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FANO MANIFOLDS AND BLOW-UPS OF LOW-DIMENSIONAL SUBVARIETIES

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Abstract. We study Fano manifolds of pseudoindex greater than one and dimension greater than five, which are blow-ups of smooth varieties along smooth centers of dimension equal to the pseudoindex of the manifold. We obtain a classification of the possible cones of curves of these manifolds, and we prove that there is only one such manifold without a fiber type elementary contraction.

1. Introduction

A smooth complex projective variety X is called *Fano* if its anticanonical bundle $-K_X$ is ample; the *index* r_X of X is the largest natural number m such that $-K_X = mH$ for some (ample) divisor H on X, while the *pseudoindex* i_X is the minimum anticanonical degree of rational curves on X.

By the Cone Theorem the cone $\mathrm{NE}(X)$ generated by the numerical classes of irreducible curves on a Fano manifold X is polyhedral. By the Contraction Theorem to each extremal ray of $\mathrm{NE}(X)$ is associated a contraction, i.e., a proper morphism with connected fibers onto a normal variety.

A natural question which arises from the study of Fano manifolds is to investigate - and possibly classify - Fano manifolds which admit an extremal contraction with special features: for example, this has been done in many cases in which the contraction is a projective bundle [1, 18, 21, 22, 23, 24], a quadric bundle [29] or a scroll [5, 16].

Recently, Bonavero, Campana, and Wiśniewski have considered the case where an extremal contraction of X is the blow-up of a smooth variety along a point, giving a complete classification [8]. The case where the center of the blow-up is a curve has shown to be much more complicated. A complete classification in case $i_X \geq 2$ has been obtained in [4], as a corollary of a more general theorem, where the classification of Fano manifolds with a contraction which is the blow-up of a manifold along a smooth subvariety of dimension $\leq i_X - 1$ is achieved. For Fano manifolds of pseudoindex $i_X = 1$ which are

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blow-ups of smooth varieties along a smooth curve, some special cases have been dealt with in the PhD thesis of Tsukioka [26] (partially published in [25]).

Considering the case when the dimension of the center of the blow-up is $i_X \geq 2$, the lowest possible dimension of the manifold is five; the cones of curves of such varieties are among those listed in [11], where the cone of curves of Fano manifolds of dimension five and pseudoindex greater than one were classified. Under the stronger assumption that $r_X \geq 2$ the complete list of Fano fivefolds which are blow-ups of smooth varieties along smooth surfaces has been given in [12].

In this paper we propose a generalization of both the results in [4] and in [12], considering Fano manifolds of dimension greater than five with a contraction which is the blow-up of a manifold along a smooth subvariety of dimension $i_X \geq 2$.

We will first give a classification of the possible cones of curves of these varieties:

Theorem 1.1. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, with a contraction $\sigma\colon X \to Y$, associated to an extremal ray R_σ, which is a smooth blow-up with center a smooth subvariety B of dimension $\dim B = i_X$. Then the possible cones of curves of X are listed in the following table, where F stands for a fiber type contraction and D_{n-3} for the blow-up of a smooth variety along a smooth subvariety of codimension three.*

ρ_X	i_X	R_1	R_2	R_3	R_4	
2		R_σ	F			(a)
2		R_σ	D_{n-3}			(b)
3	*2,3*	R_σ	F	F		(c)
3	*2*	R_σ	F	D_{n-3}		(d)
4	*2*	R_σ	F	F	F	(e)

We will then prove that there is only one Fano manifold satisfying the assumption of Theorem 1.1 whose cone of curves is as in case (b) - or, equivalently, which does not admit a fiber type contraction:

Theorem 1.2. *Let X be a Fano manifold of dimension $n \geq 6$ and pseudoindex $i_X \geq 2$, which is the blow-up of another Fano manifold Y along a smooth subvariety B of dimension i_X; assume that X does not admit a fiber type contraction. Then $Y \simeq \mathbb{G}(1,4)$ and B is a plane of bidegree $(0,1)$.*

We notice that, in view of the classification given in Theorem 1.1, Generalized Mukai Conjecture [9, 2] holds for the Fano manifolds we are considering.

Let us point out that the assumption $i_X \geq 2$ is essential for our methods, as well as for the ones used in [4], [11] and [12], on which they are based.

The proofs of Theorems 1.1 and 1.2 are contained in Sections 5 and 6. In Section 4 we consider manifolds which possess a quasi-unsplit dominating family, proving that they are as in Theorem 1.1, cases (a) and (c)-(e).

In Section 6 we consider manifolds which do not possess a family as above, proving first that their cone of curves is as in case (b), and then that the only such manifold is the blow-up of $\mathbb{G}(1,4)$ along a plane of bidegree $(0,1)$.

2. Background material

2.1. Fano-Mori contractions

Let X be a smooth Fano variety of dimension n and let K_X be its canonical divisor. By Mori's *Cone Theorem* the cone $\mathrm{NE}(X)$ of effective 1-cycles, which is contained in the $\mathbb{R}$-vector space $N_1(X)$ of 1-cycles modulo numerical equivalence, is polyhedral; a face τ of $\mathrm{NE}(X)$ is called an *extremal face* and an extremal face of dimension one is called an *extremal ray*.

To every extremal face τ one can associate a morphism $\varphi : X \to Z$ with connected fibers onto a normal variety; the morphism φ contracts those curves whose numerical classes lie in τ, and is usually called the *Fano-Mori contraction* (or the *extremal contraction*) associated to the face τ.

An extremal ray R is called *numerically effective*, or of *fiber type*, if $\dim Z < \dim X$, otherwise the ray is *non nef* or *birational*. We usually denote with $\mathrm{Exc}(\varphi) := \{x \in X \mid \dim \varphi^{-1}(\varphi(x)) > 0\}$ the *exceptional locus* of φ; if φ is of fiber type then of course $\mathrm{Exc}(\varphi) = X$. If the exceptional locus of a birational ray R has codimension one, the ray and the associated contraction are called *divisorial*; if its codimension is bigger they are called *small*.

2.2. Families of rational curves

For this subsection our main reference is [15], with which our notation is coherent; for missing proofs and details see also [2, 11].

Definition 2.1. We define a *family of rational curves* to be an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ of the scheme $\mathrm{Ratcurves}^n(X)$ (see [15, Definition 2.11]). Given a rational curve $f\colon \mathbb{P}^1 \to X$ we will call a *family of deformations* of f any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of f.

We define $\mathrm{Locus}(V)$ to be the image in X of the universal family over V via the evaluation; we say that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$.

Remark 2.2. If V is a dominating family of rational curves, then its general member is a free rational curve. In particular, by [15, II.3.7], if B is a subset of X of codimension ≥ 2, a general curve of V does not meet B.

Corollary 2.3. *Let $\sigma\colon X \to Y$ be a smooth blow-up with center B of codimension ≥ 2 and exceptional locus E, let V be a dominating family of rational curves in Y and let V^* be a family of deformations of the strict transform of a general curve of V. Then $E \cdot V^* = 0$.*

For every point $x \in \mathrm{Locus}(V)$, we will denote by V_x the subscheme of V parametrizing rational curves passing through x.

Definition 2.4. Let V be a family of rational curves on X. We say that

- V is *unsplit* if it is proper;
- V is *locally unsplit* if every component of V_x is proper for the general $x \in \mathrm{Locus}(V)$.

Proposition 2.5 ([15, IV.2.6]). *Let X be a smooth projective variety, V a family of rational curves and $x \in \mathrm{Locus}(V)$ a point such that every component of V_x is proper. Then*

(a) $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
(b) $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

Remark 2.6. The assumptions on V in [15, IV.2.6] are slightly different, but the same proof works for the statement above.

In case V is the unsplit family of deformations of an extremal rational curve of minimal degree, Proposition 2.5 gives the *fiber locus inequality*:

Proposition 2.7 ([13, 28]). *Let φ be a Fano-Mori contraction of X and E its exceptional locus; let F be an irreducible component of a (non trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1,$$

where $l = \min\{-K_X \cdot C \mid C$ is a rational curve in $F\}$. If φ is the contraction of an extremal ray R, then l is called the length of the ray.

Definition 2.8. We define a *Chow family of rational curves* $\mathcal{V}$ to be an irreducible component of $\mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles.

If V is a family of rational curves, the closure of the image of V in $\mathrm{Chow}(X)$ is called the *Chow family associated to* V. We will usually denote the Chow family associated to a family with the calligraphic version of the same letter.

Definition 2.9. We denote by $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_k$ with the following properties:

- C_i belongs to the family $\mathcal{V}^i$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_k$,

i.e., $\mathrm{Locus}(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of k cycles belonging respectively to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

We denote by $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1, \ldots, C_m$ with the following properties:

- C_i belongs to a family $\mathcal{V}^j$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_m$,

i.e., $\mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_Y$ is the set of points that can be joined to Y by a connected chain of at most m cycles belonging to the families $\mathcal{V}^1, \ldots, \mathcal{V}^k$.

Definition 2.10. Let $V^1, \ldots, V^k$ be unsplit families on X. We will say that $V^1, \ldots, V^k$ are *numerically independent* if their numerical classes $[V^1], \ldots, [V^k]$ are linearly independent in the vector space $N_1(X)$. If moreover $C \subset X$ is a curve we will say that $V^1, \ldots, V^k$ are *numerically independent from* C if the class of C in $N_1(X)$ is not contained in the vector subspace generated by $[V^1], \ldots, [V^k]$.

Lemma 2.11 ([2, Lemma 5.4])**.** *Let $Y \subset X$ be a closed irreducible subset and V an unsplit family of rational curves. Assume that curves contained in Y are numerically independent from curves in V, and that $Y \cap \mathrm{Locus}(V) \neq \emptyset$. Then for a general $y \in Y \cap \mathrm{Locus}(V)$*

(a) $\dim \mathrm{Locus}(V)_Y \geq \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V_y)$;
(b) $\dim \mathrm{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$.

Moreover, if $V^1, \ldots, V^k$ are numerically independent unsplit families such that curves contained in Y are numerically independent from curves in $V^1, \ldots, V^k$, then either $\mathrm{Locus}(V^1, \ldots, V^k)_Y = \emptyset$ *or*

(c) $\dim \mathrm{Locus}(V^1, \ldots, V^k)_Y \geq \dim Y + \sum(-K_X \cdot V^i) - k$.

Definition 2.12. We define on X a relation of *rational connectedness with respect to* $\mathcal{V}^1, \ldots, \mathcal{V}^k$ in the following way: x and y are in $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation if there exists a chain of rational curves in $\mathcal{V}^1, \ldots, \mathcal{V}^k$ which joins x and y, i.e., if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1, \ldots, \mathcal{V}^k)_x$ for some m.

To the $\mathrm{rc}(\mathcal{V}^1, \ldots, \mathcal{V}^k)$-relation it is possible to associate a fibration $\pi\colon X \dashrightarrow Z$, defined on an open subset (see [10], [15, IV.4.16]). If π is the constant map we say that X is *rc*$(\mathcal{V}^1, \ldots, \mathcal{V}^k)$*-connected.*

Definition 2.13. A *minimal horizontal dominating family* with respect to π is a family V of horizontal rational curves such that $\mathrm{Locus}(V)$ dominates Z^0 and $-K_X \cdot V$ is minimal among the families with this property.

If π is the identity map we say that V is a *minimal dominating family* for X.

Definition 2.14. Let $\mathcal{V}$ be the Chow family associated to a family of rational curves V. We say that V is *quasi-unsplit* if every component of any reducible cycle of $\mathcal{V}$ is numerically proportional to the numerical class $[V]$ of a curve of V.

We say that V is *locally quasi-unsplit* if, for a general $x \in \mathrm{Locus}(\mathcal{V})$ every component of any reducible cycle of $\mathcal{V}_x$ is numerically proportional to V

Note that any family of deformations of a rational curve whose numerical class lies in an extremal ray of $\mathrm{NE}(X)$ is quasi-unsplit.

Notation: Let S be a subset of X. We write $N_1(S) = \langle V^1, \ldots, V^k \rangle$ if the numerical class in $\mathrm{N}_1(X)$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \mathbb{Q}$ and $C_i \in V^i$. We write $\mathrm{NE}(S) = \langle V^1, \ldots, V^k \rangle$ (or $\mathrm{NE}(S) =$

$\langle R_1, \ldots, R_k \rangle$) if the numerical class in $\mathrm{N}_1(X)$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i[C_i]$, with $a_i \in \mathbb{Q}_{\geq 0}$ and $C_i \in V^i$ (or $[C_i]$ in R_i).

Lemma 2.15 ([6, Lemma 1.4.5], [19, Lemma 1], [11, Corollary 2.23])**.** *Let $Y \subset X$ be a closed subset and V a quasi-unsplit family of rational curves. Then every curve contained in* $\mathrm{Locus}(V)_Y$ *is numerically equivalent to a linear combination with rational coefficients*

$$aC_Y + bC_V,$$

where C_Y is a curve in Y, C_V belongs to the family V and $a \geq 0$.

Moreover, if Σ is an extremal face of $\mathrm{NE}(X)$*, Y is a fiber of the associated contraction and $[V]$ does not belong to Σ, then*

$$\mathrm{NE}(\mathrm{ChLocus}_m(V)_Y) = \langle \Sigma, [V] \rangle \quad \textit{for every} \quad m \geq 1.$$

Remark 2.16. In the quoted papers, the results are proved for unsplit families of rational curves, but they are true - with the same proofs - for quasi-unsplit ones.

3. Dominating families and Picard number

We collect in this section some technical results that we will need in the proof.

The first is a variation of a classical construction of Mori theory, and says that, given a family of rational curves V and a curve C contained in $\mathrm{Locus}(V_x)$ for an x such that V_x is proper we have $[C] \equiv a[V]$.

The only new remark - which already followed from the old proofs, but, to our best knowledge, was not stated - is the fact that a is a positive integer.

Lemma 3.1. *Let X be a smooth variety, V a family of rational curves on X, $x \in$ $\mathrm{Locus}(V)$ a point such that V_x is proper and C a curve contained in* $\mathrm{Locus}(V_x)$*. Then C is numerically equivalent to an integral multiple of a curve of V.*

Proof. Consider the basic diagram:

$$\begin{array}{lc} (3.1.1) & \begin{array}{ccc} p^{-1}(V_x) =: U_x & \xrightarrow{i} & X \\ {\scriptstyle p}\downarrow & & \\ V_x & & \end{array} \end{array}$$

Let C be a curve contained in $\mathrm{Locus}(V_x)$; if C is a curve parametrized by V we have nothing to prove, so we can suppose that this is not the case.

In particular we have that $i^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of p and dominates C via i; let S' be the surface

$p^{-1}(p(C'))$, let B' be the curve $p(C') \subset V_x$ and let $\nu\colon B \to B'$ be the normalization of B'. By base change we obtain the following diagram:

$$\begin{tikzcd} S_B \arrow[r, "\bar{\nu}"] \arrow[d] & U_x \arrow[r, "i"] \arrow[d, "p"] & X \\ B \arrow[r, "\nu"'] & V_x & \end{tikzcd}$$

Let now $\mu\colon S \to S_B$ be the normalization of S_B; by standard arguments (see for instance [27, 1.14]) it can be shown that S is a ruled surface over the curve B; let $j\colon S \to X$ be the composition of i, $\bar{\nu}$ and μ. Since every curve parametrized by S passes through x there exists an irreducible curve $C_x \subset S$ which is contracted by j; by [15, II.5.3.2] we have $C_x^2 < 0$, hence C_x is the minimal section of S.

Since every curve in S is algebraically equivalent to a linear combination with integral coefficients of C_x and a fiber f, and since C_x is contracted by j, every curve in $j(S)$ is algebraically equivalent in X to an integral multiple of $j_*(f)$, which is a curve of the family V; but algebraic equivalence implies numerical equivalence and so the lemma is proved. □

Corollary 3.2. *Let X be a smooth variety of dimension n and let V be a locally unsplit dominating family such that $-K_X \cdot V = n+1$. Then $X \simeq \mathbb{P}^n$.*

Proof. For a general point $x \in X$ we know that V_x is proper and $X = \mathrm{Locus}(V_x)$ by Proposition 2.5 (b). Therefore, by Lemma 3.1, for every curve C in X we have $-K_X \cdot C \geq n+1$ and we can apply [14, Theorem 1.1]. □

Remark 3.3. The corollary also followed from the arguments in the proof of [14, Theorem 1.1].

In the rest of the section we establish some bounds on the Picard number of Fano manifolds with minimal dominating families of high anticanonical degree.

Lemma 3.4. *Let X be a Fano manifold of dimension $n \geq 3$ and pseudoindex $i_X \geq 2$ with a minimal dominating family W such that $-K_X \cdot W > 2$. If X contains an effective divisor D such that $\mathrm{NE}(D) = \langle [W] \rangle$, then $\rho_X = 1$.*

Proof. The effective divisor D has positive intersection number with at least one of the extremal rays of X. Let R be such a ray, denote by φ_R the associated contraction and by V^R a family of deformations of a minimal rational curve in R.

If the numerical class of W does not belong to R, then D cannot contain curves whose numerical classes lie in R, therefore every fiber of φ_R is one-dimensional.

By Proposition 2.7 this is possible only if $l(R) \leq 2$ and therefore, since $l(R) \geq i_X$, it must be $l(R) = i_X = 2$.

Since every fiber of φ_R is one-dimensional we have, for every $x \in \mathrm{Locus}(V^R)$ that $\dim \mathrm{Locus}(V^R_x) = 1$ and therefore, by Proposition 2.5 (a) V^R is a dominating family. But, recalling that

$$2 = -K_X \cdot V^R < -K_X \cdot W,$$

we contradict the assumption that W is minimal.

It follows that $[W] \in R$, so the family W is quasi-unsplit and $D \cdot W > 0$; hence X can be written as $X = \mathrm{Locus}(\mathcal{W})_D$, and by Lemma 2.15 we have $\rho_X = 1$. $\square$

Corollary 3.5. *Let X be a Fano manifold of dimension $n \geq 3$ and pseudoindex $i_X \geq 2$ which admits a minimal dominating family W such that $-K_X \cdot W \geq n$. Then $\rho_X = 1$.*

Proof. Let $x \in X$ be a general point; every minimal dominating family is locally unsplit, hence $\mathrm{NE}(\mathrm{Locus}(W_x)) = \langle [W] \rangle$ by Lemma 2.15.

By Proposition 2.5 we have $\dim \mathrm{Locus}(W_x) \geq -K_X \cdot W - 1 \geq n-1$, so either $X = \mathrm{Locus}(W_x)$ or $\mathrm{Locus}(W_x)$ is an effective divisor verifying the assumptions of Lemma 3.4. In both cases we can conclude that $\rho_X = 1$. $\square$

Lemma 3.6. *Let X be a Fano manifold of dimension $n \geq 5$ and pseudoindex $i_X \geq 2$, with a minimal dominating family W such that $-K_X \cdot W = n - 1$; let $U \subset X$ be the open subset of points $x \in X$ such that W_x is unsplit. If a general curve C of W is contained in U, then either a component of $\mathrm{Locus}(W)_C$ is a divisor and $\rho_X = 1$ or there exists an unsplit family V such that $-K_X \cdot V = 2$, $D := \mathrm{Locus}(V)$ is a divisor and $D \cdot W > 0$.*

Proof. Let C be a general curve of W and consider $\mathrm{Locus}(W)_C$; by Lemma 2.15 and Proposition 2.5 we have $\mathrm{NE}(\mathrm{Locus}(W)_C) = \langle [W] \rangle$ and $\dim \mathrm{Locus}(W)_C \geq n - 2$.

If $X = \mathrm{Locus}(W)_C$, then clearly $\rho_X = 1$, while if $\mathrm{Locus}(W)_C$ has codimension one we conclude by Lemma 3.4.

Therefore we can assume that, for a general C in W, each component of $\mathrm{Locus}(W)_C$ has codimension two in X. The fibration $\pi\colon X \dashrightarrow Z$ associated to the open prerelation defined by W is proper, since a general fiber F coincides with $\mathrm{Locus}(W_x)$ for a general $x \in F$ and $\mathrm{Locus}(W_x)$ is closed since W is locally unsplit.

Being π proper there exists a minimal horizontal dominating family V with respect to π; since the general fiber of π has dimension $n-2$, then $\dim Z = 2$, hence for a general $x \in \mathrm{Locus}(V)$ we have $\dim \mathrm{Locus}(V_x) \leq 2$.

It follows that V is an unsplit family, which cannot be dominating by the minimality of W, so $\dim \mathrm{Locus}(V_x) \geq i_X \geq 2$, and $D = \mathrm{Locus}(V)$ is a divisor by Proposition 2.5. Since D dominates Z we have $D \cdot W > 0$. $\square$

4. Fano manifolds obtained blowing-up non Fano manifolds

We start now the proof of our results. Let us fix once and for all the setup and the notation:

4.1. *X is a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, which has a contraction $\sigma\colon X \to Y$ which is the blow-up of a manifold Y along a smooth subvariety B of dimension i_X. We denote by R_σ the extremal ray corresponding to σ, by l_σ its length and by E_σ its exceptional locus.*

Remark 4.2. The assumption on $\dim B$ is equivalent to

$$l_\sigma + i_X = n - 1.$$

In this section we will deal with Fano manifolds as in 4.1 which are obtained as a blow-up $\sigma\colon X \to Y$ of a manifold Y which is not Fano. It turns out that there is only one possibility (Corollary 4.4). We start with a slightly more general result:

Theorem 4.3. *Let X, R_σ and E_σ be as in 4.1 and assume that there exists on X an unsplit family of rational curves V such that $E_\sigma \cdot V < 0$. Then either $[V] \in R_\sigma$ or $X = \mathbb{P}_{\mathbb{P}^{n-3}\times\mathbb{P}^2}(\mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$.*

Proof. Assume $[V] \notin R_\sigma$. Since $E_\sigma \cdot V < 0$ then $\mathrm{Locus}(V) \subseteq E_\sigma$, so V is not a dominating family.

Pick $x \in \mathrm{Locus}(V)$ and let F_σ be the fiber of σ through x; we have

$$\dim E_\sigma \geq \dim \mathrm{Locus}(V_x) + \dim F_\sigma \geq i_X + l_\sigma = n - 1,$$

so all the above inequalities are equalities; in particular we have $\dim \mathrm{Locus}(V_x) = i_X$ and so, by Proposition 2.5,

$$\dim \mathrm{Locus}(V) \geq n + i_X - 1 - \dim \mathrm{Locus}(V_x) = n - 1,$$

hence $\mathrm{Locus}(V) = E_\sigma$; therefore the above (in)equalities are true for every $x \in E_\sigma$.

Considering V as a family on the smooth variety E_σ we can write, again by Proposition 2.5 (a)

$$n - 1 + i_X = \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) \geq -K_E \cdot V + n - 2,$$

therefore $-K_{E_\sigma} \cdot V \leq i_X + 1$; on the other hand

$$-K_{E_\sigma} \cdot V = -K_X \cdot V - E_\sigma \cdot V \geq i_X + 1,$$

forcing $-K_{E_\sigma} \cdot V = i_X + 1$ and $E_\sigma \cdot V = -1$.

Then on E_σ we have two unsplit dominating families of rational curves verifying the assumptions of [19, Theorem 1], hence $E \simeq \mathbb{P}^{i_X} \times \mathbb{P}^{l_\sigma}$; in particular $\rho_{E_\sigma} = 2$.

Now let R be an extremal ray of X such that $E_\sigma \cdot R > 0$; by [18, Corollary 2.15] the contraction φ_R associated to R is a $\mathbb{P}^1$-bundle; in particular, by Proposition 2.7, this implies that $i_X = 2$.

Moreover, denoted by V^R a family of deformation of a minimal rational curve in R, we have $X = \mathrm{Locus}(V^R)_{E_\sigma}$, so $\rho_X = 3$ and the description of X is obtained arguing as in the proof of Proposition 7.3 in [18]. $\square$

Corollary 4.4. *In the assumptions of Theorem 1.1 either Y is a Fano manifold or $X = \mathbb{P}_{\mathbb{P}^{n-3}\times\mathbb{P}^2}(\mathcal{O}(1,1)\oplus\mathcal{O}(2,2))$, $Y \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1)^{n-2})$ and $B \simeq \mathbb{P}^2$ is the section corresponding to the surjection $\mathcal{O}\oplus\mathcal{O}(1)^{n-2} \to \mathcal{O}$.*

Proof. If Y is not Fano, then there exists an extremal ray $R' \in \mathrm{NE}(X)$ such that $E_\sigma \cdot R' < 0$. $\square$

Remark 4.5. Note that if $X \simeq \mathbb{P}_{\mathbb{P}^{n-3}\times\mathbb{P}^2}(\mathcal{O}(1,1)\oplus\mathcal{O}(2,2))$, then $\mathrm{NE}(X)$ is generated by three extremal rays: one – the $\mathbb{P}^1$-bundle contraction – is of fiber type, while the other two are smooth blow-ups with the same exceptional locus. In particular $\mathrm{NE}(X)$ is as in Theorem 1.1, case (d).

Corollary 4.6. *Let X, R_σ and E_σ be as in 4.1. Assume that Y is a Fano manifold and that there exists on X a family of rational curves V such that $E_\sigma \cdot V < 0$. Then $-K_X \cdot V \geq l_\sigma$; moreover, if V is unsplit, then $[V] \in R_\sigma$.*

Proof. Among the irreducible components of cycles in $\mathcal{V}$ there is at least one whose family of deformations V^{E_σ} is unsplit and such that $E_\sigma \cdot V^{E_\sigma} < 0$. By Theorem 4.3 we have that $[V^{E_\sigma}] \in R_\sigma$, hence

$$-K_X \cdot V \geq -K_X \cdot V^{E_\sigma} \geq l_\sigma.$$

To prove the last assertion note that, if V is an unsplit family, we can apply Theorem 4.3 directly to V. $\square$

5. Manifolds with a dominating (quasi)-unsplit family

In this section we will describe the cone of curves of Fano manifolds as in 4.1 which admit a minimal dominating quasi-unsplit family of rational curves W, and such that the target of the blow-up $\sigma\colon X \to Y$ is a Fano manifold.

If the family W is quasi-unsplit but not unsplit, then the result can be obtained easily:

Lemma 5.1. *Assume that W is not unsplit. Then $\rho_X = 2$, $i_X = 2$ and $\mathrm{NE}(X) = \langle R_\sigma, [W]\rangle$.*

Proof. Since W is not unsplit we have $-K_X \cdot W \geq 2i_X$; moreover, by the minimality assumption we have that W is locally unsplit. Consider the associated Chow family $\mathcal{W}$ and the rc$\mathcal{W}$-fibration $\pi\colon X \dashrightarrow Z$; since a general fiber of π contains $\mathrm{Locus}(W_x)$ for some x, and by Proposition 2.5 we have $\dim \mathrm{Locus}(W_x) \geq -K_X \cdot W - 1 \geq 2i_X - 1$ we have

$$\dim Z \leq n + 1 - 2i_X \leq n - 1 - i_X = \dim F_\sigma,$$

where F_σ is a fiber of σ.

A family V^σ of deformations of a minimal curve in R_σ is thereby horizontal and dominating with respect to π; moreover, since F_σ dominates Z we have that $X = \mathrm{Locus}(\mathcal{W})_{F_\sigma}$, hence $\mathrm{NE}(X) = \langle R_\sigma, [W]\rangle$ by Lemma 2.15. $\square$

In view of Lemma 5.1, we can assume throughout the section that W is an unsplit dominating family.

Lemma 5.2. *Let X be a Fano manifold with $\rho_X = 3$. Assume that there exists an effective divisor E which is negative on one extremal ray R of $\mathrm{NE}(X)$ and is nonnegative on the other extremal rays. If $E \cdot C = 0$ for a curve $C \subset X$ whose numerical class lies in $\partial\,\mathrm{NE}(X)$, then $[C]$ is contained in a two-dimensional face of $\mathrm{NE}(X)$ which contains R.*

Proof. The divisor E is not nef. Since E is effective, also $-E$ is not nef, hence the hyperplane $\{E = 0\}$ has nonempty intersection with the interior of $\mathrm{NE}(X)$ and the statement follows. $\square$

Lemma 5.3. *Assume that there exists an extremal ray R_τ such that $[W] \notin R_\tau$ and either $E_\sigma \cdot R_\tau > 0$ or $E_\sigma \cdot W > 0$. Then every fiber of the contraction τ associated to R_τ has dimension not greater than two. In particular τ is either a fiber type contraction or a smooth blow-up of a codimension three subvariety, and in this case the exceptional locus of τ is $E_\tau = \mathrm{Locus}(W, V^\tau)_{F_\sigma}$, for some fiber F_σ of σ.*

Proof. Let F_τ be a fiber of τ. If $E_\sigma \cdot R_\tau > 0$, there exists a fiber F_σ of σ which meets F_τ; since W is dominating we have $F_\sigma \subset \mathrm{Locus}(W)_{F_\sigma}$ and therefore $F_\tau \cap \mathrm{Locus}(W)_{F_\sigma} \neq \emptyset$.

If else $E_\sigma \cdot W > 0$, then $E_\sigma \cap \mathrm{Locus}(W)_{F_\tau} \neq \emptyset$, so there exists a fiber F_σ of σ such that $F_\sigma \cap \mathrm{Locus}(W)_{F_\tau} \neq \emptyset$; equivalently, we have that $F_\tau \cap \mathrm{Locus}(W)_{F_\sigma} \neq \emptyset$.

In both cases this intersection cannot be of positive dimension, since every curve in F_τ has numerical class belonging to R_τ, while every curve in $\mathrm{Locus}(W)_{F_\sigma}$ has numerical class contained in the cone $\langle R_\sigma, [W]\rangle$ by Lemma 2.15. By our assumptions

$$\dim \mathrm{Locus}(W)_{F_\sigma} \geq \dim F_\sigma + i_X - 1 = l_\sigma + i_X - 1 = n - 2,$$

hence $\dim F_\tau \leq 2$. Proposition 2.7 implies that τ cannot be a small contraction; if it is divisorial, by the same inequality it is equidimensional with two-dimensional fibers, so it is a smooth blow-up by [3, Theorem 5.1].

In this last case, denoted by V^τ a family of deformations of a minimal curve in R_τ, we have

$$\dim \mathrm{Locus}(W, V^\tau)_{F_\sigma} \geq n - 1,$$

hence $E_\tau = \mathrm{Locus}(W, V^\tau)_{F_\sigma}$. $\square$

Lemma 5.4. *Assume that $E_\sigma \cdot W = 0$. Let $\pi\colon X \dashrightarrow Z$ be the rcW-fibration and let V be a minimal horizontal dominating family with respect to π. Then R_σ, W and V are numerically independent. In particular $\rho_X \geq 3$.*

Proof. Since $E_\sigma \cdot W = 0$, E_σ does not dominate Z, hence E_σ cannot contain $\mathrm{Locus}(V)$ and therefore $E_\sigma \cdot V \geq 0$.

Let $\mathcal{H}$ be the pull-back to X of a very ample divisor in $\mathrm{Pic}(Z)$; $\mathcal{H}$ is zero on curves in the family W and it is positive outside the indeterminacy locus of π; in particular $\mathcal{H} \cdot V > 0$ since V is horizontal and $\mathcal{H} \cdot R_\sigma > 0$ since the indeterminacy locus has codimension at least two in X.

If $[V]$ were contained in the plane spanned by R_σ and $[W]$ we could write $[V] = \alpha[V^\sigma] + \beta[W]$, but intersecting with E_σ we would get $\alpha \leq 0$, while intersecting with $\mathcal{H}$ we would get $\alpha > 0$, a contradiction which proves the lemma. $\square$

Proposition 5.5. *Assume that $E_\sigma \cdot W = 0$. Let π be the rcW-fibration and let V be a minimal horizontal dominating family with respect to π. Then V is unsplit.*

Proof. Assume first that $E_\sigma \cdot V > 0$.

If V is not unsplit we will have, by Proposition 2.5 (a) for a general $x \in \mathrm{Locus}(V)$, that

$$\dim \mathrm{Locus}(V_x) \geq 2i_X - 1 \geq 3.$$

Since $E_\sigma \cdot V > 0$, then $E_\sigma \cap \mathrm{Locus}(V_x) \neq \emptyset$, therefore $\mathrm{Locus}(V_x)$ meets a fiber F_σ of σ. Moreover, since W is dominating, $F_\sigma \subset \mathrm{Locus}(W)_{F_\sigma}$ and so the intersection $\mathrm{Locus}(V_x) \cap \mathrm{Locus}(W)_{F_\sigma}$ is not empty. By Lemma 2.11

$$\dim \mathrm{Locus}(W)_{F_\sigma} \geq l_\sigma + i_X - 1 = n - 2,$$

so $\mathrm{Locus}(W)_{F_\sigma}$ contains a curve whose class is proportional to $[V]$, a contradiction by Lemma 5.4, since $\mathrm{NE}(\mathrm{Locus}(W)_{F_\sigma}) = \langle [W], R_\sigma \rangle$.

We will now deal with the harder case $E_\sigma \cdot V = 0$, assuming by contradiction that V is not unsplit.

We claim that E_σ has non zero intersection number with at least one component of a cycle of the Chow family $\mathcal{V}$. To prove the claim, consider the $\mathrm{rc}(W, \mathcal{V})$-fibration $\pi_{W,\mathcal{V}}$; a general fiber of $\pi_{W,\mathcal{V}}$ contains $\mathrm{Locus}(V, W)_x$ for some x, so it has dimension $\geq 3i_X - 2$.

Since E_σ is not contained in the indeterminacy locus of $\pi_{W,\mathcal{V}}$ - which has codimension at least two in X - it meets some fiber G of $\pi_{W,\mathcal{V}}$ which, by semicontinuity, has dimension $\geq 3i_X - 2$. Therefore there exists a fiber F_σ of σ such that $F_\sigma \cap G \neq \emptyset$. and, for such a fiber, we have

$$\dim(F_\sigma \cap G) \geq l_\sigma + 3i_X - 2 - n \geq 2i_X - 3 \geq 1;$$

Let C be a curve in $F_\sigma \cap G$; since $C \subset F_\sigma$ we have $E_\sigma \cdot C < 0$; on the other hand, since $C \subset G$ the numerical class of C can be written as a linear combination of $[W]$ and of classes of irreducible components of cycles in $\mathcal{V}$ by [2, Corollary 4.2]. Since $E_\sigma \cdot W = 0$ we see that E_σ cannot have zero intersection number with all the components of cycles in $\mathcal{V}$ and the claim is proved.

So in $\mathcal{V}$ there exists a reducible cycle $\Gamma = \sum_{i=1}^{k} \Gamma_i$ such that $E_\sigma \cdot \Gamma_1 < 0$. Then there exists an unsplit family T on which E_σ is negative and such that $[\Gamma_1] = [T] + [\Delta]$, with Δ an effective rational 1-cycle.

Since Y is a Fano manifold, by Corollary 4.6 we have that $[T] \in R_\sigma$ and $-K_X \cdot T \geq l_\sigma$; therefore, for a general $x \in \mathrm{Locus}(V)$, by Proposition 2.5 (b)

$$\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1 = -K_X \cdot (T + \Delta + \sum_{i=2}^{k} \Gamma_i) - 1 \geq l_\sigma + i_X - 1 = n - 2.$$

If $\dim \mathrm{Locus}(V_x) \geq n-1$, then $X = \mathrm{Locus}(W)_{\mathrm{Locus}(V_x)}$ and $\rho_X = 2$ against Lemma 5.4; therefore $\dim \mathrm{Locus}(V_x) = -K_X \cdot V - 1 = n-2$, hence V is a dominating family by Proposition 2.5, $\Gamma = \Gamma_1 + \Gamma_2$, $\Delta = 0$, $[\Gamma_1] \in R_\sigma$ and $-K_X \cdot \Gamma_2 = i_X$.

Pick a general $x \in \mathrm{Locus}(V)$ and let $D := \mathrm{Locus}(W)_{\mathrm{Locus}(V_x)}$. We have $\dim D \geq n-1$ by Lemma 2.11; moreover, since $\mathrm{N}_1(D) = \langle [W], [V] \rangle$ and $\rho_X \geq 3$ by Lemma 5.4, we cannot have $D = X$, hence D is an effective divisor.

We will now reach a contradiction by showing that D has zero intersection number with every extremal ray of X.

Let $\overline{V}$ be any unsplit family whose numerical class is not contained in the plane spanned by $[W]$ and $[V]$; we cannot have $\dim \mathrm{Locus}(\overline{V}_x) = 1$, otherwise $\overline{V}$ would be dominating of anticanonical degree 2, against the minimality of V. This implies that $D \cdot \overline{V} = 0$ since $\mathrm{N}_1(D) = \langle [W], [V] \rangle$ implies that $D \cap \mathrm{Locus}(\overline{V}_x) = \emptyset$.

It follows that $D \cdot \Gamma_2 = 0$ and that D is trivial on every extremal ray not lying in the plane $\langle [V], [W] \rangle$. Since $[V] = [\Gamma_1] + [\Gamma_2]$ and $[\Gamma_1] \in R_\sigma$, which is a ray not contained in the plane spanned by $[W]$ and $[V]$ we have that also $D \cdot V = 0$.

To conclude it is now enough to observe that we must have $D \cdot W = 0$, otherwise $\mathrm{ChLocus}_2(W)_{\mathrm{Locus}(V_x)} = X$, forcing again $\rho_X = 2$. We have thus reached a contradiction, since the effective divisor D has to be trivial on the whole $\mathrm{NE}(X)$. ☐

Proposition 5.6. *Assume that E_σ is trivial on every unsplit dominating family of rational curves of X. Then the cone of curves of X is generated by R_σ and two other extremal rays; one of them is of fiber type and it is spanned by the numerical class of W, the other is birational and the associated contraction is a smooth blow-up of a codimension three subvariety.*

Proof. Let π be the rc$W$-fibration, and let V be a minimal horizontal dominating family with respect to π. By Proposition 5.5 we know that V is unsplit.

We claim that V is not a dominating family. Assume by contradiction that $\overline{\mathrm{Locus}(V)} = X$.

If F_σ is any fiber of σ we have, by Lemma 2.11,

$$\dim \mathrm{Locus}(V, W)_{F_\sigma} \geq \dim F_\sigma + 2i_X - 2 = l_\sigma + 2i_X - 2 \geq n - 1.$$

Notice that, by the assumptions on the intersection numbers, we have $\mathrm{Locus}(V, W)_{F_\sigma} \subseteq E_\sigma$, and therefore $\mathrm{Locus}(V, W)_{F_\sigma} = E_\sigma$; in particular it follows from the above inequalities that $i_X = 2$.

We can repeat the same arguments to show that also $\mathrm{Locus}(W, V)_{F_\sigma} = E_\sigma$; hence every curve contained in E_σ is numerically equivalent to a linear combination

$$a[V^\sigma] + b[V] + c[W]$$

with a, b, $c \geq 0$ by Lemma 2.15, and therefore $\mathrm{NE}(E_\sigma) = \langle R_\sigma, [V], [W]\rangle$. In particular E_σ has nonpositive intersection with every curve it contains.

Let R_ϑ be an extremal ray such that $E_\sigma \cdot R_\vartheta > 0$; by [18, Corollary 2.15] the associated contraction $\vartheta\colon X \to Y$ is a $\mathbb{P}^1$-bundle; the associated family V^ϑ is dominating and unsplit and $E_\sigma \cdot V^\vartheta > 0$, a contradiction. We have thus proved that V is not dominating.

Consider the $\mathrm{rc}(W, V)$-fibration $\pi'\colon X \dashrightarrow Z'$; Z' has positive dimension since by Lemma 5.4 we have $\rho_X \geq 3$.

A general fiber F' of π' contains $\mathrm{Locus}(V, W)_x$ for some $x \in \mathrm{Locus}(V)$, hence $\dim F' \geq 2i_X - 1$ and thus

$$\dim Z' \leq n + 1 - 2i_X \leq l_\sigma.$$

Let X^0 be the open subset of X on which π' is defined; since $\dim(X \setminus X^0) \leq n - 2$, a general fiber F_σ of σ is not contained in the indeterminacy locus of π'. Moreover, curves in F_σ are not contracted by π', since, by Lemma 5.4, $[V]$, $[W]$ and R_σ are numerically independent. Hence $\pi'|_{F_\sigma \cap X^0}\colon F_\sigma \cap X^0 \to Z'$ is a finite morphism and we have $\dim Z' \geq \dim F_\sigma = l_\sigma$ and the above inequalities are equalities.

It follows that $i_X = 2$, $\dim Z' = l_\sigma$ and F_σ dominates Z'; this implies that X is $rc(W, V, V^\sigma)$-connected (V^σ is the family of deformations of a minimal curve in R_σ). More precisely $X = \mathrm{ChLocus}_m(W, V)_{F_\sigma}$ for some m and so, by Lemma 2.15, the numerical class of every curve in X can be written as

$$\alpha[V^\sigma] + \beta[W] + \gamma[V],$$

with $\alpha \geq 0$. This implies that the plane $\langle [V], [W]\rangle$ is extremal in $\mathrm{NE}(X)$.

By Corollary 4.6 we have that E_σ is nonnegative on the rays different from R_σ, hence, by Lemma 5.2 $[W]$ is in an extremal face with R_σ. Since $[W]$ is also in an extremal face with $[V]$ it follows that $[W]$ spans an extremal ray of $\mathrm{NE}(X)$, whose associated contraction is of fiber type.

Let R_τ be the extremal ray of $\mathrm{NE}(X)$ which lies in the face contained in the plane spanned by $[V]$ and $[W]$. We have $E_\sigma \cdot R_\tau > 0$, otherwise E_σ would be nonpositive on the whole cone. By Lemma 5.3 the associated contraction τ is either of fiber type with fibers of dimension ≤ 2 or a smooth blow-up.

In the first case, the family of deformations V^τ of a minimal curve in R_τ would be a dominating family on which E_σ is positive. Moreover, since by Proposition 2.7, taking into account that $\dim F_\tau \leq 2$ for every fiber of τ, we have $K_X \cdot V^\tau \leq 3 < 2i_X$ this family would also be unsplit, against our assumptions.

It follows that τ a smooth blow-up of a codimension three subvariety.

We claim that $E_\tau \cdot W > 0$. If $\mathrm{Locus}(V) \subset E_\tau$, then this follows from the fact that V is horizontal dominating with respect to the contraction of the ray spanned by $[W]$. If $E_\tau \cdot W = 0$, then we will have $E_\tau \cdot V < 0$, hence $\mathrm{Locus}(V) \subset E_\tau$ and the claim is proved.

It follows that V^τ is horizontal dominating with respect to the contraction of the ray spanned by $[W]$, so we can replace V by V^τ in the first part of the proof and get that X is $rc(W, V^\tau, V^\sigma)$-connected.

Let $\tau\colon X \to X'$ be the blow-down contraction; X' is then rationally connected with respect to the images of curves in W and in V^σ; since $\rho_{X'} = 2$ the images of curves in W are not numerically proportional to the images of curves in V^σ.

Let F_τ be a general fiber of τ, let $A = \tau(\mathrm{Locus}(V^\sigma)_{F_\tau})$ and $B = \tau(\mathrm{Locus}(W)_{F_\tau})$. Every curve in A is numerically proportional to the image of a curve of V^σ and every curve in B is numerically proportional to the image of a curve of W, hence $\dim(A \cap B) = 0$. Since F_τ is general and W is dominating we have $\dim B = \dim \mathrm{Locus}(W)_{F_\tau} \geq 2i_X - 1 = 3$, hence $\dim A \leq n - 3 = l_\sigma = \dim F_\sigma$.

This implies that every fiber of σ meeting F_τ is contained in E_τ, hence that $E_\tau \cdot R_\sigma = 0$. Now we can show that $\mathrm{NE}(X) = \langle [W], R_\sigma, R_\tau \rangle$. Assume by contradiction that there exists another extremal ray R; since $E_\tau \cdot R_\tau < 0$, $E_\tau \cdot W > 0$ and $E_\tau \cdot R_\sigma = 0$ we have $E_\tau \cdot R < 0$, but, by Lemma 5.3, $E_\tau = \mathrm{Locus}(W, V^\tau)_{F_\sigma}$ for some fiber F_σ of σ, hence, by Lemma 2.15, $\mathrm{NE}(E_\tau) = \langle [W], R_\sigma, R_\tau \rangle$. □

Theorem 5.7. *Let X be a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, with a contraction $\sigma\colon X \to Y$ which is the blow-up of a Fano manifold Y along a smooth subvariety B of dimension i_X. If X admits a dominating unsplit family of rational curves W, then the possible cones of curves of X are listed in the following table, where R_σ is the ray corresponding to σ, F stands for a fiber type contraction and D_{n-3} for a divisorial contraction whose exceptional locus is mapped to a subvariety of codimension three.*

ρ_X	i_X	R_1	R_2	R_3	R_4
2		R_σ	F		
3	*2,3*	R_σ	F	F	
3	*2*	R_σ	F	D_{n-3}	
4	*2*	R_σ	F	F	F

In particular Generalized Mukai Conjecture (*see* [9, 2]) *holds for X.*

Proof. Let V^σ be a family of deformations of a minimal rational curve in R_σ.

By Proposition 5.6 we can assume that $E_\sigma \cdot W > 0$; therefore the family V^σ is horizontal and dominating with respect to the rc$W$-fibration $\pi\colon X \dashrightarrow Z$.

It follows that a general fiber F' of the the $\mathrm{rc}(W, V^\sigma)$-fibration $\pi' \colon X \dashrightarrow Z'$ contains $\mathrm{Locus}(W)_{F_\sigma}$ for some fiber F_σ of σ, and therefore, by Lemma 2.11,

$$\dim F' \geq \dim \mathrm{Locus}(W)_{F_\sigma} \geq l_\sigma + i_X - 1 = n - 2,$$

hence $\dim Z' \leq 2$.

If $\dim Z' = 0$, then X is $\mathrm{rc}(W, V^\sigma)$-connected and $\rho_X = 2$; denote by R_ϑ the extremal ray of $\mathrm{NE}(X)$ different from R_σ. We claim that in this case $[W] \in R_\vartheta$. In fact, if this were not the case, R_ϑ would be a small ray by [11, Lemma 2.4], but in our assumptions we have $E \cdot R_\vartheta > 0$, against Lemma 5.3.

We can thus conclude that in this case $\mathrm{NE}(X) = \langle R_\sigma, R_\vartheta \rangle$ and that R_ϑ is of fiber type.

If $\dim Z' > 0$, take V' to be a minimal horizontal dominating family for π'; by [2, Lemma 6.5] we have $\dim \mathrm{Locus}(V'_x) \leq 2$, and therefore, by Proposition 2.5 (a)

$$-K_X \cdot V' \leq \dim \mathrm{Locus}(V'_x) + 1 \leq 3,$$

so V' is unsplit and $i_X \leq 3$.

Consider now the $\mathrm{rc}(W, V^\sigma, V')$-fibration $\pi'' \colon X \dashrightarrow Z''$: its fibers have dimension $\geq n - 1$ and so $\dim Z'' \leq 1$.

If $\dim Z'' = 0$ we have that X is $\mathrm{rc}(W, V^\sigma, V')$-connected and $\rho_X = 3$; by Lemma 5.3 every extremal ray of X has an associated contraction which is either of fiber type or divisorial.

The classes $[V^\sigma]$ and $[W]$ lie on an extremal face $\Sigma = \langle R_\sigma, R \rangle$ of $\mathrm{NE}(X)$, since, otherwise, by [11, Lemma 2.4], X would have a small contraction, against Lemma 5.3. Let $\mathcal{H}$ be the pull back via π of a very ample divisor on Z.

We know that $\mathcal{H} \cdot W = 0$ and $\mathcal{H} \cdot R_\sigma > 0$, since V^σ is horizontal and dominating with respect to π. It follows that $[W] \in R$ (and so R is of fiber type), since otherwise the exceptional locus of R would be contained in the indeterminacy locus of π, and thus the associated contraction would be small, contradicting again Lemma 5.3.

Assume that there exists an extremal ray R' not belonging to Σ such that its associated contraction is of fiber type. This ray must lie in a face of $\mathrm{NE}(X)$ with R by [11, Lemma 5.4].

If $E \cdot R' > 0$, we can exchange the role of R and R' and repeat the previous argument, therefore R' lies in a face with R_σ and $\mathrm{NE}(X) = \langle R_\sigma, R, R' \rangle$.

If $E \cdot R' = 0$, there cannot be any extremal ray in the half-space of $\mathrm{NE}(X)$ determined by the plane $\langle R', R_\sigma \rangle$ and not containing R, otherwise this ray would have negative intersection with E, contradicting Theorem 4.3. So again $\mathrm{NE}(X) = \langle R_\sigma, R, R' \rangle$.

We can thus assume that every ray not belonging to Σ is divisorial. Let R' be such a ray, denote by E' its exceptional locus, and by W' a family of deformations of a minimal rational curve in R'.

Recalling that, for a fiber F' of the rc(W, V^σ)-fibration π' we have $\dim F' \geq n-2$ we can write $E' = \mathrm{Locus}(W')_{F'}$. By Lemma 2.15 it follows that $\mathrm{NE}(E') = \langle R_\sigma, R, R' \rangle$. In particular E' cannot be trivial on Σ, otherwise it would be nonpositive on the whole $\mathrm{NE}(X)$.

We claim that R and R' lie on an extremal face of $\mathrm{NE}(X)$: if $E' \cdot R > 0$ the family W' is horizontal and dominating with respect to π and so R' and R are in a face by [11, Lemma 5.4]. If else $E' \cdot R = 0$ we have $E' \cdot R_\sigma > 0$. It follows that, if R and R' do not span an extremal face, there is an extremal ray R'' (in the half-space determined by $\langle R, R' \rangle$ and not containing R_σ) on which the divisor E' is negative. The exceptional locus of R'' must then be contained in E', contradicting the fact that $\mathrm{NE}(E') = \langle R_\sigma, R, R' \rangle$.

So we have proved that every ray not belonging to Σ lies in a face with R, and this implies that such a ray is unique and $\mathrm{NE}(X) = \langle R_\sigma, R, R' \rangle$.

Recalling that $E' = \mathrm{Locus}(W')_{F'}$ and that $\dim F' \geq n-2$ we have that every fiber of the contraction φ' associated to R' has dimension two; it follows that $i_X = 2$ and that φ' is a smooth blow-up of a codimension three subvariety by [3, Theorem 5.1].

Finally, if $\dim Z'' = 1$ consider a minimal horizontal dominating family V'' for π'': in this case $\rho_X = 4$, $i_X = 2$ and both V' and V'' are dominating. Let F_σ be a fiber of σ: then we can write $X = \mathrm{Locus}(V', V'')_{\mathrm{Locus}(W)_{F_\sigma}}$. By Lemma 2.15 every curve in X can be written with positive coefficients with respect to V^σ and W; but W, V' and V'' play a symmetric role, so we can conclude that $\mathrm{NE}(X) = \langle R_\sigma, [W], [V'], [V''] \rangle$, and all the three rays different from R_σ are of fiber type. $\square$

6. Manifolds without a dominating quasi-unsplit family

In this section we will show that the only Fano manifold as in 4.1 which does not admit a dominating quasi-unsplit family of rational curves is the blow-up of $\mathbb{G}(1,4)$ along a plane of bidegree $(0,1)$ (Theorem 6.7). In view of Theorem 5.7 this will conclude the proof of Theorem 1.1 and prove Theorem 1.2.

From now on we will thus work in the following setup:

6.1. *X is a Fano manifold of pseudoindex $i_X \geq 2$ and dimension $n \geq 6$, which does not admit a quasi-unsplit dominating family of rational curves and has a contraction $\sigma\colon X \to Y$ which is the blow-up of a manifold Y along a smooth subvariety B of dimension i_X. We denote by R_σ the extremal ray corresponding to σ, by l_σ its length and by E_σ its exceptional locus.*

In view of Corollary 4.4 we can assume that Y is a Fano manifold. We need some preliminary work to establish some properties of families of rational curves on X and Y.

Lemma 6.2. *Assume that $\rho_X = 2$. Let W' be a minimal dominating family of rational curves for Y. Then $-K_Y \cdot W' \geq n-1$.*

Proof. Let W^* be a family of deformations of the strict transform of a general curve of W'. The family W^* is dominating and therefore, by 6.1, not quasi-unsplit. Moreover, by Corollary 2.3, we have $E_\sigma \cdot W^* = 0$, hence there exists a component Γ_1^* of a reducible cycle Γ^* in $\mathcal{W}^*$ such that $E_\sigma \cdot \Gamma_1^* < 0$.

By Corollary 4.6 we have $-K_X \cdot \Gamma_1^* \geq l_\sigma$, and therefore

$$-K_Y \cdot W' = -K_X \cdot W^* \geq l_\sigma + i_X = n - 1. \qquad \square$$

Proposition 6.3. *Let X, Y, R_σ and E_σ be as in 6.1. Then there does not exist on X any locally unsplit dominating family W such that $E_\sigma \cdot W > 0$.*

Proof. Assume that such a family W exists; we will derive a contradiction showing that in this case $n = 5$.

First of all we prove that $i_X = 2$ and that X is rationally connected with respect to the Chow family $\mathcal{W}$ associated to W and to V^σ, the family of deformations of a general curve of minimal degree in R_σ.

Since $E_\sigma \cdot W > 0$, for a general $x \in X$, the intersection $E_\sigma \cap \mathrm{Locus}(W_x)$ is nonempty. On the other hand, the fact that $E_\sigma \cdot V^\sigma < 0$ yields that the families W and V^σ are numerically independent, and therefore, for every fiber F_σ of σ and for a general $x \in X$, we have $\dim(\mathrm{Locus}(W_x) \cap F_\sigma) \leq 0$.

Now, if we denote by F_σ a fiber of σ which meets $\mathrm{Locus}(W_x)$, it follows that

$$2i_X - 1 \leq -K_X \cdot W - 1 \leq \dim \mathrm{Locus}(W_x) \leq n - \dim F_\sigma \leq n - l_\sigma = i_X + 1,$$

whence $i_X = 2$, $\dim \mathrm{Locus}(W_x) = i_X + 1 = 3$ and $-K_X \cdot W = 2i_X = 4$.

In particular $\dim(E_\sigma \cap \mathrm{Locus}(W_x)) = 2 = \dim B$, hence $\sigma(E_\sigma \cap \mathrm{Locus}(W_x)) = B$ and every fiber of σ meets $\mathrm{Locus}(W_x)$.

Let x and y be two general points in X; every fiber of σ meets both $\mathrm{Locus}(W_x)$ and $\mathrm{Locus}(W_y)$, so the points x and y can be connected using two curves in W and a curve of V^σ. This implies that X is $\mathrm{rc}(\mathcal{W}, V^\sigma)$-connected.

Our next step consists in proving that $\rho_X = 2$, showing that the numerical class of every irreducible component of any cycle of $\mathcal{W}$ lies in the plane Π spanned in $\mathrm{N}_1(X)$ by $[W]$ and R_σ.

Let $x \in X$ be a general point; by Lemma 2.11 we have

$$\dim \mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W_x)} \geq l_\sigma + 2i_X - 2 \geq n - 1,$$

therefore $E_\sigma = \mathrm{Locus}(V^\sigma)_{\mathrm{Locus}(W_x)}$ and $\mathrm{N}_1(E_\sigma) = \Pi$ by Lemma 2.15.

We have already proved that $-K_X \cdot W = 4$ and $i_X = 2$; therefore every reducible cycle of $\mathcal{W}$ has exactly two irreducible components, and the families of deformations of these components are unsplit.

Let $\Gamma_1 + \Gamma_2$ be a reducible cycle of $\mathcal{W}$; without loss of generality we can assume that $E_\sigma \cdot \Gamma_1 > 0$. Denote by W^1 a family of deformations of Γ_1; being unsplit, the family W^1 cannot be dominating, hence for every $x \in \mathrm{Locus}(W^1)$ we have $\dim \mathrm{Locus}(W_x^1) \geq 2$ by Proposition 2.5. Since $E_\sigma \cap \mathrm{Locus}(W_x^1) \neq \emptyset$ it follows that $\dim(E_\sigma \cap \mathrm{Locus}(W_x^1)) \geq 1$ for every $x \in \mathrm{Locus}(W^1)$, so $[W^1] \in \Pi$, and consequently also $[W^2] \in \Pi$; it follows that $\rho_X = 2$.

Let now T_Y be a minimal dominating family of rational curves for Y and let T be the family of deformations of the strict transform of a general curve of T_Y. By Lemma 6.2 we have $-K_X \cdot T = -K_Y \cdot T_Y \geq n-1$.

By this last inequality, the intersection $\mathrm{Locus}(W_x) \cap \mathrm{Locus}(T_x)$ for a general $x \in X$ has positive dimension; since T is numerically independent from W – recall that $E_\sigma \cdot T = 0$ and $E_\sigma \cdot W > 0$ – the family T cannot be locally quasi-unsplit.

Therefore, in the associated Chow family $\mathcal{T}$, there exists a reducible cycle $\Lambda = \Lambda_1 + \Lambda_2$ such that a family of deformations T^1 of Λ_1 is dominating and numerically independent from T.

The family T^1, being dominating, cannot be unsplit, hence $-K_X \cdot T^1 \geq 4$; moreover, since T^1 is also numerically independent from T we have $E_\sigma \cdot T^1 > 0$. It follows that $E_\sigma \cdot \Lambda_2 < 0$ and so $-K_X \cdot \Lambda_2 \geq l_\sigma$ by Lemma 4.6. Therefore

$$-K_Y \cdot T_Y = -K_X \cdot T \geq l_\sigma + 2i_X = n+1$$

so $Y \simeq \mathbb{P}^n$ by Corollary 3.2.

The center B of σ cannot be a linear subspace of Y, since otherwise $i_X + l_\sigma = n+1$; take l to be a proper bisecant of B and let $\widetilde{l}$ be its strict transform: we have

$$2 = i_X \leq -K_X \cdot \widetilde{l} = n+1-2l_\sigma = 4 - l_\sigma,$$

hence $l_\sigma = 2$ and $n = 5$. □

Corollary 6.4. *Let X, Y, R_σ and E_σ be as in 6.1. Then there does not exist any family of rational curves V independent from R_σ such that V_x is unsplit for some $x \in E$ and such that $E \subseteq \overline{\mathrm{Locus}(V)}$.*

Proof. Assume by contradiction that such a family exists.

First of all we prove that V cannot be unsplit. If this is the case, since on X there are no unsplit dominating families it must be $\overline{\mathrm{Locus}(V)} = \mathrm{Locus}(V) = E$. Moreover, by Proposition 2.5 (a) we have $\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V$ for every $x \in \mathrm{Locus}(V)$. We apply Lemma 2.11 (a) and Proposition 2.7 to get that $\dim \mathrm{Locus}(V)_{F_\sigma} = n-1$ for every fiber F_σ of σ. It follows that $E = \mathrm{Locus}(V)_{F_\sigma}$ and therefore $\mathrm{NE}(E) = \langle R_\sigma, [V] \rangle$ by Lemma 2.15.

Since V is a dominating unsplit family for the smooth variety E, by Proposition 2.5 (b) we have $-K_E \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$, hence, by adjunction, $E \cdot V < 0$; since V is numerically independent from R_σ it follows from Theorem 4.3 that Y is not a Fano manifold, a contradiction.

Since V is not unsplit we have $-K_X \cdot V \geq 2i_X$ and therefore, by Proposition 2.5 (b), for a point $x \in E$ such that V_x is unsplit, we have

$$\dim \mathrm{Locus}(V_x) \geq -K_X \cdot V - 1 \geq 2i_X - 1.$$

On the other hand, since V is numerically independent from R_σ, we have, for any fiber F_σ of σ, that $\dim \mathrm{Locus}(V_x) \cap F_\sigma \leq 0$, hence $\dim \mathrm{Locus}(V_x) \leq n - l_\sigma = i_X + 1$.

It follows that $i_X = 2$, $-K_X \cdot V = 4$ and $\dim \mathrm{Locus}(V_x) = 3$; the last two equalities, by Proposition 2.5, imply that V is dominating.

Moreover, since $-K_X \cdot V = 4$, the family V is also locally unsplit, otherwise we would have a dominating family of lower degree, hence unsplit.

Since $E \cap \mathrm{Locus}(V_x)$ is not empty and we cannot have $\mathrm{Locus}(V_x) \subset E$ – recall that V_x is unsplit and V is independent from R_σ, so $\mathrm{Locus}(V_x)$ can meet fibers of σ only in points – it follows that $E \cdot V > 0$ and we can apply Proposition 6.3. □

Remark 6.5. If $C_Y \subset Y$ is a rational curve which meets the center B of the blow-up in k points and is not contained in it, then $-K_Y \cdot C_Y \geq n-1+(k-1)l_\sigma$.

Proof. Let C be the strict transform of C_Y: then the statement follows from the canonical bundle formula

$$-K_X = -\sigma^* K_Y - l_\sigma E,$$

which yields

$$-K_Y \cdot C_Y = -K_X \cdot C + l_\sigma E \cdot C \geq i_X + k l_\sigma \geq n - 1 + (k-1) l_\sigma. \qquad \square$$

Corollary 6.6. *Let W_Y be a minimal dominating family for Y and assume that $-K_Y \cdot W_Y = n-1$. Assume that there exists a reducible cycle Γ in $\mathcal{W}_Y$ which meets B. Then $\Gamma \subset B$ and $\mathrm{NE}(B) = \langle [W_Y] \rangle$.*

Proof. Let Γ_i be a component of Γ: we know that $-K_Y \cdot \Gamma_i < n-1$, so the whole cycle Γ has to be contained in B by Remark 6.5.

Let W_Y^i be a family of deformations of Γ_i; the pointed locus $\mathrm{Locus}(W_Y^i)_b$ is contained in B for every $b \in B$, again by Remark 6.5, hence

$$-K_Y \cdot W_Y^i \leq \dim \mathrm{Locus}(W_Y^i)_b \leq \dim B = i_X \leq i_Y,$$

where the last inequality follows from [7, Theorem 1, (iii)].

Therefore W_Y^i is unsplit and $B = \mathrm{Locus}(W_Y^i)_b$, hence $\mathrm{NE}(B) = \langle [W_Y^i] \rangle$ by Lemma 2.15. It follows that all the components Γ_i of Γ are numerically proportional, and thus they are all numerically proportional to W_Y. □

We are now ready to prove the following:

Theorem 6.7. *Let X be a Fano manifold of dimension $n \geq 6$ and pseudoindex $i_X \geq 2$, which is the blow-up of another Fano manifold Y along a smooth subvariety B of dimension i_X; assume that X does not admit a quasi-unsplit dominating family of rational curves. Then $Y \simeq \mathbb{G}(1,4)$ and B is a plane of bidegree $(0,1)$.*

Proof. The proof is quite long and complicated; we will divide it into different steps, in order to make our procedure clearer.

Step 1. *A minimal dominating family of rational curves on Y has anticanonical degree $n-1$.*

Let W_Y be a minimal dominating family of rational curves for Y, and let W be the family of deformations of the strict transform of a general curve of W_Y.

Apply [4, Lemma 4.1] to W (note that in the proof of that lemma the minimality of W is not needed). The first case in the lemma cannot occur by Corollary 6.4, so there exists a reducible cycle $\Gamma = \Gamma_\sigma + \Gamma_V + \Delta$ in $\mathcal{W}$ with $[\Gamma_\sigma]$ belonging to R_σ, Γ_V belonging to a family V, independent from R_σ, such that V_x is unsplit for some $x \in E_\sigma$, and Δ an effective rational 1-cycle. In particular

$$-K_X \cdot W \geq -K_X \cdot (\Gamma_\sigma + \Gamma_V + \Delta) \geq l_\sigma + i_X = n - 1. \tag{6.7.2}$$

By the canonical bundle formula and Corollary 2.3 we have that

$$-K_Y \cdot W_Y = -K_X \cdot W \geq n - 1.$$

If $-K_Y \cdot W_Y = n+1$, then Y is a projective space by Corollary 3.2. The center of σ cannot be a linear subspace, otherwise as in the proof of Proposition 6.3 we can show that $l_\sigma = 2$ and $n = 5$, against the assumptions.

We can thus assume that $-K_Y \cdot W_Y \leq n$.

Note that, by (6.7.2), the reducible cycle Γ has only two irreducible components Γ_σ and Γ_V; moreover the class of Γ_σ is minimal in R_σ, hence $E_\sigma \cdot \Gamma_\sigma = -1$, and $-K_X \cdot V \leq i_X + 1$. In particular V is an unsplit family.

Recalling that $E_\sigma \cdot W = 0$ we get $E_\sigma \cdot \Gamma_V = 1$. Geometrically, a general curve of V is the strict transform of a curve in W_Y which meets B in one point; moreover, since a curve of W_Y not contained in B cannot meet B in more than one point by Remark 6.5, we have that

$$\sigma(\operatorname{Locus}(V) \setminus E_\sigma) = \operatorname{Locus}(W_Y)_B \setminus B. \tag{6.7.3}$$

Assume that $-K_Y \cdot W_Y = n$; in this case $\rho_Y = 1$ by Corollary 3.5.

For a general point $y \in Y$, we have that $\operatorname{Locus}(W_Y)_y$ is an effective, hence ample, divisor, so it meets B. In particular we have $\dim \operatorname{Locus}(W_Y)_B = n$, and by (6.7.3) this implies that V is dominating, against the assumptions since V is unsplit. This completes Step 1.

Notice that $-K_Y \cdot W_Y = n - 1$ implies that all inequalities in (6.7.2) are equalities. In particular it follows that $-K_X \cdot V = i_X$.

Step 2. *The strict transforms of curves in a minimal dominating family of rational curves on Y which meet B fill up a divisor on X.*

Let x be a point in $E_\sigma \cap \operatorname{Locus}(V)$ and let F_σ be the fiber of σ containing x; since $\dim F_\sigma + \dim \operatorname{Locus}(V_x) \leq n$ we have

$$\dim \operatorname{Locus}(V_x) \leq n - l_\sigma = i_X + 1.$$

By Proposition 2.5 (a) we have that $\dim \operatorname{Locus}(V) \geq n-2$; since V is an unsplit family it cannot be dominating, so we need to show that $\dim \operatorname{Locus}(V) \neq n-2$.

Assume by contradiction that $\dim \operatorname{Locus}(V) = n-2$; in this case, by Proposition 2.5 (b), for every $x \in \operatorname{Locus}(V)$ we have $\dim \operatorname{Locus}(V_x) = i_X + 1$, so for every $x \in X$ the intersection $\operatorname{Locus}(V_x) \cap E_\sigma$ dominates B.

Consider a point $x \in \mathrm{Locus}(V) \setminus E_\sigma$, denote by y its image $\sigma(x)$ and consider $\mathrm{Locus}(\mathcal{W}_Y)_y$: since $\mathrm{Locus}(V_x) \cap E_\sigma$ dominates B, we have $B \subset \mathrm{Locus}(\mathcal{W}_Y)_y$. But cycles in $\mathcal{W}_Y$ passing through y and meeting B are irreducible by Corollary 6.6, so $B \subseteq \mathrm{Locus}(W_Y)_y$ and by Lemma 3.1 the numerical class of every curve in B is proportional to $[W_Y]$. This fact together with Corollary 6.6 allows us to conclude that B does not meet any reducible cycle of $\mathcal{W}_Y$.

We claim that a general curve C of W_Y is contained in the open subset U of points $y \in Y$ such that $(W_Y)_y$ is proper. If this were not true, then $\mathrm{Locus}(W_Y) \setminus U$ should have codimension one, and so there would exist a family W_Y^1 of deformations of an irreducible component of a cycle of $\mathcal{W}_Y$ whose locus is a divisor; moreover this divisor should have positive intersection number with W_Y.

This last condition would imply that $\mathrm{Locus}(W_Y^1)$ has nonempty intersection with B, since the numerical class of any curve in B is an integral multiple of $[W_Y]$, but we have proved that B does not meet any reducible cycle of $\mathcal{W}_Y$, so we have reached a contradiction that proves the claim.

Therefore we can apply Lemma 3.6 and get that a component of $\mathrm{Locus}(W_Y)_C$ is a divisor, call it D_C, such that $D_C \cdot W_Y > 0$ and moreover $\rho_Y = 1$, since in the other case of the quoted lemma we would find a family of rational curves of anticanonical degree two meeting B, against Remark 6.5.

Being $\rho_Y = 1$ the effective divisor D_C is ample, hence it meets B; therefore for a general curve C in W_Y there exists another curve of W_Y which meets both B and C; in other words, a general curve of W_Y meets $\mathrm{Locus}(W_Y)_B$, a contradiction, since $\mathrm{Locus}(W_Y)_B$ has codimension two in Y by (6.7.3).

Step 3. *The Picard number of Y is one.*

By (6.7.3) we have that $\dim \mathrm{Locus}(W_Y)_B = \dim \mathrm{Locus}(V) = n - 1$. This implies that B contains curves whose numerical class is proportional to $[W_Y]$, otherwise by Lemma 2.11 we would have $\dim \mathrm{Locus}(W_Y)_B = n$.

If B does not meet any reducible cycle of $\mathcal{W}_Y$ we can argue as in the claim in Step 2 and conclude that $\rho_Y = 1$.

If else B meets a reducible cycle of $\mathcal{W}_Y$, then, by Corollary 6.6, every curve in B is numerically proportional to $[W_Y]$, hence $\mathrm{NE}(\mathrm{Locus}(W_Y)_B) = \langle [W_Y] \rangle$ and we conclude that $\rho_Y = 1$ by Lemma 3.4.

Step 4. *The numerical classes of the strict transforms of curves in a minimal dominating family of rational curves on Y which meet B are extremal in* $\mathrm{NE}(X)$.

Let $D = \mathrm{Locus}(V)$; by Step 2 D is a divisor. Since $E_\sigma \cdot W = 0$ and $\mathrm{Pic}(X) = \langle E_\sigma, D \rangle$ we have $D \cdot W > 0$.

Therefore $\mathrm{Locus}(W, V)_x = \mathrm{Locus}(V)_{\mathrm{Locus}(W_x)}$ is nonempty for a general $x \in X$, and so has dimension $\geq n - 2 + i_X - 1 \geq n - 1$ by Lemma 2.11. It follows that $i_X = 2$ and $D = \mathrm{Locus}(W, V)_x$.

The last equality, by Lemma 2.15, yields that every curve in D is numerically equivalent to a linear combination $a[W] + b[V]$ with $a \geq 0$.

This implies that $\mathrm{NE}(D)$ is contained in the cone spanned by $[V]$ and by an extremal ray R of $\mathrm{NE}(X)$. Since $E_\sigma \cdot W = 0$ and $E_\sigma \cdot V > 0$ it must be $E_\sigma \cdot R < 0$, so $R = R_\sigma$ and $\mathrm{NE}(D) \subseteq \langle R_\sigma, [V]\rangle$.

Let R_τ be the extremal ray of $\mathrm{NE}(X)$ different from R_σ and denote by τ the associated contraction. The contraction τ is birational, since X does not admit quasi-unsplit dominating families of rational curves, therefore its fibers have dimension at least two by Proposition 2.7.

We claim that $[V] \in R_\tau$; if we assume that this is not the case, then $D \cap \mathrm{Exc}(\tau) = \emptyset$, since otherwise D will meet a fiber F_τ of τ, hence $\dim D \cap F_\tau \geq 1$, contradicting $\mathrm{NE}(D) \subseteq \langle R_\sigma, [V]\rangle$.

It follows that $D \cdot R_\tau = 0$, so $D \cdot R_\sigma > 0$ (and thus $\mathrm{NE}(D) = \langle R_\sigma, [V]\rangle$, since fibers of σ have dimension $l_\sigma = n - 1 - i_X = n - 3 \geq 3$, hence $\dim(D \cap F_\sigma) > 0$ for every fiber F_σ of σ).

Notice also that the effective divisor E_σ must be positive on R_τ.

Let F_σ and F_τ be two meeting fibers of the contractions σ and τ respectively; we have $\dim(F_\sigma \cap F_\tau) = 0$, hence

$$n \geq \dim F_\sigma + \dim F_\tau \geq l_\sigma + l_\tau.$$

Therefore, recalling that $i_X = 2$ and thus $l_\sigma = n - 3$, we have $l_\tau \leq 3$, so $\dim \mathrm{Exc}(\tau) \geq n - 2$ by Proposition 2.7.

In particular, if F_σ is a fiber of σ meeting $\mathrm{Exc}(\tau)$ we have

$$\dim(F_\sigma \cap \mathrm{Exc}(\tau)) \geq l_\sigma - 2 \geq 1.$$

Let C be a curve in $F_\sigma \cap \mathrm{Exc}(\tau)$; since $D \cdot R_\sigma > 0$ we have $D \cap C \neq \emptyset$, hence $D \cap \mathrm{Exc}(\tau) \neq \emptyset$, a contradiction that proves the extremality of $[V]$.

Step 5. *The contraction of X different from σ is the blow-up of $\mathbb{P}^n$ along a smooth subvariety of codimension three.*

Since $[V] \in R_\tau$ we have $D = \mathrm{Locus}(V) \subset \mathrm{Exc}(\tau)$; being τ birational it follows that $D = \mathrm{Exc}(\tau)$ and τ is divisorial; we will denote from now on the exceptional divisor by E_τ.

Since $E_\tau = \mathrm{Locus}(W, V)_x$ for a general $x \in X$ every fiber of τ meets $\mathrm{Locus}(W_x)$, so from $\dim(F_\tau \cap \mathrm{Locus}(W_x)) = 0$ we derive

$$\dim F_\tau \leq n - \dim \mathrm{Locus}(W_x) \leq 2.$$

On the other hand, by Proposition 2.7, we have $\dim F_\tau \geq 2$ for every fiber of τ, hence $\tau|_{E_\tau}$ is equidimensional; we can apply [3, Theorem 5.1] to get that $\tau\colon X \to Z$ is a smooth blow-up.

Let T_Z be a minimal dominating family of rational curves for Z and T^* a family of deformations of the strict transform of a general curve of T_Z.

Among the families of deformations of the irreducible components of cycles in $\mathcal{T}^*$ there is at least one family which is dominating and locally unsplit; call it T.

By Proposition 6.3 we have $E_\sigma \cdot T = 0$, therefore T is numerically proportional to W; If $-K_X \cdot T < -K_X \cdot W$, then the images in Y of the curves in T would be a dominating family for Y of degree smaller than the degree of W_Y, a contradiction, hence $-K_X \cdot T \geq -K_X \cdot W = n - 1$.

Notice also that, since $E_\tau \cdot T^* = 0$ and $\mathrm{Pic}(X)$ is generated by E_σ and E_τ we cannot have $T = T^*$. In particular $-K_X \cdot T^* \geq -K_X \cdot T + i_X$. It follows that

$$-K_Z \cdot T_Z = -K_X \cdot T^* \geq -K_X \cdot T + i_X \geq n + 1,$$

so $Z \simeq \mathbb{P}^n$ by Corollary 3.2 and T_Z is the family of lines in Z.

Step 6. *Conclusion.*

Take $l_\sigma - 2$ general sections $H_i \in |\tau^*\mathcal{O}_{\mathbb{P}^n}(1)|$; their intersection $\mathcal{I}$ is a Fano manifold of dimension five with two blow-up contractions of length two $\sigma_{|\mathcal{I}} \colon \mathcal{I} \to Y'$ and $\tau_{|\mathcal{I}} \colon \mathcal{I} \to \mathbb{P}^5$.

By the classification in [11] two cases are possible: either the center of $\tau_{|\mathcal{I}}$ is a Veronese surface or it is a cubic scroll contained in a hyperplane. The first case can be excluded observing that, in our case, the degree of E_σ on a minimal curve in R_τ is one, since $E_\sigma \cdot W = 0$ and $E_\sigma \cdot R_\sigma = -1$.

It follows that Y' is a del Pezzo manifold of degree five, i.e., a linear section of $\mathbb{G}(1,4)$; Y has Y' as an ample section, and therefore Y is $\mathbb{G}(1,4)$ by [17, Proposition A.1]. The center of $\sigma_{|\mathcal{I}} \colon \mathcal{I} \to Y'$ is a plane of bidegree $(0,1)$ by [20, Theorem XLI]. □

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