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ON QUASI-STABLE EXCHANGE IDEALS

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Abstract. We introduce, in this article, the quasi-stable exchange ideal for associative rings. If I is a quasi-stable exchange ideal of a ring R, then so is $M_n(I)$ as an ideal of $M_n(R)$. As an application, we prove that every square regular matrix over quasi-stable exchange ideal admits a diagonal reduction by quasi invertible matrices. Examples of such ideals are given as well.

1. Introduction

Following Ara (cf. [1]), an ideal I of a ring R is an exchange ideal provided that for every $x \in I$ there exist an idempotent $e \in I$ and elements $r, s \in I$ such that $e = xr = x + s - xs$. Clearly, an ideal I of a ring R is an exchange ideal if and only if for any $x \in I$, there exists an idempotent $e \in xR$ such that $1-e \in (1-x)R$. Exchange ideal plays a key role in the direct sum decomposition theory of exchange rings. Many authors have studied such ideals, e.g., [1] and [12].

So as to investigate directly infinite rings, we introduce a new class of exchange ideals, i.e., quasi-stable exchange ideals of a ring R. If I is a quasi-stable exchange ideal of a ring R, we will show that $M_n(I)$ is a quasi-stable exchange ideal of $M_n(R)$. As is well known, every square matrix over a unit-regular ring admits a diagonal reduction. Ara et. al. extended this result and proved that every square regular matrix over a separative exchange ring admits a diagonal reduction by invertible matrices (cf. [2]). It is interesting to investigate diagonal reduction of matrices over an ideal of a ring R even though there exist some square matrices over R which can not be reduced. As an application, we prove that every square regular matrix over quasi-stable exchange ideal admits a diagonal reduction by quasi invertible matrices. These also give nontrivial generalizations of [4, Theorem 16] and [6, Theorem 11].

Throughout, all rings are associative with identity, all ideals are two-sided ideals and all modules are right unitary modules. We use $M_n(R)$ to denote

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the ring of $n \times n$ matrices over R with identity I_n. $GL_n(R)$ denotes the n-dimensional general linear group of R. Set $GL_n(I) = GL_n(R) \bigcap \big(I_n + M_n(I)\big)$. An element $x \in R$ is regular provided that $x = xyx$ for a $y \in R$. $\Gamma(I)$ stands for the set of all products of a left invertible element and a right invertible element in $1 + I$, i.e., $\{uv \in R \mid \exists\ s, t \in 1 + I \text{ such that } su = 1, vt = 1\}$.

2. Equivalent characterizations

Definition 2.1. Let I be an ideal of a ring R. We say that I is a right quasi-stable ideal if $aR + bR = R$ with $a \in I, b \in R$ implies that there exists $y \in R$ such that $a + by \in \Gamma(I)$. We say that I is a left quasi-stable ideal if $Ra + Rb = R$ with $a \in I, b \in R$ implies that there exists $z \in R$ such that $a + zb \in \Gamma(I)$. An ideal I of a ring R is a quasi-stable ideal in case it is both right and left quasi-stable ideal.

Let $J(R)$ be the Jacobson radical of rings R. If $ax + b = 1$ with $a \in J(R), x, b \in R$, then $b \in U(R)$. Hence, $a + b \cdot b^{-1} = 1 + J(R) \in \Gamma\big(J(R)\big)$. Thus, $J(R)$ is a right quasi-stable exchange ideal. The purpose of this section is to investigate several equivalent characterizations of right quasi-stable ideals. The left quasi-stable ideals have analogous results.

Theorem 2.2. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is a right quasi-stable ideal.*
(2) *Every element in I is a product of an idempotent in I and an element in $\Gamma(I)$.*

Proof. (1)⇒(2) Given any $x \in I$, there exists $y \in I$ such that $x = xyx$. Since $xy + (1 - xy) = 1$ with $x \in I$, we have $z \in R$ such that $x + (1 - xy)z = w \in \Gamma(I)$. So $x = xyx = xy\big(x + (1 - xy)z\big) = ew$, where $e = xy \in I$ is an idempotent.

(2)⇒(1) Suppose that $ax + b = 1$ with $a \in I, x, b \in R$. Then $b \in 1 + I$. Since I is an exchange ideal of R, by [1, Lemma 1.1], we have an idempotent $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. Hence $axt + e = 1$, and then $(1 - e)axt + e = 1$. So $(1 - e)a \in I$ is regular. Thus we have an idempotent $f \in I$ and a $w \in \Gamma(I)$ such that $(1 - e)a = fw$. So $fwxt + e = 1$, and then $fwxt(1 - f) + e(1 - f) = 1 - f$. We infer that $f + e(1 - f) = 1 - fwxt(1 - f)$. Hence, $(1 - e)a + e(1 - f)w = fw + e(1 - f)w = \big(1 - fwxt(1 - f)\big)w$. As a result, $a + bs\big((1 - f)w - a\big) = \big(1 + fwxt(1 - f)\big)^{-1}w \in \Gamma(I)$. Therefore I is a right quasi-stable ideal. □

Corollary 2.3. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is a right quasi-stable ideal.*
(2) *Whenever $ax + b = 1$ with $a, x \in I, b \in 1 + I$, there exists $y \in R$ such that $a + by \in \Gamma(I)$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $x \in I$ be regular. Then we have $y \in I$ such that $x = xyx$. Since $xy + (1 - xy) = 1$ with $x, y \in I, 1 - xy \in 1 + I$, by hypothesis, there exists $z \in R$ such that $x + (1 - xy)z = w \in \Gamma(I)$. Thus, $x = xyx = xy\big(x + (1 - xy)z\big) = ew$, where $e = xy \in I$ is an idempotent. According to Theorem 2.2, we obtain the result. □

Recall that an ideal I of a ring R has stable range one provided that $aR + bR = R$ with $a \in I, b \in R$ implies that there exists $y \in R$ such that $a + by \in GL_1(R)$. We recall a simple known result.

Lemma 2.4. *Given $ax + b = 1, a, x, b \in R$, then the following hold*:

(1) *If $u(a+by) = 1$, then $\big(x+(1-xy)ub\big)\big(a+y(1-xa)\big) = 1$. If $(a+by)u = 1$, then $\big(a + y(1 - xa)\big)\big(x + (1 - xy)ub\big) = 1$.*
(2) *If $(x+zb)v = 1$, then $\big(x+(1-xa)z\big)\big(a+bv(1-za)\big) = 1$. If $v(x+zb) = 1$, then $\big(a + bv(1 - za)\big)\big(x + (1 - xa)z\big) = 1$.*

Proof. Straightforward. □

Proposition 2.5. *Let I be an exchange ideal of a ring R. If I has stable range one, then I is a right quasi-stable ideal*

Proof. Assume that $ax + b = 1$ with $a, x \in I, b \in 1 + I$. Then $\big(a + (1 - a)b\big)(x + b) + (1 - a)b\big(1 - (x + b)\big) = 1$, where $a + (1 - a)b \in 1 + I$. Since I has stable range one, we have $y \in R$ such that $\big(a + (1 - a)b\big) + (1 - a)b\big(1 - (x + b)\big)y \in GL_1(I)$. That is, $a+(1-a)b\big(1+(1-(x+b))y\big) \in GL_1(I)$. As $a(x+b)+(1-a)b = 1$, we can find $z \in R$ such that $x+b+z(1-a)b \in GL_1(I)$, i.e., $x+\big(1+z(1-a)\big)b \in GL_1(I)$. By using Lemma 2.4 again, we have $t \in R$ such that $a+bt \in GL_1(I)$. Therefore I is a right quasi-stable ideal, as desired. □

It follows from Lemma 2.4 that stable range one for ideals is right and left symmetric. Recall that a ring R is perfect in case $R/J(R)$ is a division ring and idempotents lift modulo $J(R)$. Consequently, every ideal of a perfect ring is quasi-stable.

Proposition 2.6. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *I is a right quasi-stable ideal.*
(2) *For any regular $a, b \in I$, $aR = bR$ implies that there exists $w \in \Gamma(I)$ such that $a = bw$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $aR = bR$ with regular $a, b \in I$. Then we have $x, y \in R$ such that $ax = b$ and $a = by$. Assume that $b = bb'b$. Replacing $b'by$ with y, we may assume that $y \in I$. From $yx + (1 - yx) = 1$, we have $z \in R$ such that $y + (1 - yx)z = w \in \Gamma(I)$. Hence $a = by = b\big(y + (1 - yx)z\big) = bw$, as required.

(2) $\Rightarrow$ (1) For any regular $x \in I$, there exists an idempotent $e \in I$ such that $xR = eR$. So $x = ew$ for some $w \in \Gamma(I)$. Therefore I is a right quasi-stable ideal by Theorem 2.2. $\square$

3. Extensions of matrices

A natural problem asks whether quasi-stable exchange ideal of a ring is invariant under matrix extension. In this section, we give this problem an affirmative answer. In the sequel, we say that the pair (a, b) is an I-unimodular row in case $ax + by = 1$ for some $x \in I, y \in R$. The I-unimodular row (a, b) is called I-reducible if there exists $z \in R$ such that $a + bz \in \Gamma(I)$.

Lemma 3.1. *Let (a, b) be a I-unimodular row in a ring R. Let $u, v \in GL_1(I)$ and $c \in R$. Then $(vau + vbc, vb)$ is also I-unimodular row. Furthermore, (a, b) is I-reducible if and only if so is $(vau + vbc, vb)$.*

Proof. Since (a, b) is an I-unimodular row in a ring R, we have $x \in I, y \in R$ such that $ax + by = 1$. Hence $(vau + vbc)(u^{-1}xv^{-1}) + vb(y - cu^{-1}x)v^{-1} = 1$. Clearly, $u^{-1}xv^{-1} \in I$. So $(vau + vbc, vb)$ is an I-unimodular row. Assume that (a, b) is I-reducible. Then we have $y \in R$ such that $a + by \in \Gamma(I)$. Choose $z = yu - c$. Then we see that $(vau + vbc) + (vb)z = v(a + by)u \in \Gamma(I)$; hence, $(au + vbc, vb)$ is I-reducible. Conversely, assume that there exists $z \in R$ such that $vau+vbc+vbz \in \Gamma(I)$. Then $v\big(a+b(c+z)u^{-1}\big)u \in \Gamma(I)$. As $u, v \in GL_1(I)$, $a + b(c + z)u^{-1} \in \Gamma(I)$. Therefore (a, b) is I-reducible. $\square$

Theorem 3.2. *Let I be a right quasi-stable exchange ideal of a ring R. Then $M_n(I)$ is a right quasi-stable exchange ideal of $M_n(R)$ for all $n \in \mathbb{N}$.*

Proof. By [1, Theorem 1.4], $M_n(I)$ is an exchange ideal of $M_n(R)$. We now induct on n. Assume inductively that the result holds for n. It will suffice to show that the result holds for $n + 1$. Suppose that

$(*)$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1(n+1)} \\ b_{21} & b_{22} & \cdots & b_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(n+1)1} & b_{(n+1)2} & \cdots & b_{(n+1)(n+1)} \end{pmatrix}$$
$$+ \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & \cdots & c_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & \cdots & c_{(n+1)(n+1)} \end{pmatrix} = I_{n+1}$$

in $M_{n+1}(R)$, where

$$\begin{pmatrix} a_{11} & \cdots & a_{1(n+1)} \\ a_{21} & \cdots & a_{2(n+1)} \\ \vdots & \ddots & \vdots \\ a_{(n+1)1} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & b_{1(n+1)} \\ b_{21} & \cdots & b_{2(n+1)} \\ \vdots & \ddots & \vdots \\ b_{(n+1)1} & \cdots & b_{(n+1)(n+1)} \end{pmatrix} \in M_{n+1}(I).$$

Then $a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1(n+1)}b_{(n+1)1}+c_{11}=1$ with $a_{11}\in I$. As I is a quasi-stable exchange ideal of R, we have $z_1\in R$ such that $a_{11}+(a_{12}b_{21}+\cdots+a_{1n}b_{n1}+c_{11})z_1\in\Gamma(I)$. Since

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ b_{21}z_1 & 1 & 0 & \cdots & 0\\ b_{31}z_1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ b_{(n+1)1}z_1 & 0 & 0 & \cdots & 1 \end{pmatrix}\in GL_{n+1}(I),$$

by virtue of Lemma 3.1, $(*)$ is $M_{n+1}(I)$-reducible if and only if this is so for the $M_{n+1}(I)$-unimodular row with elements

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ b_{21}z_1 & 1 & 0 & \cdots & 0\\ b_{31}z_1 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ b_{(n+1)1}z_1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$
$$+\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)}\\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)}\\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}\begin{pmatrix} z_1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)}\\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)}\\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

So we assume that $a_{11}\in\Gamma(I)$. From $c_{21},\dots,c_{(n+1)1}\in I$, we have $a_{ij}\in I$ (either $i\neq 1$ or $j\neq 1$) in $(*)$. Write $a_{11}=uv, su=1, vt=1, s,t\in 1+I$. Then $sa_{11}t=1$, and so

$$\begin{pmatrix} s & 0 & 0 & \cdots & 0\\ 1-a_{11}ts & a_{11}t & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\times\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n+1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & a_{(n+1)3} & \cdots & a_{(n+1)(n+1)} \end{pmatrix}$$

$$\times\begin{pmatrix} t & 1-tsa_{11} & 0 & \cdots & 0 \\ 0 & sa_{11} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1(n+1)} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2(n+1)} \\ d_{31} & d_{32} & d_{33} & \cdots & d_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{(n+1)1} & d_{(n+1)2} & d_{(n+1)3} & \cdots & d_{(n+1)(n+1)} \end{pmatrix},$$

where

$$\begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1-a_{11}ts & a_{11}t & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{11}t & 1-a_{11}ts & 0 & \cdots & 0 \\ 0 & s & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1},$$

$$\begin{pmatrix} t & 1-tsa_{11} & 0 & \cdots & 0 \\ 0 & sa_{11} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} sa_{11} & 0 & 0 & \cdots & 0 \\ 1-tsa_{11} & t & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1} \in GL_{n+1}(I).$$

Thus $(*)$ is $M_{n+1}(I)$-reducible if and only if this is so for the $M_{n+1}(I)$-unimodular row with elements

$$\begin{pmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1(n+1)} \\ d_{21} & d_{22} & d_{23} & \cdots & d_{2(n+1)} \\ d_{31} & d_{32} & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{(n+1)1} & d_{3(n+1)} & * & \cdots & d_{(n+1)(n+1)} \end{pmatrix},$$

$$\begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1-a_{11}ts & a_{11}t & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

In $(*)$, we may assume that $d_{ij} \in I$ (either $3 \leq i \leq n+1$ or $3 \leq j \leq n+1$) and $d_{12} = sa_{11}(1 - tsa_{11}) + sa_{12}sa_{11}, d_{21} = (1 - a_{11}ts)a_{11}t + a_{11}ta_{21}t, d_{22} = \big((1 - a_{11}ts)a_{11} + a_{11}ta_{21}\big)(1 - tsa_{11}) + \big((1 - a_{11}ts)a_{12} + a_{11}ta_{22}\big)sa_{11} \in I$. By Lemma 3.1 again, $(*)$ is $M_{n+1}(I)$-reducible if and only if this is so for the $M_{n+1}(I)$-unimodular row with elements

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 1 - a_{11}ts & a_{11}t & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1(n+1)} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2(n+1)} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3(n+1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{(n+1)1} & c_{(n+1)2} & c_{(n+1)3} & \cdots & c_{(n+1)(n+1)} \end{pmatrix}.$$

So we may assume that $a_{11} = 1, a_{1i} = 0 = a_{i1}$ $(2 \leq i \leq n+1)$ in $(*)$. Furthermore, we may assume that $(*)$ is in the following form:

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} \begin{pmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_n \end{pmatrix},$$

$D \in M_n(I)$ and $\left(\begin{smallmatrix} e_{11} & E_{12} \\ E_{21} & E_{22} \end{smallmatrix}\right) \in M_{n+1}(I)$. This infers that $DE_{22} + C_{22} = I_n$. By the induction hypothesis, $M_n(I)$ is a quasi-stable exchange ideal of $M_n(R)$. So we can find $Z_2 \in M_n(R)$ such that $D + C_{22}Z_2 \in \Gamma\big(M_n(I)\big)$. Thus, we pass to the $M_{n+1}(I)$-unimodular row with elements

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & D \end{pmatrix} + \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0_{1\times n} \\ 0_{n\times 1} & Z_2 \end{pmatrix}, \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

In addition, we have $C_{12} \in M_{1\times n}(I)$. It suffices to prove that $M_{n+1}(I)$-unimodular row with elements

$$\begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & D + C_{22}Z_2 \end{pmatrix} \text{ and } \begin{pmatrix} c_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is $M_{n+1}(I)$-reducible. Write $D + C_{22}Z_2 = UV$, $SU = I_n$, $VT = I_n$, $S, T \in I_n + M_n(I)$. Thus,

$$\begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & D + C_{22}Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & U \end{pmatrix} \begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & V \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & S \end{pmatrix} \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & U \end{pmatrix} = I_{n+1},$$

$$\begin{pmatrix} 1 & C_{12}Z_2 \\ 0_{n\times 1} & V \end{pmatrix} \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & T \end{pmatrix} \begin{pmatrix} 1 & -C_{12}Z_2T \\ 0_{n\times 1} & I_2 \end{pmatrix} = I_{n+1},$$

$$\begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & S \end{pmatrix}, \begin{pmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & T \end{pmatrix} \begin{pmatrix} 1 & -C_{12}Z_2T \\ 0_{n\times 1} & I_2 \end{pmatrix} \in I_{n+1} + M_{n+1}(I).$$

This implies that $\begin{pmatrix} 1 & C_{12}Z_2 \\ 0 & D+C_{22}Z_2 \end{pmatrix} \in \Gamma\big(M_{n+1}(I)\big)$, as required. □

Corollary 3.3. *Let I be a right quasi-stable exchange ideal of a ring R. Then every regular $n \times n$ matrix over I is a product of an idempotent $n \times n$ matrix over I and an matrix in $\Gamma\big(M_n(I)\big)$.*

Proof. Since I is a right quasi-stable exchange ideal of R, by Theorem 3.2, $M_n(I)$ is a right quasi-stable exchange ideal of $M_n(R)$. Therefore we complete the proof from Theorem 2.2. □

Let $FP(I)$ denote the set of finitely generated projective right R-module P such that $P = PI$.

Lemma 3.4. *Let I be an exchange ideal of a ring R. If $P \in FP(I)$. Then there exist idempotents $e_1, \ldots, e_n \in I$ such that $P \cong e_1R \oplus \cdots \oplus e_nR$.*

Proof. See [1, Proposition 1.5]. □

Lemma 3.5. *Let I be a quasi-stable exchange ideal of a ring R. For any regular $a, b \in I$, $aR \cong bR$ implies that $a = w_1bw_2$ for some $w_1, w_2 \in \Gamma(I)$.*

Proof. Suppose that $\psi : aR \cong bR$. Then one easily checks that $Ra = R\psi(a)$ and $\psi(a)R = bR$. As $a \in I$, we have $\psi(a) \in Ra \subseteq I$. Since I is a right quasi-stable ideal, it follows by Proposition 2.6 that there exists $w_2 \in \Gamma(I)$ such that $bw_2 = \psi(a)$. Likewise, we have $w_1 \in \Gamma(I)$ such that $a = w_1\psi(a)$. Therefore $a = w_1bw_2$, where $w_1, w_2 \in \Gamma(I)$. □

We use A^T to denote the transpose of the matrix A. We now derive the main result of this article.

Theorem 3.6. *Let I be a quasi-stable exchange ideal of a ring R. Then every square regular matrix over I admits a diagonal reduction by quasi invertible matrices.*

Proof. Given any regular $A \in M_n(I)$, we have an idempotent matrix $E \in M_n(I)$ such that $AR^{n\times 1} = E^{n\times 1}R^{n\times 1}$, where $R^{n\times 1} = \{(x_1, \ldots, x_n)^T \mid x_1, \ldots, x_n \in R\}$. Clearly, $ER^{n\times 1} \in FP(I)$. By Lemma 3.4, there exist idempotents $e_1, \ldots, e_n \in I$ such that $ER^{n\times 1} \cong e_1R \oplus \cdots \oplus e_nR \cong \mathrm{diag}(e_1, \ldots, e_n)R^{n\times 1}$ as right R-modules. Set $R^{1\times n} = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in R\}$. Then $AR^{n\times 1} \bigotimes_R R^{1\times n} \cong \mathrm{diag}(e_1, \ldots, e_n)R^{n\times 1} \bigotimes_R R^{1\times n}$. So $AM_n(R) \cong \mathrm{diag}(e_1, \ldots, e_n)M_n(R)$. Therefore the result follows. □

Let I be an ideal of a ring R. We use $TM_n(R)$ to denote the ring of all $n \times n$ lower triangular matrices over R and $TM_n(I)$ to denote the ideal of all $n \times n$ lower triangular matrices over I.

Lemma 3.7. *Let I be an ideal of a ring R, and let $n \in \mathbb{N}$. If $u_{ii} \in \Gamma(I)$ $(1 \leq i \leq n), u_{ij} \in I$ $(j < i, 1 \leq i, j \leq n)$ and $u_{ij} = 0$ $(i < j, 1 \leq i, j \leq n)$. Then $(u_{ij})_{n\times n} \in \Gamma\big(TM_n(I)\big)$.*

Proof. Straightforward. □

Proposition 3.8. *Let I be a right quasi-stable exchange ideal of a ring R, and let $n \in \mathbb{N}$. Then $TM_n(I)$ is a right quasi-stable exchange ideal of $TM_n(R)$.*

Proof. Obviously, $TM_n(I)$ is an exchange ideal of $TM_n(R)$. Given

$$\begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & a_n \end{pmatrix}\begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & x_n \end{pmatrix} + \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & b_n \end{pmatrix} = I_n$$

with $\left(\begin{smallmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & a_n \end{smallmatrix}\right), \left(\begin{smallmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & x_n \end{smallmatrix}\right) \in TM_n(I)$, then for each i $(1 \leq i \leq n)$ we get $a_{ii}x_{ii} + b_{ii} = 1$ with $a_{ii} \in I, x_{ii}, b_{ii} \in R$. As I is a right quasi-stable ideal, we can find $y_i \in R$ such that $a_{ii} + b_{ii}y_i \in \Gamma(I)$. Clearly, $b_{ii} \in 1 + I$ and $a_{ij}, b_{ij} \in I$ $(j < i, 1 \leq i, j \leq n)$. By virtue of Lemma 3.7, we get

$$\begin{aligned} &\begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & a_n \end{pmatrix} + \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & b_n \end{pmatrix}\begin{pmatrix} y_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y_n \end{pmatrix} \\ = &\begin{pmatrix} a_{11} + b_{11}y_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ * & \cdots & a_{nn} + b_{nn}y_n \end{pmatrix} \in \Gamma\big(TM_n(I)\big), \end{aligned}$$

as required. □

4. Examples

The aim of this section is to construct several examples of quasi-stable ideals. A natural problem asks that if right quasi-stable ideal is right and left symmetric. So far, we can not answer this question. Now we establish an interesting properties of such ideals, which is an extension of [4, Lemma 14].

Proposition 4.1. *Let I be a right quasi-stable ideal of a ring R. Then for any regular $x \in I$, there exist an idempotent $e \in R$, a right invertible $u \in 1 + I$, a left invertible $v \in 1 + I$ such that $x = euv$.*

Proof. Assume that $A = (a_{ij}) \in GL_2(R) \cap \left(\begin{smallmatrix} 1+I & 1+I \\ I & 1+I \end{smallmatrix}\right)$, where $a_{12} \in \Gamma(I)$. Write $a_{12} = uv, su = 1, vt = 1, s, t \in 1 + I$. Then $sa_{12}t = 1$. Clearly, we have

$$\begin{pmatrix} s & 0 \\ 1 - a_{12}ts & a_{12}t \end{pmatrix} = \begin{pmatrix} a_{12}t & 1 - a_{12}ts \\ 0 & s \end{pmatrix}^{-1},$$

$$\begin{pmatrix} sa_{12} & 0 \\ 1 - tsa_{12} & t \end{pmatrix} = \begin{pmatrix} t & 1 - tsa_{12} \\ 0 & sa_{12} \end{pmatrix}^{-1} \in GL_2(I).$$

So we get

$$\begin{pmatrix} s & 0\\ 1-a_{12}ts & a_{12}t\end{pmatrix} A \begin{pmatrix} sa_{12} & 0\\ 1-tsa_{12} & t\end{pmatrix} = \begin{pmatrix} * & 1\\ * & *\end{pmatrix}$$
$$\in GL_2(R)\cap \begin{pmatrix} 1+I & 1+I\\ I & 1+I\end{pmatrix}.$$

We infer that

$$A=\begin{pmatrix} s & 0\\ 1-a_{12}ts & a_{12}t\end{pmatrix}^{-1}\begin{pmatrix} * & 1\\ * & *\end{pmatrix}\begin{pmatrix} sa_{12} & 0\\ 1-tsa_{12} & t\end{pmatrix}^{-1}.$$

Therefore

$$A^{-1}=\begin{pmatrix} sa_{12} & 0\\ 1-tsa_{12} & t\end{pmatrix}\begin{pmatrix} * & 1\\ * & *\end{pmatrix}^{-1}\begin{pmatrix} s & 0\\ 1-a_{12}ts & a_{12}t\end{pmatrix}.$$

From $\left(\begin{smallmatrix} * & 1\\ * & *\end{smallmatrix}\right)\in GL_2(R)\cap\left(\begin{smallmatrix} 1+I & 1+I\\ I & 1+I\end{smallmatrix}\right)$, we can find $u\in GL_1(I)$ such that

$$\begin{pmatrix} 1 & 0\\ * & 1\end{pmatrix}\begin{pmatrix} * & 1\\ * & *\end{pmatrix}\begin{pmatrix} 1 & 0\\ * & 1\end{pmatrix}=\begin{pmatrix} 0 & 1\\ u & 0\end{pmatrix}=\begin{pmatrix} 0 & u^{-1}\\ 1 & 0\end{pmatrix}^{-1},$$

where $\left(\begin{smallmatrix} 1 & 0\\ * & 1\end{smallmatrix}\right)\in GL_2(R)\cap\left(\begin{smallmatrix} 1 & 0\\ -1+I & 1\end{smallmatrix}\right)$. Thus

$$\begin{pmatrix} * & 1\\ * & *\end{pmatrix}^{-1}=\begin{pmatrix} 1 & 0\\ * & 1\end{pmatrix}\begin{pmatrix} 0 & u^{-1}\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & 0\\ * & 1\end{pmatrix}=\begin{pmatrix} * & u^{-1}\\ * & *\end{pmatrix}.$$

So we deduce that

$$\begin{aligned} A^{-1} &= \begin{pmatrix} sa_{12} & 0\\ 1-tsa_{12} & t\end{pmatrix}\begin{pmatrix} * & u^{-1}\\ * & *\end{pmatrix}\begin{pmatrix} s & 0\\ 1-a_{12}ts & a_{12}t\end{pmatrix}\\ &= \begin{pmatrix} * & sa_{12}u^{-1}a_{12}t\\ * & *\end{pmatrix}.\end{aligned}$$

As $u\in 1+I$, we have $u^{-1}\in 1+I$. Set $w=sa_{12}u^{-1}a_{12}t$. As $sa_{12}t=1$, we see that $sa_{12}\in 1+I$ is right invertible and $u^{-1}a_{12}t\in 1+I$ is left invertible.

Assume that $B=(b_{ij})\in GL_2(I)$. Write $B^{-1}=(c_{ij})$. Then $B^{-1}\in GL_2(I)$; hence, $c_{12}R+c_{11}R=R$ with $c_{12}\in I$. As I is a right quasi-stable ideal, we can find $y\in R$ such that $c_{12}+c_{11}y\in\Gamma(I)$. Obviously, $y\in 1+I$, and so

$$B^{-1}\begin{pmatrix} 1 & y\\ 0 & 1\end{pmatrix}=\begin{pmatrix} * & c_{12}+c_{11}y\\ * & *\end{pmatrix}\in GL_2(R)\cap\begin{pmatrix} 1+I & 1+I\\ I & 1+I\end{pmatrix}.$$

By the consideration above, we can find some $w_1\in 1+I$ such that

$$\begin{pmatrix} 1 & -y\\ 0 & 1\end{pmatrix}B=\left(B^{-1}\begin{pmatrix} 1 & y\\ 0 & 1\end{pmatrix}\right)^{-1}=\begin{pmatrix} * & c_{12}+c_{11}y\\ * & *\end{pmatrix}^{-1}=\begin{pmatrix} * & w_1\\ * & *\end{pmatrix},$$

where w_1 is the product of a right invertible element and a left invertible element $v\in 1+I$.

Given $ax+b=1$ with $a,x\in I, b\in R$, then $\left(\begin{smallmatrix} 1 & x\\ -a & b\end{smallmatrix}\right)=\left(\begin{smallmatrix} 1-xa & x\\ -a & 1\end{smallmatrix}\right)^{-1}\in GL_1(I)$. By the proceeding discussion, we can find $z\in R$ such that $\left(\begin{smallmatrix} 1 & z\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x\\ -a & b\end{smallmatrix}\right)=$

$\begin{pmatrix} * & w_2 \\ * & * \end{pmatrix}$, where $w_2 \in 1+I$ is the product of a right invertible element and a left invertible element $v \in 1+I$. Therefore $x+zb = w_2$.

For any regular $x \in I$, it follows from $xy+(1-xy)=1$ that $w := x+(1-xy)z \in 1+I$ is the product of a right invertible element and a left invertible element $v \in 1+I$. Set $e = xy \in I$. Then $x = xy\big(x+(1-xy)z\big) = ew$, where $e = e^2 \in I$ is an idempotent. Therefore we complete the proof. □

Recall that an ideal I of a ring R is regular provided that for any $x \in I$ there exists $y \in I$ such that $x = xyx$. We say that a ring R is right quasi-stable in case it is a right quasi-stable ideal as itself.

Proposition 4.2. *Let I be a regular ideal of a ring R. If eRe is a right quasi-stable ring for all idempotents $e \in I$, then I is a right quasi-stable exchange ideal of R.*

Proof. By [1, Example], I is an exchange ideal. Given $ax+b=1$ with $a \in I, x, b \in R$, then $a = aa'a$ for a $a' \in R$. Set $c = a'ax$. Then $ac+b=1$ with $a, c \in I, b \in 1+I$. As $a, c, 1-b \in I$. In view of [7, Lemma 3.2], there exists an idempotent $e \in I$ such that $a, x, 1-b \in eRe$. Hence, $(1-b)(1-e) = 0$, and so $b(1-e) = 1-e$. In addition, $(1-b)e = 1-b$; hence, $b = be+1-e$. Thus, $ax+be = e$. This implies that $be \in eRe$, and so $ebe = be$. Since $ax+ebe = e$, by hypothesis, we can find some $u, v, s, t \in eRe$ such that $a+ebeye = uv, su = e, vt = e$ for a $y \in R$. Thus, $a+beye+1-e = (u+1-e)(v+1-e)$, and so $a+b(eye+1-e) = (u+1-e)(v+1-e)$, where $(s+1-e)(u+1-e) = 1, (v+1-e)(t+1-e) = 1$ and $s+1-e, t+1-e \in 1+I$. Therefore I is a right quasi-stable ideal of R, as desired. □

Corollary 4.3. *Let I be a regular ideal of a ring R. If $aR+bR = R$ with $a \in 1+I, b \in R$ implies that there exists $y \in R$ such that $a+by \in R$ is right or left invertible, then I is a quasi-stable exchange ideal of R.*

Proof. Let $e \in I$ be an idempotent. In view of [5, Lemma 4.1], eRe is one-sided unit-regular. For any $x \in eRe$, by [3, Theorem 4], there exist an idempotent $f \in eRe$ and a right or left $u \in eRe$ such that $x = eu$. This implies that eRe is a right quasi-stable ring from Theorem 2.2. According to Proposition 4.2, I is a right quasi-stable exchange ideal. By the symmetry of one-sided unit-regularity, we establish the result. □

Recall that an ideal I of a regular ring R satisfies the comparability axiom provided that for any $x, y \in I$, either $xR \lesssim yR$ or $yR \lesssim xR$ (cf. [10]). Let I be an ideal of a regular ring R. If I satisfies the comparability axiom, we note that $aR+bR = R$ with $a \in 1+I, b \in R$ implies that $a+by \in R$ is right or left invertible for a $y \in R$.

Corollary 4.4. *Let I be a regular ideal of a ring R. If I satisfies the comparability axiom, then I is quasi-stable.*

Proof. Clearly, $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in R$ is right or left invertible. Therefore we complete the proof by Corollary 4.3. □

By [8, Corollary 9.15], every regular, right self-injective ring satisfies general comparability. We now extend this result to right injective ideals of regular rings.

Proposition 4.5. *Let I be a regular ideal of a ring R. If I is an injective right R-module, then I is a quasi-stable ideal of R.*

Proof. Since I is regular, I is an exchange ideal. As I is injective, there exists a splitting exact sequence $0 \to I \hookrightarrow R \to R/I \to 0$. Thus, we have a right R-module $C \cong R/I$ such that $R = I \oplus C$. Thus, $I = eR$ for some idempotent $e \in I$. Let $f \in I$ be an idempotent. Then we have an inclusion $i : fR \hookrightarrow eR$. Construct a R-morphism $\varphi : eR \to fR$ given by $\varphi(er) = fer$ for any $r \in R$. It is easy to verify that $\varphi i = 1_{fR}$. This implies that the exact sequence $0 \to fR \hookrightarrow eR \to eR/fR \to 0$ splits. Thus, we have a right R-module $D \cong eR/fR$ such that $eR = fR \oplus D$. Since eR is injective, so is fR. For any $m \in Z(fR)$, there exists some $z \in R$ such that $m = mzm$. Hence, $r(m) = (1 - zm)R$. As $r(m) \bigcap zmR = 0$, we get $zmR = 0$; hence, $m = mzm = 0$. That is, $Z(fR) = 0$, i.e., fR is nonsingular. In view of [8, Corollary 1.23], $fRf \cong \text{End}_R(fR)$ is a regular, right self-injective ring. According to [8, Corollary 9.15], eRe satisfies general comparability. Let $x \in fRf$, we can find an idempotent $g \in fRf$ and a related unit $w \in fRf$ such that $x = gw$. As $w \in fRf$ is a related unit, there exists an idempotent $g \in fRf$ such that $gw \in g(fRf)$ is right invertible and $(f-g)w \in (f-g)(fRf)$ is left invertible. Thus, $w = \big((f-g)w+g\big)(gw+f-g)$. According to Theorem 2.2, eRe is a right quasi-stable ring. According to Proposition 4.2, I is a right quasi-stable ideal. Analogously, we show that I is a left quasi-stable ideal. Therefore I is quasi-stable, as desired. □

Let R be a regular ring, and let $a \in R$. If RaR is injective, it follows from Proposition 4.5 and Theorem 2.2 that a is the product of an idempotent, a left invertible element and a right invertible element.

Example 4.6. Let R be regular, and let

$$I = \{x \in R \mid xR \text{ is injective}\}.$$

Then I is a quasi-stable ideal.

Proof. It is directly proved that I is an ideal of R. For any $a \in I$, there exists an idempotent $e \in I$ such that $a \in eRe$ from [6, Lemma 3.2]. As eR is injective, it follows from [8, corollary 1.23] that eRe is a regular, right self-injective ring. Thus, it satisfies related comparability. Hence, there exists an idempotent $f \in eRe$ and a related unit $w \in ere$ such that $a = eu$. This implies that $a = e(u + 1 - e)$, where $e \in I$ is an idempotent and $u + 1 - e \in \Gamma(I)$. According to Theorem 2.2, I is a right quasi-stable ideal. Similarly, we show that I is a left quasi-injective ideal, as asserted. □

5. Directly finite ideals

We say that an ideal I of a ring R is directly finite provided that for any $a, b \in I$, $(1+a)(1+b) = 1$ implies that $(1+b)(1+a) = 1$. An ideal I of a ring R is said to be of bounded index provided that there exists some $n \in \mathbb{N}$ such that $x^n = 0$ for any nilpotent element $x \in I$. Let R be a regular ring, and let $I = \{x \in R \mid \text{End}_R(xR) \text{ is of bounded index}\}$. Then I is a directly finite, quasi-stable exchange ideal.

Lemma 5.1. *Let I be a directly finite, right quasi-stable exchange ideal of a ring R. Suppose that $AX + B = I_n$ with $A, X \in M_n(I), B \in M_n(R)$. Then*

1. *There exists some $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$.*
2. *There exists some $Z \in M_n(R)$ such that $X + ZB \in GL_n(I)$.*

Proof. (1) Since I is directly finite, one easily checks that $\Gamma(I) = GL_1(I)$. By iteration of the process of Theorem 3.2 and replacing the elements in $\Gamma(I)$ by invertible elements in $1 + I$, we can find some $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$.

(2) By (1), there is $Y \in M_n(R)$ such that $A + BY \in GL_n(I)$. In view of Lemma 2.4, one directly verifies that $\big(X + (X_n - XY)(A + BY)^{-1}B\big)^{-1} = A + Y(I_n - XA)$. Check $Z = (X_n - XY)(A + BY)^{-1}$. Then $X + ZB \in GL_n(I)$, as asserted. □

Theorem 5.2. *Let I be a directly finite, right quasi-stable exchange ideal of a ring R. Then for any regular $A \in M_n(I)$ there exist $U, V \in GL_n(I)$ such that $UAV = \text{diag}(e_1, \ldots, e_n)$ for some idempotents $e_1, \ldots, e_n \in I$.*

Proof. Given any regular matrix $A \in M_n(I)$, there exists $E = E^2 \in M_n(I)$ such that $AM_n(R) = EM_n(R)$. Similarly to Theorem 3.4, we have idempotents $e_1, \ldots, e_n \in I$ such that $\varphi : AM_n(R) \cong \text{diag}(e_1, \ldots, e_n)M_n(R)$. Then $M_n(R)A = M_n(R)\varphi(A), \varphi(A)M_n(R) = \text{diag}(e_1, \ldots, e_n)M_n(R)$. One directly verifies that there exist some $X, Y \in M_n(I)$ such that $XA = \varphi(A)$ and $A = Y\varphi(A)$. Since $YX + (I_n - YX) = I_n$, it follows by Lemma 5.1 that there exists some $Z \in M_n(R)$ such that $U := X + Z(I_n - YX) \in GL_n(I)$. Hence $UA = \big(X + Z(I_n - YX)\big)A = XA = \varphi(A)$. Likewise, we can find some $V \in GL_n(I)$ such that $\varphi(A)V = \text{diag}(e_1, \ldots, e_n)$. Therefore $UAV = \text{diag}(e_1, \ldots, e_n)$, as asserted. □

Let I be an ideal of a ring R. Set $B(I) = \{e \in I \mid e = e^2 \text{ and } ex = xe \text{ for any } x \in I\}$. We say that I is an abelian ideal in case every idempotent in I is in $B(I)$. For example, every semicommutative ideal of a ring is an abelian ideal.

Corollary 5.3. *Let I be an abelian exchange ideal of a ring R. Then for any regular $A \in M_n(I)$ there exist $U, V \in GL_n(I)$ such that $UAV = \text{diag}(e_1, \ldots, e_n)$ for some idempotents $e_1, \ldots, e_n \in I$.*

Proof. For any regular $x \in I$, we have $y \in I$ such that $x = xyx$ and $y = yxy$. Since I is an abelian exchange ideal of R, we have $x = x^2y = yx^2$, and then $x = xy(1 + x - xy)$. Set $e = xy$ and $u = 1 + x - xy$. Then $e = e^2 \in I$ and $u(1 + y - xy) = 1$. Hence $u \in \Gamma(I)$. Thus I is a right quasi-stable ideal by Theorem 2.2.

Suppose that $(1-x)(1-y) = 1$ with $x, y \in I$; hence, $(1-y)(1-x) \in 1+I$ is an idempotent. Since I is an abelian ideal of R, $\big(1-(1-y)(1-x)\big)x = x\big(1-(1-y)(1-x)\big)$. Furthermore, we get $(1-y)(1-x)(1-x) = (1-x)(1-y)(1-x)$. So $(1-y)(1-x) = (1-y)(1-x)(1-x)(1-y) = (1-x)(1-y)(1-x)(1-y) = 1$. That is, I is a directly finite ideal. Therefore we complete the proof from Theorem 5.2. □

Recall that an ideal I of a ring R is periodic provided that for any $x \in I$, there exists $n(x) \in \mathbb{N}$ such that $x = x^{n(x)+1}$.

Corollary 5.4. *Let I be a periodic ideal of a ring R. Then for any regular $A \in M_n(I)$ there exist $U, V \in GL_n(I)$ such that $UAV = \mathrm{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in I$.*

Proof. For any idempotent $e \in I$ and any idempotent $x \in I$, we have $(ex - exe)^2 = 0$. So we deduce that $ex = exe$ because I is an periodic ideal. Likewise, we have $xe = exe$; hence, $ex = xe$. This means that I is an abelian ideal of R. On the other hand, I is a strongly π-regular ideal; hence, it is an exchange ideal. So the proof is true by Corollary 5.3. □

Example 5.5. Let V be a countably generated infinite-dimensional vector space over a division D, and let $(a_{ij}) \in M_n\big(\mathrm{End}_D(V)\big)$ with all $\dim_D(a_{ij}V) < \infty$. Then there exist $U, V \in GL_n\big(\mathrm{End}_D(V)\big)$ such that $UAV = \mathrm{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in \mathrm{End}_D(V)$.

Proof. Let $I = \{x \in \mathrm{End}_D(V) \mid \dim_D(xV) < \infty\}$. Obviously, I is an ideal of $\mathrm{End}_D(V)$. For any idempotent $e \in I$, eRe is unit-regular; hence, I has stable range one. This implies that I is directly finite. In view of Proposition 2.5, I is a quasi-stable ideal. According to Theorem 5.2, the result follows. □

Let R be an exchange ring, and let $(a_{ij}) \in M_n(R)$. If each $Ra_{ij}R$ has stable range one, analogously, we conclude that there exist $(u_{ij}), (v_{ij}) \in GL_n(R)$ such that $(u_{ij})(a_{ij})(v_{ij}) = \mathrm{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in R$. In addition, $Ru_{ij}R, Rv_{ij}R$ $(i \neq j), R(1 - u_{ii})R$ and $R(1 - v_{ii})R$ all have stable range one.

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