's with restrictions

$$\widehat{\mu} \geq \sum_{i=1}^{n} \widehat{\mu}_i \tag{2.1}$$

and

$$\widehat{\mu}_i \geq \sum_{j=1}^{n_i} \widehat{\mu}_{ij},\ i = 1,\ 2,\ \cdots,\ n, \tag{2.2}$$

etc.

For simplicity we find the RMLE's in the case of three stages.

For three stages, the log likelihood function is

$$
\begin{aligned}
\ln L(U_2, \sigma^2, X_2) = &- \frac{p(1 + n + \sum_{i=1}^{n} n_i)}{2}\ \ln 2\pi - \frac{p(1 + n + \sum_{i=1}^{n} n_i)}{2}\ \ln \sigma^2\\
&- \frac{1}{2\sigma^2} \sum_{k=1}^{p} (x_{(k)} - \mu)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{k=1}^{p} (x_{i(k)} - \mu_i)^2\\
&- \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{p} (x_{ij(k)} - \mu_{ij})^2
\end{aligned} \tag{2.3}
$$

where

$$U_2' = (\mu, \mu_1, \cdots, \mu_n, \mu_{11}, \mu_{12}, \cdots, \mu_{n1}, \mu_{n2}, \cdots, \mu_{nn_n}),$$
$$X_2' = (x_{(1)}, \cdots, x_{1(1)}, \cdots, x_{11(1)}, \cdots, x_{nn_n(p)}).$$

It can be shown clearly that the log likelihood function is a strictly concave functions. We consider various special cases.

Case 1 . If $\bar{x} \geq \sum_{i=1}^{n} \bar{x}_i$ and $\bar{x}_i \geq \sum_{j=1}^{n_i} \bar{x}_{ij}$ for all i, then

$$\begin{aligned} &\widehat{\mu} = \bar{x}, \\ &\widehat{\mu}_i = \bar{x}_i,\ i = 1,\ 2,\ \cdots,\ n, \\ &\widehat{\mu}_{ij} = \bar{x}_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i. \end{aligned} \tag{2.4}$$

Case 2 . If $\bar{x} \geq \sum_{i=1}^{n} \bar{x}_i$ but $\bar{x}_i < \sum_{j=1}^{n_i} \bar{x}_{ij}$ for some i, say, $i = 1,\ 2,\ \cdots,\ l$, then

$$\begin{aligned} &\widehat{\mu} = \bar{x}, \\ &\widehat{\mu}_i = \bar{x}_i,\ i = l+1,\ 2,\ \cdots,\ n, \\ &\widehat{\mu}_{ij} = \bar{x}_{ij},\ i = l+1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i, \\ &\widehat{\mu}_i = \bar{x}_i - (\bar{x}_i - \sum_{j=1}^{n_i} \bar{x}_{ij})/(n_i+1), i = 1, 2,\ \cdots, l, \\ &\widehat{\mu}_{ij} = \bar{x}_{ij} + (\bar{x}_i - \sum_{j=1}^{n_i} \bar{x}_{ij})/(n_i+1), i = 1, 2,\ \cdots, l, j = 1, 2,\ \cdots, n_i. \end{aligned} \tag{2.5}$$

If (2.5) satisfies (2.1) and (2.2), we are done.
However, since $\widehat{\mu}_i > \bar{x}_i,\ i = 1,\ 2,\ \cdots,\ l$, (2.1) may fail.
If (2.1) fails, proceed as in Case 4.

Case 3 . If $\bar{x} < \sum_{i=1}^{n} \bar{x}_i$ and $\bar{x}_i \geq \sum_{j=1}^{n_i} \bar{x}_{ij}$ for all i, then

$$\begin{aligned} &\widehat{\mu} = \bar{x} - (\bar{x} - \sum_{i=1}^{n} \bar{x}_i)/(n+1), \\ &\widehat{\mu}_i = \bar{x}_i + (\bar{x} - \sum_{i=1}^{n} \bar{x}_i)/(n+1), i = 1, 2,\ \cdots, n, \\ &\widehat{\mu}_{ij} = \bar{x}_{ij},\ i = 1,\ 2,\ \cdots,\ n,\ j = 1,\ 2,\ \cdots,\ n_i. \end{aligned} \tag{2.6}$$

If (2.6) satisfies (2.1) and (2.2), we are done. However, since $\widehat{\mu}_i < \bar{x}_i$, it may that (2.2) fails to hold for some i. If (2.2) fails for some i, proceed as in Case 4.

Case 4 . If $\bar{x} < \sum_{i=1}^{n} \bar{x}_i$ and $\bar{x}_i < \sum_{j=1}^{n_i} \bar{x}_{ij}$ for $i = 1,\ 2,\ \cdots,\ l$, or if (2.1) fails for Case 2, or if (2.2) fails for Case 3, then (2.3) can be converted as follows.

$$\begin{aligned}
\ln L(U_2, \sigma^2, X_2) = &-\frac{p(1+n+\sum_{i=1}^{n} n_i)}{2} \ln 2\pi - \frac{p(1+n+\sum_{i=1}^{n} n_i)}{2} \ln \sigma^2 \\
&- \frac{1}{2\sigma^2} \sum_{k=1}^{p} (x_{(k)} - \sum_{i=1}^{l} \sum_{j=1}^{n_i} \mu_{ij} - \sum_{i=l+1}^{n} \mu_i)^2 \\
&- \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{k=1}^{p} (x_{i(k)} - \sum_{j=1}^{n_i} \mu_{ij})^2 - \frac{1}{2\sigma^2} \sum_{i=l+1}^{n} \sum_{k=1}^{p} (x_{i(k)} - \mu_i)^2 \\
&- \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{p} (x_{ij(k)} - \mu_{ij})^2.
\end{aligned} \tag{2.7}$$

Setting the partial derivative with respect to each parameter equal to zero, we obtain the following estimators

$$\begin{aligned}
&\widehat{\mu}_{ij} = (\bar{x} + \bar{x}_i + \bar{x}_{ij}) - (\widehat{\mu} + \widehat{\mu}_i),\ i = 1,\ 2,\ \cdots,\ l,\ j = 1,\ 2,\ \cdots,\ n_i, \\
&\widehat{\mu}_{ij} = \bar{x}_{ij},\ i = l+1,\ 2,\ \cdots,\ l,\ j = 1,\ 2,\ \cdots,\ n_i, \\
&\widehat{\mu}_i = (n_i \bar{x} + n_i \bar{x}_i + \sum_{j=1}^{n_i} \bar{x}_{ij})/(n_i + 1) - \frac{n_i}{n_i + 1} \widehat{\mu},\ i = 1,\ 2,\ \cdots,\ l, \\
&\widehat{\mu}_i = (\bar{x} + \bar{x}_i) - \widehat{\mu},\ i = l+1,\ \cdots,\ n, \\
&\widehat{\mu} = (S + T)/[\sum_{i=1}^{l} (n_i/(n_i + 1)) + (n - l + 1)],
\end{aligned}$$

where

$$\begin{aligned}
S &= \sum_{i=1}^{l} [n_i \bar{x} + n_i \bar{x}_i + \sum_{j=1}^{n_i} \bar{x}_{ij})/(n_i + 1)], \\
T &= \sum_{i=l+1}^{n} (\bar{x} + \bar{x}_i).
\end{aligned} \tag{2.8}$$

Also, we have

$$\hat{\sigma}^2 = \frac{1}{p(1+n+\sum_{i=1}^{n} n_i)} [\sum_{k=1}^{p} (x_{(k)} - \widehat{\mu})^2 + \sum_{i=1}^{n} \sum_{k=1}^{p} (x_{i(k)} - \widehat{\mu})^2 + \sum_{i=1}^{n} \sum_{j=1}^{n_i} \sum_{k=1}^{p} (x_{ij(k)} - \widehat{\mu_{ij}})^2].$$

By the very way these solutions were found it follows that (2.1) is satisfied and (2.2) is satisfied for $i = 1,\ 2,\ \cdots,\ l$. However, it may be that (2.2) is violated for one or more values of $i > 1$.

If, after performing the procedure described in Case 4, variables $1,\ 2,\ \cdots,\ l$ are violators, we find that variables $l+1$ and $l+2$ have become violators.

Assume that U and A denote column vectors and that $f(\cdot)$ is a real valued function defined on an appropriate subset of R^n.

Theorem 2.1 (Richard and Richard (1976)) Assume that $f(U)$ is a strictly concave function and that U_k maximizes $f(U)$ subject to the constraints

$$A_i'U \leq b_i,\ i = 1,\ 2,\ \cdots,\ k.$$

If the additional constraint

$$A_{k+1}'U \leq b_k + 1$$

is imposed, then

$$\begin{aligned}&(1)\ \text{If } A_{k+1}'U_k \leq b_k + 1, U_k + 1 = U_k : \\ &(2)\ \text{If } A_{k+1}'U_k > b_k + 1, A_k + 1'U_k + 1 = b_k + 1.\end{aligned}$$

Corollary 2.2 (Richard and Richard, 1976) Theorem 2.1 still holds if the constraints $A_i'U \leq b_i$ are replaced by $A_i'U = b_i$ for $i \in B$ where $B \subset \{1,\ 2,\ \cdots k\}$.

Assume that U indicates the RMLE. Moreover, denote the constraint in (2.1) by C_0, and the constraint in (2.2) by $C_1,\ C_2,\ \cdots,\ C_n$, respectively. We shall say a constraint is imposed on a RMLE if equality actually holds in that constraint in the RMLE. Let F denote the set of constraints imposed in constructing U, i.e., $F = \{C_i :$ equality holds in C_i for $U\}$.

Theorem 2.3 Let $\tilde{U}$ denote the intermediate solution for the RMLE's obtained by imposed constraints C where $C \subset F$. Then if $\tilde{U}$ violates $C_i,\ C_i \in F$.

Proof: Consider first the case where $\tilde{U}$ violates C_0 and $C_0 \notin F$. Then

$$\widehat{\mu} > \sum_{i=1}^{n} \widehat{\mu}_i,$$

and hence

$$\widetilde{\mu} = \widehat{\mu} = \bar{x}.$$

Since $C \subset F$, $\widehat{U}$ is obtained by imposing lower levels, so that $\widehat{\mu}_i \geq \widetilde{\mu}_i,\ i = 1,\ 2,\ \cdots,\ n$. This leads to an immediate contradiction.

Next, consider the case where $\tilde{U}$ violates $C_i\ (i > 0)$. If $C_0 \notin F$, i.e., $\widehat{\mu} > \sum_{i=1}^{n} \mu_i$, then $\tilde{U}$ and $\widehat{U}$ may be obtained by considering collections of unrelated two-stage problems. In this case,

$$\widetilde{\mu}_{ij} = \bar{x}_{ij} \text{ and } \widetilde{\mu}_i = \bar{x}_i,$$

so if $\tilde{U}$ violates $C_i\ (i > 0)$, then $C_i \in F$.

Thus, it now suffices to consider the case where $C_0 \in F$. In fact, it suffices to consider $C_0 \in C$.

To see this, let U' denote the RMLE obtained by imposing C_0 in addition to those constraints in C. In this event, since we must have $\mu' \geq \bar{x}$,

$$\mu'_i = (\bar{x} + \bar{x}_i) - \mu' \leq \bar{x}_i = \widetilde{\mu}_i.$$

However, $\mu'_{ij} = \widetilde{\mu}_{ij} = \bar{x}_{ij}$, so that U' violates C_i whenever $\tilde{U}$ does.

Now let us construct the new intermediate solution, say, U^* by imposing all constraints in F with the exception of C_i. Note that in obtaining $hatU$, it would no matter if $\leq$ were replaced by $=$ for those constraints in any subset of F. Thus if

$$\sum_{j=1}^{n_i} \mu^*{}_{ij} > \mu^*{}_i, \tag{2.9}$$

the Corollary 2.1 guarantees that $C_i \in F$.

Moreover, it will suffice to show that (2.9) holds when we impose only one additional constraint belonging to F, since we can repeat the argument as many times as needed and then use the above reasoning. Let the violators be relabeled, i.e., $C = \{C_0,\ C_1,\ \cdots,\ C_l\}$, and let $C_l + 1$ be the additional constraints imposed in constructing U^*. Then we may assume that $\tilde{U}$ violates $C_l + 1$, since if no other constraint except C_i was violated, repeated use of the Corollary 2.2 and (2.9) would ensure that $C_i \in F$.

It will suffice to show that $\mu^*{}_i \leq \widetilde{\mu}_i$, since then (2.9) must hold. Also, by the formulas of $\mu^*{}_i$ and $\widetilde{\mu}_i$, it will suffice to show that

$$\mu^* \geq \widetilde{\mu}.$$

Assume that $\mu^* < \widetilde{\mu}$. Then $\sum_{j=1}^{n_i} \widetilde{\mu}_{l+1j} < \widetilde{\mu}_l + 1$. This is inconsistent with the assumption that $\tilde{U}$ violates $C_l + 1$. □

From Theorem 2.3, the optimal strategy is to repeat the procedure of Case 4 taking variables $1,\ 2,\ \cdots,\ l,\ l+1,\ l+2$ as violators.

Of course it may happen that new violators are found in this case as well.

However, since n is a finite number, we can be sure that this procedure will lead to a solution after a finite number of repetitions.

Similarly, the techniques used here would apply to four or more stages.

3. Simulated results

In this section, we compare the proposed RMLE's with the unrestricted maximum likelihood estimators in the sense of the biases and mean squared errors of these estimators through the Mote Carlo methods. For each set of parameters 10,000 trials were run.

Table 3.1 Bias and MSE for $\widehat{\mu}$ - two stages

μ	Bias	MSE
50	1.89 (0.13)	38.6 (1.35)
100	2.73 (0.18)	74.2 (1.52)
150	3.51 (0.19)	117.3 (2.02)
200	3.78 (0.22)	147.6 (2.78)

Table 3.2 Bias and MSE for $\widehat{\mu}_i$ - two stages

μ	μ_i	Bias	MSE
100	253540	-0.62 (0.09) -0.87 (0.10) -1.09 (0.12)	20.7 (0.81) 31.1 (0.82) 34.6 (0.97)
200	507080	-1.01 (0.12) -1.42 (0.15) -1.53 (0.17)	46.8 (1.21) 65.6 (1.38) 67.1 (1.62)

Table 3.3 Bias and MSE for $\widehat{\mu}, \widehat{\mu}_i, \widehat{\mu}_{ij}$ - three stages

Level	Parameter	Bias	MSE
Top (μ)	150	5.12 (0.02)	109.51 (3.77)
	200	5.63 (0.24)	135.10 (4.69)
	50	0.74 (0.12)	27.32 (0.91)
Middle (μ_i)	50	0.79 (0.13)	28.77 (0.94)
	100	0.51 (0.17)	45.17 (1.52)
	150	-0.07 (0.19)	66.56 (2.09)
	20	-1.08 (0.09)	16.42 (0.50)
Bottom (μ_{ij})	20	-0.92 (0.09)	16.68 (0.52)
	30	-1.43 (0.11)	23.71 (0.84)
	30	-1.40 (0.11)	24.87 (0.83)
	35	-1.31 (0.12)	31.98 (0.94)
	65	-2.23 (0.15)	52.30 (1.68)
	50	-1.59 (0.14)	42.13 (1.29)
	100	-3.12 (0.20)	82.26 (2.78)

Table 3.4 Parameters used in 3-stage Monte Carlo study

μ	μ_1	μ_{11}	μ_{12}	μ_2	μ_{21}	μ_{22}
150	100	35	65	50	20	30
200	150	100	50	50	20	30

We see that a relatively large savings in MSE can be had in exchange for a relatively small loss in bias. In the case for two stages, we know that the estimator for μ has a positive bias while the estimators for μ_{ij} have a negative bias. However, the Monte Carlo results indicate that the bias of $\widehat{\mu}_i$ is rather small relative to the bias in $\widehat{\mu}$ and $\widehat{\mu}_{ij}$ with a considerable savings in MSE.

From Table 3.1 to Table 3.3, we see that a substantial saving in MSE is obtained at the expense of a small loss in bias when we use RMLE's instead of the unrestricted MLE's in every case.

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