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Reference priors for two parameter exponential stress-strength model

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Abstract

In this paper, we develop the noninformative priors for the reliability in a stress-strength model where a strength X and a stress Y have independent exponential distributions with different scale parameters and a common location parameter. We derive the reference priors and prove the propriety of joint posterior distribution under the general prior including the reference priors. Through the simulation study, we show that the proposed reference priors match the target coverage probabilities in a frequentist sense.

Keywords: Exponential distribution, nonregular case, reference prior, stress-strength model.

1. Introduction

The exponential distribution plays an important role in the field of life testing and reliability. The reasons for using the exponential distribution assumption in reliability applications can be found in the early work of Davis (1952), Epstein and Sobel (1953), and others. Further justification, in the form of theoretical arguments to support the use of the exponential distribution as the failure law of complex equipment, is presented in the book by Barlow and Proschan (1975) and Lawless (2003).

The exponential distribution with location and scale parameters is a nonregular family of distribution. This distribution is very useful in describing data with a threshold time. This distribution is easy to deal with and has received considerable theoretical attention. The maximum likelihood estimates of this model is easy to obtain (Lawless, 2003). This distribution is also called two parameter exponential distribution.

The problem of making inference about stress-strength model has received a considerable attention in literature. When a stress exceeds a strength, an item will be failed to function.

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The probability that an item functions properly is the measure of confidence of the item. So, to make statistical inference about this probability is very important.

Let X be the random variable which describes the life span of an item, and Y be the random variable which affects the lifetime of an item giving some stress. Then, the probability of functioning can be written by

$$R = P(Y < X).$$

About statistical inference for R related with two parameter exponential distribution, Beg (1980) obtained the minimun variance unbasied estimatior (M.V.U.E.) of R when X and Y are independent exponential random variables with unequal scale and unequal location parameters. Bai and Hong (1992) discussed estimation of R in the exponential case with common location parameter. Baklizi and El-Masri (2004) considered a bayesian estimation of reliability R, where X and Y have independent exponential distributions with the common location parameter. Krishnamoorthy, Mukherjee and Guo (2007) proposed the test and interval estimation procedures based on the generalized variable approach for the reliability R when X and Y have independent exponential distributions with unequal scale and unequal location parameters.

The studies, mentioned above except Baklizi and El-Masri (2004), are related with frequentist's approach. A Bayesian approach with noninformative prior is an attractive method in statistical inferences. There may be a situation when one forces to use noninformative prior, because of lack of prior information. Another reason is that a noninformative prior is an objective prior. So, if one uses a noninformative prior, it doesn't need to study the robustness of inference under the hyper parameters. But, because a noninformative prior is frequently improper density, one must prove the propriety of posterior density when one adopts a noninformative prior.

Despite of this point, many authors have tried to develop noninformative priors in various statistical models. The performance of a noninformative prior in statistical inference was good. There were two efficient ways of developing noninformative priors. One is the reference prior of Bernardo (1979) which is extended by Berger and Beranrdo (1989, 1992) to a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. The other is a probability matching prior initiated by Welch and Peers (1963) which matches the posterior quantile of parameter of interest to frequentist coverage probability asymptotically. Lately, Stein (1985, 1994), Tibshirani (1989), DiCiccio and Stern (1994), Datta and Ghosh (1995), Datta (1996), Kim *et al.* (2009a, 2009b) and Mukerjee and Ghosh (1997) developed probability matching priors and studied their properties in many statistical models.

A statistical model with threshold parameter or guarantee time parameter is an useful model when lifetime of an item is always longer than a given threshold time. To analyze these data, we need inclusion of such a parameter in the model. Occasionally, this model is natural extension of a regular family of distribution. Specially, in Bayesian analysis, objective priors like reference priors or probability matching priors were developed when the family of distribution is regular. But, But related with a nonregular distribution, there were only a little work developing objective priors. Ghosal and Samanta (1995, 1997) developed the reference priors for the case of one parameter families of discontinuous densities in the sense of Bernardo (1979). Ghosal (1997, 1999) proposed the general method to find the reference priors for the multiparameter nonregular cases when, except a threshold parameter, the distribution under consideration is regular with respect to the rest of parameters. Based on his results, recently, Kang *et al.* (2008, 2010) developed the reference priors for exponential and half-normal distributions and showed that the proposed reference prior matches the target coverage probabilities very well.

In this paper, we want to develop the reference priors for R on the purpose of objective Bayesian analysis, when X and Y have two independent two parameter exponential distributions with the common location parameter. As examples of this model, when X and Y are lifetimes of two devices with same guarantee time which are distributed as two parameter exponential distribution, R is the probability of one device failing before the other. Another example is that, when X represents the strength of a chamber against a pressure and Y is a operating pressure which gives stress to the chamber, R is the probability of operating properly with operating pressure.

Consider X and Y have independent two parameter exponential distributions with scale parameters λ_1 and λ_2, respectively and a common location parameter μ. Then the probability density functions of X and Y are given by

$$f(x|\mu,\lambda_1) = \lambda_1^{-1}\exp\left\{-\frac{x-\mu}{\lambda_1}\right\}, x \ge \mu, \mu \ge 0, \lambda_1 > 0, \tag{1.1}$$

and

$$f(y|\mu,\lambda_2) = \lambda_2^{-1}\exp\left\{-\frac{y-\mu}{\lambda_2}\right\}, y \ge \mu, \mu \ge 0, \lambda_2 > 0, \tag{1.2}$$

respectively. The reliability $R = P(Y < X)$ is given by $\lambda_1/(\lambda_1+\lambda_2)$. The present paper focuses on the reference priors for the reliability parameter R.

The outline of the remaining sections is as follows. In Section 2, we develop reference priors for the reliability parameter. In Section 3, we provide the propriety of the posterior distribution for the general prior including the reference priors. In Section 4, simulated frequentist coverage probabilities under the derived priors are given.

2. The reference priors

In this section, we will develop the reference priors for different groups of orderings of importance by following Ghosal (1997).

Let X_i, $i = 1,\cdots,n_1$, denote observations from two parameter exponential distribution with parameters μ and λ_1, and Y_i , $i = 1,\cdots,n_2$, denote observations from two parameter exponential distribution with parameters μ and λ_2. Then likelihood function is given by

$$f(\mathbf{x},\mathbf{y}|\mu,\lambda_1,\lambda_2) = \lambda_1^{-n_1}\lambda_2^{-n_2}\exp\left\{-\frac{1}{\lambda_1}\sum_{i=1}^{n_1}(x_i-\mu) - \frac{1}{\lambda_2}\sum_{i=1}^{n_2}(y_i-\mu)\right\}, \tag{2.1}$$

where $\mu \ge 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Let

$$\theta_1 = \frac{\lambda_1}{\lambda_1+\lambda_2} \text{ and } \theta_2 = \lambda_1^{n_1}\lambda_2^{n_2}.$$

Then with this reparameterization, the likelihood function is given by

$$f(\mathbf{x},\mathbf{y}|\mu,\theta_1,\theta_2)=\theta_2^{-1}\exp\left\{-\theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{\frac{n_2}{n_1+n_2}}\sum_{i=1}^{n_1}(x_i-\mu)\right.$$
$$\left.-\ \theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{-\frac{n_1}{n_1+n_2}}\sum_{i=1}^{n_2}(y_i-\mu)\right\}. \tag{2.2}$$

From the likelihood function (2.2), for, $j,k=1,2$, let

$$J_{jk}(\mu,\theta_1,\theta_2)=\int\int g_{\theta_j}(\mathbf{x},\mathbf{y};\mu,\theta_1,\theta_2)g_{\theta_k}(\mathbf{x},\mathbf{y};\mu,\theta_1,\theta_2)d\mathbf{x}d\mathbf{y},$$

where $g_{\theta_j}=\partial g/\partial\theta_j$ and $g=f^{\frac{1}{2}}$. Let $F(\mu,\theta_1,\theta_2)=\{4J_{jk}(\mu,\theta_1,\theta_2)\}, j,k=1,2$. Then this matrix is given by

$$F(\mu,\theta_1,\theta_2)=Diag\left\{\frac{n_1n_2}{(n_1+n_2)(1-\theta_1)^2\theta_1^2},\frac{1}{(n_1+n_2)\theta_2^2}\right\}. \tag{2.3}$$

And let

$$c(\mu,\theta_1,\theta_2)=E_{\mu,\theta_1,\theta_2}[\partial\log f/\partial\mu]$$
$$=n_1\theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{\frac{n_2}{n_1+n_2}}+n_2\theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{-\frac{n_1}{n_1+n_2}}.$$

Then the conditional reference prior for μ given θ_1,θ_2 is

$$\pi(\mu|\theta_1,\theta_2)=c(\mu,\theta_1,\theta_2)$$
$$=n_1\theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{\frac{n_2}{n_1+n_2}}+n_2\theta_2^{-\frac{1}{n_1+n_2}}\left(\frac{1-\theta_1}{\theta_1}\right)^{-\frac{n_1}{n_1+n_2}}. \tag{2.4}$$

We firstly derived the reference prior based on Berger and Bernardo (1992) algorithm of the regular case when θ_1 is parameter of interest. The reference prior is developed by considering a sequence of compact subsets of the parameter space and taking the limit of a sequence of priors as these compact subsets fill out of the parameter space. The compact subsets were taken to be Cartesian products of sets of the form

$$\theta_1\in[a_1,b_1],\theta_2\in[a_2,b_2].$$

Here the limit of a_1,a_2 will tend to 0 and b_1,b_2 will tend to ∞. A subscripted Q denotes a function that is constant and does not depend on any parameter but any Q may depend on the ranges of the parameters. For the derivation of the reference prior, let $h_1=4J_{11}(\mu,\theta_1,\theta_2)$ and $h_2=4J_{22}(\mu,\theta_1,\theta_2)$, then

$$h_1=\frac{n_1n_2}{(n_1+n_2)(1-\theta_1)^2\theta_1^2}\text{ and }h_2=\frac{1}{(n_1+n_2)\theta_2^2}.$$

Note that

$$\int_{a_2}^{b_2}h_2^{1/2}d\theta_2=\int_{a_2}^{b_2}(n_1+n_2)^{1/2}\theta_2^{-1}d\theta_2=(n_1+n_2)^{1/2}Q_1.$$

It follows that

$$\pi_2^l(\theta_2|\theta_1) = Q_1^{-1}\theta_2^{-1}.$$

Now

$$\begin{aligned} E^l\{\log h_1|\theta_2\} &= \int_{a_2}^{b_2} Q_1^{-1}\theta_2^{-1} \log\left[\frac{n_1 n_2}{(n_1+n_2)(1-\theta_1)^2\theta_1^2}\right] d\theta_2 \\ &= \log\left[\frac{n_1 n_2}{(n_1+n_2)(1-\theta_1)^2\theta_1^2}\right]. \end{aligned}$$

It follows that

$$\pi_1^l(\theta_1) \propto \exp[E^l\{\log h_1|\theta_2\}/2] = \exp\{Q_2/2\}(1-\theta_1)^{-1}\theta_1^{-1}.$$

Therefore the reference prior is

$$\pi(\theta_1,\theta_2) = \lim_{l\to\infty} \frac{\pi_2^l(\theta_2|\theta_1)\pi_1^l(\theta_1)}{\pi_2^l(\theta_{20}|\theta_{10})\pi_1^l(\theta_{10})} \propto (1-\theta_1)^{-1}\theta_1^{-1}\theta_2^{-1},$$

where θ_{10} and θ_{20} are an inner points of the interval $(0,\infty)$. Therefore the reference prior for (μ,θ_1,θ_2), when θ_1 is parameter of interest, is given by

$$\begin{aligned} \pi_1(\mu,\theta_1,\theta_2) &\propto \pi(\theta_1,\theta_2)\pi(\mu|\theta_1,\theta_2) \\ &\propto (1-\theta_1)^{-1}\theta_1^{-1}\theta_2^{-1} \\ &\quad\times \left[n_1\theta_2^{-\frac{1}{n_1+n_2}} \left(\frac{1-\theta_1}{\theta_1}\right)^{\frac{n_2}{n_1+n_2}} + n_2\theta_2^{-\frac{1}{n_1+n_2}} \left(\frac{1-\theta_1}{\theta_1}\right)^{-\frac{n_1}{n_1+n_2}} \right]. \end{aligned} \tag{2.5}$$

Note that under original parametrization $(\mu,\lambda_1,\lambda_2)$, the reference prior is

$$\pi_1(\mu,\lambda_1,\lambda_2) \propto \lambda_1^{-1}\lambda_2^{-1}(n_1\lambda_1^{-1} + n_2\lambda_2^{-1}). \tag{2.6}$$

Also as F is independent of μ, the prior for (θ_1,θ_2) given μ will not depend on μ. Hence the reference prior for (μ,θ_1,θ_2) when (θ_1,θ_2) is a parameter of interest is given by

$$\begin{aligned} \pi_2(\mu,\theta_1,\theta_2) &\propto (1-\theta_1)^{-1}\theta_1^{-1}\theta_2^{-1} \\ &\quad\times \left[n_1\theta_2^{-\frac{1}{n_1+n_2}} \left(\frac{1-\theta_1}{\theta_1}\right)^{\frac{n_2}{n_1+n_2}} + n_2\theta_2^{-\frac{1}{n_1+n_2}} \left(\frac{1-\theta_1}{\theta_1}\right)^{-\frac{n_1}{n_1+n_2}} \right]^{\frac{1}{2}}. \end{aligned} \tag{2.7}$$

When (θ_1,θ_2) is a parameter of interest, the reference prior for (μ,θ_1,θ_2) based on an appropriate penalty term of Ghosh and Mukerjee (1992) and also see Ghosal (1997) is given by

$$\pi_3(\mu,\theta_1,\theta_2) = [\det F(\mu,\theta_1,\theta_2)]^{\frac{1}{2}} \propto (1-\theta_1)^{-1}\theta_1^{-1}\theta_2^{-1}. \tag{2.8}$$

Note that under original parametrization $(\mu,\lambda_1,\lambda_2)$, the reference prior is

$$\pi_3(\mu,\lambda_1,\lambda_2) \propto \lambda_1^{-1}\lambda_2^{-1}. \tag{2.9}$$

We will consider the reference priors (2.6) and (2.9) in the next section, because we know that the reference prior based on an appropriate term is more efficient.

3. Implementation of the Bayesian procedure

We investigate the propriety of posteriors for a general class of priors which include the reference priors (2.6) and (2.9). We consider the class of priors

$$\pi_g(\mu, \lambda_1, \lambda_2) \propto \lambda_1^{-a}\lambda_2^{-b}(n_1\lambda_1^{-1} + n_2\lambda_2^{-1})^c, \tag{3.1}$$

where $a > 0, b > 0$ and $c \geq 0$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of $(\mu, \lambda_1, \lambda_2)$ under the general prior (3.1) is proper if $n_1 + a + c - 2 > 0, n_2 + b + c - 1 > 0$ or $n_1 + a + c - 1 > 0, n_2 + b + c - 2 > 0$.

Proof: Under the general prior (3.1), the joint posterior for $\mu, \lambda_1, \lambda_2$ given $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}\pi(\mu, \lambda_1, \lambda_2|\mathbf{x}, \mathbf{y}) &\propto \lambda_1^{-n_1-a}\lambda_2^{-n_2-b}(n_1\lambda_1^{-1} + n_2\lambda_2^{-1})^c \\ &\quad \times \exp\left\{-\frac{1}{\lambda_1}\sum_{i=1}^{n_1}(x_i - \mu) - \frac{1}{\lambda_2}\sum_{i=1}^{n_2}(y_i - \mu)\right\}.\end{aligned} \tag{3.2}$$

Then integrating with respect to μ in (3.2), we have the posterior

$$\begin{aligned}\pi(\lambda_1, \lambda_2|\mathbf{x}, \mathbf{y}) &\propto \lambda_1^{-n_1-a}\lambda_2^{-n_2-b}(n_1\lambda_1^{-1} + n_2\lambda_2^{-1})^c(n_1\lambda_1^{-1} + n_2\lambda_2^{-1})^{-1} \\ &\quad \times \left[\exp\left\{-\frac{n_1}{\lambda_1}(\bar{x} - z) - \frac{n_2}{\lambda_2}(\bar{y} - z)\right\} - \exp\left\{-\frac{n_1}{\lambda_1}\bar{x} - \frac{n_2}{\lambda_2}\bar{y}\right\}\right],\end{aligned} \tag{3.3}$$

where $\bar{x} = \sum_{i=1}^{n_1} x_i/n_1,\ \bar{y} = \sum_{i=1}^{n_2} y_i/n_2,\ z = min\{x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}\}$. From the posterior density (3.3), we obtain the following inequality.

$$\begin{aligned}\pi(\lambda_1, \lambda_2|\mathbf{x}, \mathbf{y}) &\leq k_1\lambda_1^{-n_1-a-c}\lambda_2^{-n_2-b-c}(n_1\lambda_1^{-1} + n_2\lambda_2^{-1})^{-1} \\ &\quad \times \left[\exp\left\{-\frac{n_1}{\lambda_1}(\bar{x} - z) - \frac{n_2}{\lambda_2}(\bar{y} - z)\right\} - \exp\left\{-\frac{n_1}{\lambda_1}\bar{x} - \frac{n_2}{\lambda_2}\bar{y}\right\}\right] \\ &\leq k_2\lambda_1^{-n_1-a-c+1}\lambda_2^{-n_2-b-c} \\ &\quad \times \left[\exp\left\{-\frac{n_1}{\lambda_1}(\bar{x} - z) - \frac{n_2}{\lambda_2}(\bar{y} - z)\right\} - \exp\left\{-\frac{n_1}{\lambda_1}\bar{x} - \frac{n_2}{\lambda_2}\bar{y}\right\}\right],\end{aligned} \tag{3.4}$$

where k_1 and k_2 are constants. Thus the function (3.4) is proper if $n_1 + a + c - 2 > 0, n_2 + b + c - 1 > 0$. This completes the proof. □

Theorem 3.2 Under the reference prior π_1 and π_2, the marginal posterior density of θ_1 is given by

$$\begin{aligned}\pi(\theta_1|\mathbf{x}, \mathbf{y}) \propto (1 - \theta_1)^{-n_2-1}\theta_1^{n_2-1} &\left\{\left[n_1\bar{x} + n_2\bar{y}\left(\frac{\theta_1}{1 - \theta_1}\right) - \left(n_1 + n_2\frac{\theta_1}{1 - \theta_1}\right)z\right]^{-(n_1+n_2)}\right. \\ &\left. - \left[n_1\bar{x} + n_2\bar{y}\left(\frac{\theta_1}{1 - \theta_1}\right)\right]^{-(n_1+n_2)}\right\}.\end{aligned}$$

Under the reference prior π_3, the marginal posterior density of θ_1 is given by

$$\pi(\theta_1|\mathbf{x},\mathbf{y}) \propto (1-\theta_1)^{-n_2-1}\theta_1^{n_2-1}\left(n_1+n_2\frac{\theta_1}{1-\theta_1}\right)^{-1}\left\{\left[n_1\bar{x}+n_2\bar{y}\left(\frac{\theta_1}{1-\theta_1}\right)\right.\right.$$
$$\left.\left. -\left(n_1+n_2\frac{\theta_1}{1-\theta_1}\right)z\right]^{-(n_1+n_2-1)} - \left[n_1\bar{x}+n_2\bar{y}\left(\frac{\theta_1}{1-\theta_1}\right)\right]^{-(n_1+n_2-1)}\right\}.$$

Note that normalizing constant for the marginal density of θ_1 requires only one dimensional integration. Therefore we can easily compute the marginal posterior density of θ_1 and the marginal moment of θ_1.

4. Numerical study

Since a reference prior is frequently a probability matching prior in regular distribution, we want to know whether the proposed reference priors give similar results in nonregular distribution or not. We investigate the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of θ_1 under the noninformative prior π_1 and π_3 given in Section 3 for several configurations $(\mu, \lambda_1, \lambda_2)$ and (n_1, n_2). That is to say, the frequentist coverage of a $100\alpha\%th$ posterior quantile should be close to α. This is done numerically. Table 4.1 gives numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the proposed priors. The computation of these numerical values is based on the following algorithm for any fixed true $(\mu, \lambda_1, \lambda_2)$ and any prespecified value α. Here α is 0.05 (0.95). Let $\theta_1^{\pi}(\alpha|\mathbf{x},\mathbf{y})$ be the posterior α-quantile of θ_1 given $\mathbf{x}$ and $\mathbf{y}$. That is to say, $F(\theta_1^{\pi}(\alpha|\mathbf{x},\mathbf{y})|\mathbf{x},\mathbf{y}) = \alpha$, where $F(\cdot|\mathbf{x},\mathbf{y})$ is the marginal posterior distribution of θ_1. Then the frequentist coverage probability of this one sided credible interval of θ_1 is

$$P_{(\mu,\lambda_1,\lambda_2)}(\alpha;\theta_1) = P_{(\mu,\lambda_1,\lambda_2)}(0 < \theta_1 < \theta_1^{\pi}(\alpha|\mathbf{x},\mathbf{y})). \tag{4.1}$$

The estimated $P_{(\mu,\lambda_1,\lambda_2)}(\alpha;\theta_1)$ when $\alpha = 0.05(0.95)$ is shown in Table 4.1. In particular, for fixed $(\mu, \lambda_1, \lambda_2)$, we take 10,000 independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the model (2.2).

For the cases presented in Table 4.1, we see that the reference prior π_3 matches the target coverage probability much more accurately than the reference prior π_1 for values of $(\mu, \theta_1, \lambda_1)$ and values of (n_1, n_2). In particular, the reference prior π_3 meets very well the target coverage probabilities in small samples. Note that the results of tables are not much sensitive to change of the values of $(\mu, \theta_1, \lambda_1)$. Thus we recommend to use the reference prior π_3 in the sense of asymptotic frequentist coverage property.

5. Concluding remarks

We have found reference priors for the reliability in a stress-strength model where stress and strength distributed as two parameter exponential distribution. We derived the reference priors when θ_1 is parameter of interest, and (θ_1, θ_2) are parameters of interest. We showed that the reference prior π_3 perform better than the reference prior π_1 in matching the target coverage probabilities. Thus we recommend the use of the reference prior π_3 for Bayesian inference of this model.

Table 4.1 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for θ_1

			$\mu = 1$		$\mu = 10$	
θ_1	λ_1	(n_1, n_2)	π_1	π_3	π_1	π_3
0.1	0.1	5,5	0.035 (0.897)	0.048 (0.949)	0.037 (0.890)	0.049 (0.946)
		5,10	0.032 (0.887)	0.047 (0.947)	0.033 (0.893)	0.046 (0.946)
		10,10	0.043 (0.921)	0.053 (0.950)	0.037 (0.921)	0.048 (0.950)
		10,20	0.038 (0.916)	0.051 (0.948)	0.033 (0.918)	0.045 (0.946)
		20,20	0.040 (0.933)	0.049 (0.951)	0.037 (0.932)	0.048 (0.950)
	1.0	5,5	0.034 (0.890)	0.044 (0.945)	0.039 (0.892)	0.053 (0.946)
		5,10	0.037 (0.890)	0.051 (0.950)	0.037 (0.893)	0.049 (0.950)
		10,10	0.039 (0.922)	0.051 (0.955)	0.039 (0.920)	0.050 (0.950)
		10,20	0.038 (0.921)	0.050 (0.952)	0.037 (0.917)	0.049 (0.949)
		20,20	0.043 (0.936)	0.053 (0.953)	0.041 (0.932)	0.050 (0.949)
	10.0	5,5	0.034 (0.866)	0.045 (0.930)	0.038 (0.894)	0.049 (0.950)
		5,10	0.027 (0.869)	0.038 (0.931)	0.033 (0.892)	0.049 (0.948)
		10,10	0.036 (0.908)	0.048 (0.939)	0.041 (0.920)	0.054 (0.952)
		10,20	0.033 (0.908)	0.045 (0.942)	0.034 (0.916)	0.049 (0.947)
		20,20	0.036 (0.926)	0.047 (0.947)	0.036 (0.932)	0.046 (0.952)
0.3	0.1	5,5	0.055 (0.907)	0.051 (0.946)	0.056 (0.906)	0.051 (0.947)
		5,10	0.051 (0.920)	0.050 (0.949)	0.050 (0.912)	0.050 (0.945)
		10,10	0.051 (0.927)	0.052 (0.946)	0.051 (0.931)	0.052 (0.951)
		10,20	0.051 (0.928)	0.053 (0.945)	0.049 (0.931)	0.051 (0.948)
		20,20	0.047 (0.935)	0.050 (0.947)	0.046 (0.939)	0.049 (0.949)
	1.0	5,5	0.051 (0.908)	0.048 (0.947)	0.056 (0.910)	0.052 (0.948)
		5,10	0.052 (0.921)	0.052 (0.948)	0.048 (0.913)	0.048 (0.946)
		10,10	0.047 (0.929)	0.048 (0.946)	0.048 (0.928)	0.050 (0.950)
		10,20	0.046 (0.933)	0.049 (0.948)	0.044 (0.930)	0.046 (0.946)
		20,20	0.047 (0.939)	0.050 (0.951)	0.051 (0.940)	0.054 (0.951)
	10.0	5,5	0.055 (0.903)	0.052 (0.940)	0.051 (0.910)	0.047 (0.949)
		5,10	0.045 (0.905)	0.045 (0.938)	0.049 (0.918)	0.049 (0.950)
		10,10	0.052 (0.919)	0.053 (0.940)	0.046 (0.928)	0.047 (0.949)
		10,20	0.045 (0.932)	0.048 (0.948)	0.049 (0.933)	0.051 (0.950)
		20,20	0.049 (0.937)	0.051 (0.951)	0.047 (0.939)	0.050 (0.951)
0.5	0.1	5,5	0.073 (0.927)	0.053 (0.947)	0.071 (0.928)	0.049 (0.949)
		5,10	0.062 (0.934)	0.051 (0.949)	0.060 (0.931)	0.050 (0.946)
		10,10	0.060 (0.940)	0.052 (0.949)	0.057 (0.937)	0.048 (0.945)
		10,20	0.054 (0.942)	0.049 (0.949)	0.056 (0.943)	0.051 (0.951)
		20,20	0.056 (0.948)	0.052 (0.951)	0.058 (0.948)	0.053 (0.952)
	1.0	5,5	0.071 (0.930)	0.050 (0.948)	0.071 (0.925)	0.048 (0.945)
		5,10	0.062 (0.935)	0.052 (0.946)	0.063 (0.937)	0.052 (0.950)
		10,10	0.058 (0.938)	0.050 (0.947)	0.059 (0.944)	0.051 (0.952)
		10,20	0.055 (0.945)	0.050 (0.951)	0.054 (0.942)	0.049 (0.949)
		20,20	0.057 (0.946)	0.052 (0.950)	0.051 (0.945)	0.048 (0.951)
	10.0	5,5	0.075 (0.921)	0.056 (0.940)	0.070 (0.929)	0.051 (0.950)
		5,10	0.059 (0.937)	0.050 (0.949)	0.064 (0.932)	0.054 (0.947)
		10,10	0.064 (0.943)	0.057 (0.951)	0.058 (0.939)	0.049 (0.949)
		10,20	0.055 (0.942)	0.049 (0.948)	0.054 (0.944)	0.049 (0.950)
		20,20	0.058 (0.943)	0.054 (0.947)	0.054 (0.945)	0.049 (0.951)
0.7	0.1	5,5	0.094 (0.946)	0.054 (0.950)	0.085 (0.947)	0.050 (0.952)
		5,10	0.075 (0.946)	0.055 (0.948)	0.065 (0.944)	0.044 (0.947)
		10,10	0.068 (0.954)	0.050 (0.953)	0.068 (0.951)	0.052 (0.950)
		10,20	0.061 (0.948)	0.050 (0.946)	0.060 (0.952)	0.048 (0.951)
		20,20	0.061 (0.957)	0.050 (0.955)	0.061 (0.953)	0.050 (0.951)
	1.0	5,5	0.087 (0.948)	0.052 (0.953)	0.086 (0.944)	0.050 (0.948)
		5,10	0.070 (0.946)	0.050 (0.948)	0.071 (0.947)	0.050 (0.949)
		10,10	0.071 (0.952)	0.051 (0.951)	0.073 (0.952)	0.054 (0.951)
		10,20	0.065 (0.953)	0.055 (0.953)	0.060 (0.952)	0.049 (0.951)
		20,20	0.063 (0.953)	0.051 (0.951)	0.065 (0.955)	0.054 (0.952)
	10.0	5,5	0.102 (0.944)	0.062 (0.948)	0.087 (0.948)	0.050 (0.952)
		5,10	0.070 (0.952)	0.051 (0.954)	0.071 (0.953)	0.049 (0.955)
		10,10	0.067 (0.953)	0.049 (0.952)	0.068 (0.950)	0.047 (0.948)
		10,20	0.065 (0.952)	0.053 (0.951)	0.064 (0.951)	0.053 (0.949)
		20,20	0.067 (0.954)	0.052 (0.952)	0.065 (0.952)	0.053 (0.951)

Table 4.1 Continued.

			$\mu = 1$		$\mu = 10$	
θ_1	λ_1	(n_1, n_2)	π_1	π_3	π_1	π_3
0.9	0.1	5,5	0.109 (0.958)	0.054 (0.946)	0.108 (0.963)	0.050 (0.950)
		5,10	0.073 (0.960)	0.048 (0.951)	0.073 (0.957)	0.049 (0.949)
		10,10	0.083 (0.963)	0.050 (0.952)	0.078 (0.962)	0.050 (0.950)
		10,20	0.065 (0.955)	0.047 (0.948)	0.065 (0.956)	0.049 (0.950)
		20,20	0.069 (0.958)	0.050 (0.950)	0.068 (0.959)	0.050 (0.950)
	1.0	5,5	0.111 (0.963)	0.052 (0.951)	0.107 (0.966)	0.052 (0.954)
		5,10	0.079 (0.958)	0.054 (0.950)	0.079 (0.958)	0.051 (0.949)
		10,10	0.078 (0.963)	0.051 (0.950)	0.078 (0.961)	0.047 (0.948)
		10,20	0.064 (0.958)	0.049 (0.951)	0.068 (0.958)	0.051 (0.950)
		20,20	0.073 (0.961)	0.052 (0.951)	0.072 (0.959)	0.051 (0.950)
	10.0	5,5	0.111 (0.961)	0.054 (0.948)	0.114 (0.962)	0.053 (0.949)
		5,10	0.080 (0.960)	0.051 (0.953)	0.076 (0.959)	0.051 (0.950)
		10,10	0.082 (0.966)	0.050 (0.955)	0.078 (0.962)	0.047 (0.950)
		10,20	0.065 (0.959)	0.051 (0.951)	0.065 (0.952)	0.048 (0.945)
		20,20	0.065 (0.960)	0.047 (0.951)	0.070 (0.960)	0.051 (0.950)

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