

GOTTLIEB SUBSETS WITH RESPECT TO A MORPHISM IN THE CATEGORY OF PAIRS

JIYEAN KIM AND KEE YOUNG LEE

ABSTRACT. We introduce the concept of cyclic morphisms with respect to a morphism in the category of pairs as a generalization of the concept of cyclic maps and we use the concept to obtain certain sets of homotopy classes in the category of pairs. For these sets, we get complete or partial answers to the following questions: (1) Is the concept the most general concept in the class of all concepts of generalized Gottlieb subsets introduced by many authors until now? (2) Are they homotopy invariants in the category of pairs? (3) When do they have a group structure?

1. Introduction

In his paper, “Generalized Gottlieb group” [13], K. Varadarajan introduced the concept of cyclic maps as a generalization of cyclic homotopies introduced by D. H. Gottlieb [2]. He also studied the concept of cocyclic maps as a dual concept of cyclic maps to fit well in the general set-up of Eckmann-Hilton duality. He studied the role of cyclic maps and cocyclic maps in the set up of Eckmann-Hilton duality. In [6, 8], Lee and Woo introduced the concept of the G-sequence under the homotopy sequence and its generalization. Moreover, they introduced the concept of cyclic morphisms with respect to a continuous map as a generalization of the concept of cyclic maps in [7].

It is our purpose in this paper to apply these concepts to objects in the category of pairs to induce further generalizations of cyclic maps appeared in the set up of Eckmann-Hilton duality and make further studies of the generalized Gottlieb subgroup. We introduce the concept of “cyclic morphism with respect to a morphism” in the category of pairs as a generalization of the cyclic map and we use the concept to define certain sets of homotopy classes in the category of pairs. We study the conditions for the sets to be homotopy invariants or groups. We also use the sets to study the role of cyclic morphisms in the category of pairs.

Received May 8, 2009.

2000 *Mathematics Subject Classification.* Primary 55Q05; Secondary 55P30.

Key words and phrases. category of pairs, cyclic map, cyclic morphism, generalized Gottlieb subset.

This work was supported by a Korea University Grant.

In Section 2, we review the concepts of cyclic maps [13] and cyclic morphisms with respect to a continuous map in the category of pairs studied in [7].

In Section 3, we generalize the concept of cyclic morphisms with respect to a continuous map to that of cyclic morphisms with respect to a morphism and study the sets of homotopy classes of cyclic morphisms with respect to a morphism in the category of pairs.

In Section 4, we study properties of cyclic morphisms with respect to a morphism in the category of pairs. Especially, we characterize the cyclic morphisms with respect to a morphism by certain sets of homotopy classes in the category of pairs. Moreover, we find some conditions such that the sets are homotopy invariants in the category of pairs.

In Section 5, we find some sufficient conditions to ensure the set to be a group.

Throughout this paper, all spaces will be connected, based and of the homotopy type of CW-complexes. Hence the exponential law of function spaces holds and all base points denoted by $*$ are non-degenerate.

2. Preliminaries

In this paper, we shall consider two kinds of well-known categories in [4].

First, we shall consider the category of topological spaces with base points. The objects are pairs $(X, *)$ with X a space and $*$ in X , and the maps are maps of these spaces which carry base point into base point. All homotopies considered will keep base points fixed. $\Pi(A, B)$ is the collection of homotopy classes of maps of A into B . In the case $A = S^n$, the n -sphere, $\Pi(A, B)$ is a group, namely the n -th homotopy group. It is also well known that if A has a co-H-structure, $\Pi(A, B)$ has a group structure for all B and if B has a H-structure, $\Pi(A, B)$ has a group structure for all A . The *reduced suspension* of X , denoted ΣX , is the space obtained from $X \times I$ by identifying $X \times 0 \cup X \times 1 \cup * \times I$ to $*$, where I is the unit interval $I = [0, 1]$. More generally, we denote the n -folds suspension of the spaces X by $\Sigma^n X \equiv \Sigma(\Sigma^{n-1} X)$. If $\alpha : X \rightarrow Y$ is a map, then the n -suspension map $\Sigma^n \alpha : \Sigma^n X \rightarrow \Sigma^n Y$ of α is the map defined by $\Sigma^n \alpha(\langle x, t_1, \dots, t_n \rangle) = \langle \alpha(x), t_1, \dots, t_n \rangle$, where $\langle x, t_1, \dots, t_n \rangle$ is an element of $\Sigma^n X$ which is the equivalence class of $(x, t_1, \dots, t_n) \in X \times I^n$. It is well-known fact that $S^n = \Sigma^n(S^0)$. The cone of X , denoted by CX , is the space obtained from $X \times I$ by pinching $X \times 1 \cup * \times I$ to $*$. Clearly X is embedded in CX by the natural inclusion map $x \rightarrow (x, 0)$. If we now squeeze $X \subset CX$ to $*$ we obtain the second definition of ΣX which is obviously equivalent to the first. We can summarize by saying we have the following “exact” sequence of spaces:

$$* \rightarrow X \rightarrow CX \rightarrow \Sigma X \rightarrow *.$$

There is a natural map $\phi' : \Sigma X \rightarrow \Sigma X \vee \Sigma X$ defined by

$$\phi'(x, t) = \begin{cases} (\langle x, 2t \rangle, *) & \text{if } 0 \leq t \leq 1/2 \\ (*, \langle x, 2t - 1 \rangle) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

It is easy to show that ϕ' is a co-H-structure on ΣX . Thus for $n \geq 1$, the set $\Pi(\Sigma^n X, B) \equiv \Pi_n(X, B)$ is a group.

As the dual concept of the suspension of a space, we here recall the concept of the loop space. Let us denote the collection of mappings of X into Y with the compact-open topology by Y^X . The *loop space* of X , ΩX , is the subspace of X^I characterized by $\ell \in \Omega X$ such that $\ell(0) = \ell(1) = *$. There is a natural map

$$\phi : \Omega X \times \Omega X \rightarrow \Omega X$$

defined by

$$\phi(\ell_1, \ell_2)(t) = \begin{cases} \ell_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \ell_2(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Continuity of ϕ follows from a standard argument in the compact open topology. It is easy to see that ϕ is a H-structure on ΩX .

Given a map $f : \Sigma A \rightarrow B$, we define the adjoint map of f $\kappa(f) = \bar{f} : A \rightarrow \Omega B$ by

$$\bar{f}(a)(t) = f\langle a, t \rangle.$$

A standard topological argument shows that \bar{f} is continuous if and only if f is continuous. Thus κ is one to one correspondence between $B^{\Sigma A}$ and $(\Omega B)^A$.

Any map $g : A' \rightarrow A$ induces a transformation $g^* : \Pi(A, B) \rightarrow \Pi(A', B)$ defined by $g^*([f]) = [f \circ g]$, where the bracket $[\]$ denotes the homotopy class of the enclosed map. Clearly $(gh)^* = h^*g^*$. Similarly, $h : B \rightarrow B'$ induces a transformation $h_* : \Pi(A, B) \rightarrow \Pi(A, B')$ defined by $h_*([f]) = [h \circ f]$. The identity map on X will be denoted by 1_X or id_X , $*$ shall stand for the base point of a space and also the map which takes everything into the base point.

Second, we shall consider the category of pairs [4]. The category of pairs is the category in which the "object" are maps $(A, *) \rightarrow (B, *)$ and a "map" from α to β is a pair of maps (f_1, f_2) such that the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha} & A_2 \\ \downarrow f_1 & & \downarrow f_2 \\ B_1 & \xrightarrow{\beta} & B_2 \end{array}$$

is commutative, i.e., $\beta f_1 = f_2 \alpha$. We shall call the maps in this category just "morphisms" to distinguish from maps between spaces. Moreover, we shall sometimes denote the morphism by the formula:

$$(f_1, f_2) : \alpha(A_1 \rightarrow A_2) \rightarrow \beta(B_1 \rightarrow B_2)$$

or more simply,

$$(f_1, f_2) : \alpha \rightarrow \beta.$$

Two morphisms $(f_1, f_2), (g_1, g_2) : \alpha \rightarrow \beta$ are called *homotopic* if there is a morphism $(H_1, H_2) : \alpha \times 1_I \rightarrow \beta$ such that H_1 is a homotopy between f_1 and

g_1 and H_2 is a homotopy between f_2 and g_2 , where 1_I is the identity map of the unit interval I into itself.

The set $\Pi(\alpha, \beta)$ is the family of all homotopy classes of morphisms from α to β in the category of pairs. In particular, $\Pi_n(\alpha, \beta) = \Pi(\Sigma^n \alpha, \beta)$ is a group if $n \geq 1$ and abelian $n \geq 2$. If $\alpha = i_n : \Sigma^{n-1} A \rightarrow C\Sigma^{n-1} A$ is the natural inclusion, $\Pi(\alpha, \beta)$ is denoted by $\Pi_n(A, \beta)$. If β is inclusion and $A = S^0$, we get the ordinary relative homotopy groups. Furthermore, if $\beta : * \rightarrow B$, $\Pi_n(A, \beta) = \Pi_n(A, B)$ and if $\beta : B \rightarrow *$, $\Pi_n(A, \beta) = \Pi_{n-1}(A, B)$.

A map $f : A \rightarrow X$ is said to be *cyclic* [13] if there exists a map $H : A \times X \rightarrow X$ such that the diagram

$$\begin{array}{ccc} A \times X & \xrightarrow{H} & X \\ j \uparrow & & \uparrow \nabla \\ A \vee X & \xrightarrow{f \vee 1} & X \vee X \end{array}$$

is homotopy commutative, where j is the inclusion map and ∇ is the folding map.

We denote the set of all homotopy classes of cyclic maps from A to X by $G(A, X)$ [13], that is, $G(A, X) = \{[f] \in \Pi(A, X) \mid f \text{ is a cyclic map}\}$, equivalently, $G(A, X) = \omega_*(\Pi(A, X^X))$, where $\omega : X^X \rightarrow X$ is the evaluation map. In particular, $G(\Sigma^n A, X)$ is denoted by $G_n(A, X)$. Clearly, $\omega_*(\Pi_n(A, X^X)) = G_n(A, X)$.

The subgroup $G_n(A, X)$ is a generalization of $G(A, X)$ and the Gottlieb group $G_n(X)$ [2]. In fact $G_0(A, X) = G(A, X)$ and $G_n(S^0, X) = G_n(X)$.

Let $h : B \rightarrow X$ be a map. A map $f : A \rightarrow X$ is called a *cyclic map with respect to $h : B \rightarrow X$* [10] if there exists a map $H : A \times B \rightarrow X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{H} & X \\ j \uparrow & & \uparrow \nabla \\ A \vee B & \xrightarrow{f \vee h} & X \vee X \end{array}$$

is homotopy commutative.

Here, we review the concept of cyclic morphisms with respect to a continuous map in the category of pairs studied in [7].

Let $h : X \rightarrow B_1$ be a map. A map $(f_1, f_2) : \alpha \rightarrow \beta$ is called a *cyclic morphism with respect to h* if there exists a map $(H_1, H_2) : \alpha \times 1_X \rightarrow \beta$ such that $(H_1, H_2)|_\alpha = (f_1, f_2)$ and $(H_1, H_2)|_{1_X} = (h, \beta h)$, that is, the following

diagram commutes:

$$\begin{array}{ccc}
 A_1 \vee X & \xrightarrow{f_1 \vee h} & B_1 \vee B_1 \\
 \downarrow j & & \downarrow \nabla \\
 A_1 \times X & \xrightarrow{H_1} & B_1 \\
 \downarrow \alpha \times 1 & & \downarrow \beta \\
 A_2 \times X & \xrightarrow{H_2} & B_2 \\
 \uparrow j & & \uparrow \nabla \\
 A_2 \vee X & \xrightarrow{f_2 \vee \beta h} & B_2 \vee B_2.
 \end{array}$$

The morphism (H_1, H_2) is called an *affiliated morphism* of (f_1, f_2) with respect to h . If $h : B_1 \rightarrow B_1$ is the identity, then (f_1, f_2) is called just a *cyclic morphism*.

We define the subset $\mathcal{G}^h(\alpha, \beta)$ of $\Pi(\alpha, \beta)$ as the set of homotopy classes of cyclic morphisms with respect to h . That is,

$$\mathcal{G}^h(\alpha, \beta) = \{[f_1, f_2] \in \Pi(\alpha, \beta) \mid (f_1, f_2) \text{ is a cyclic morphism with respect to } h\}.$$

We denote $\mathcal{G}^h(\Sigma^n \alpha, \beta)$ by $\mathcal{G}_n^h(\alpha, \beta)$, where $\Sigma^n \alpha : \Sigma^n A_1 \rightarrow \Sigma^n A_2$ is the suspension map of α . In particular, if $i_n : \Sigma^n A \rightarrow C\Sigma^n A$ is the natural inclusion, then we denote $\mathcal{G}^h(i_n, \beta)$ by $\mathcal{G}_n^h(A, \beta)$. Moreover, we denote $\mathcal{G}_n^h(A, \beta)$ by $\mathcal{G}_n(A, \beta)$ if $h : B_1 \rightarrow B_1$ is the identity map. $\mathcal{G}_n(A, \beta)$ is a generalization of $G_n^{Rel}(B_2, B_1)$ because $\mathcal{G}_n(S^0, i) = G_n^{Rel}(B_2, B_1)$, where $i : B_1 \rightarrow B_2$ are the inclusion.

Define $\bar{\beta} : (B_1^X, h) \rightarrow (B_2^X, \beta h)$ by $\bar{\beta}(g) = \beta g$, where $\beta : B_1 \rightarrow B_2$ is a map and let $\omega_1 : B_1^X \rightarrow B_1$ and $\omega_2 : B_2^X \rightarrow B_2$ be evaluation maps. Then $(\omega_1, \omega_2) : \bar{\beta} \rightarrow \beta$ is a morphism and it induces a map $(\omega_1, \omega_2)_* : \Pi(\alpha, \bar{\beta}) \rightarrow \Pi(\alpha, \beta)$.

Theorem 2.1 ([7]). *Let $\beta : B_1 \rightarrow B_2$ be a map and $\bar{\beta} : (B_1^X, h) \rightarrow (B_2^X, \beta h)$ be the map mentioned above. Then $(\omega_1, \omega_2)_* \Pi(\alpha, \bar{\beta}) = \mathcal{G}^h(\alpha, \beta)$.*

By above theorem, if α is a suspension map, then $\mathcal{G}^h(\alpha, \beta)$ is a group. In particular, $\mathcal{G}_n^h(\alpha, \beta)$ is a group, for $n \geq 1$.

Let $(g_1, g_2) : \gamma \rightarrow \alpha$ be a morphism. If $(f_1, f_2) : \alpha \rightarrow \beta$ is a cyclic morphism with respect to h , then the composition $(f_1, f_2) \circ (g_1, g_2) : \gamma \rightarrow \beta$ is a cyclic morphism with respect to h . Thus the induced map $(g_1, g_2)^* : \Pi(\alpha, \beta) \rightarrow \Pi(\gamma, \beta)$ carries $\mathcal{G}^h(\alpha, \beta)$ into $\mathcal{G}^h(\gamma, \beta)$. Moreover, if $(g_1, g_2) : \alpha \rightarrow \gamma$, then the induced map $(g_1, g_2)_* : \Pi(\alpha, \beta) \rightarrow \Pi(\alpha, \gamma)$ carries $\mathcal{G}^h(\alpha, \beta)$ into $\mathcal{G}^{g_1 h}(\alpha, \gamma)$. If $\alpha \times 1_X : A_1 \times X \rightarrow A_2 \times X$ is a cofibration, then $\mathcal{G}^h(\alpha, \beta)$ is determined by the homotopy class of h . That is, if h is homotopic to h' , then $\mathcal{G}^h(\alpha, \beta) = \mathcal{G}^{h'}(\alpha, \beta)$. Suppose α is a cofibration. Then if $(g_1, g_2) : \gamma \rightarrow \alpha$ is a homotopy equivalence, $(g_1, g_2)^* : \mathcal{G}^h(\alpha, \beta) \rightarrow \mathcal{G}^h(\gamma, \beta)$ is an isomorphism. Similarly, if $(g_1, g_2) : \alpha \rightarrow \gamma$ is a homotopy equivalence, then $(g_1, g_2)_* : \mathcal{G}^h(\alpha, \beta) \rightarrow \mathcal{G}^{g_1 h}(\alpha, \gamma)$ is an isomorphism. Therefore we have the following theorem.

Theorem 2.2 ([7]). *The subgroup $\mathcal{G}_n(A, \beta)$ of $\Pi_n(A, \beta)$ is a homotopy invariant with respect to two variables.*

3. Cyclic morphisms with respect to a morphism and their homotopy classes

In this section, we introduce the concept of cyclic morphisms with respect to a morphism and study the set of their homotopy classes in the category of pairs.

Definition. Let $\alpha : A_1 \rightarrow A_2$, $\beta : B_1 \rightarrow B_2$ and $\eta : X \rightarrow Y$ be objects and $(h_1, h_2) : \eta \rightarrow \beta$ be a morphism in the category of pairs. A map $(f_1, f_2) : \alpha \rightarrow \beta$ is called a *cyclic morphism with respect to (h_1, h_2)* if there exists a map $(H_1, H_2) : \alpha \times \eta \rightarrow \beta$ such that $(H_1, H_2)|_\alpha = (f_1, f_2)$ and $(H_1, H_2)|_\eta = (h_1, h_2)$, that is, the following diagram commutes:

$$\begin{array}{ccc}
 A_1 \vee X & \xrightarrow{f_1 \vee h_1} & B_1 \vee B_1 \\
 \downarrow j & & \downarrow \nabla \\
 A_1 \times X & \xrightarrow{H_1} & B_1 \\
 \downarrow \alpha \times \eta & & \downarrow \beta \\
 A_2 \times Y & \xrightarrow{H_2} & B_2 \\
 \uparrow j & & \uparrow \nabla \\
 A_2 \vee Y & \xrightarrow{f_2 \vee h_2} & B_2 \vee B_2.
 \end{array}$$

In this case, the morphism (H_1, H_2) is called (h_1, h_2) -affiliated morphism of (f_1, f_2) or (f_1, f_2) has (h_1, h_2) -affiliated morphism (H_1, H_2) .

Remark 1. If $\eta = id_X$, then it is clear that (f_1, f_2) is a cyclic morphism with respect to (h_1, h_2) if and only if (f_1, f_2) is a cyclic morphism with respect to h_1 . If $B_2 = *$, then it is clear that (f_1, f_2) is a cyclic morphism with respect to (h_1, h_2) if and only if f_1 is a cyclic map with respect to h_1 . In general, if (f_1, f_2) is a cyclic morphism with respect to (h_1, h_2) , then f_1 is cyclic with respect to h_1 and f_2 is cyclic with respect to h_2 . Moreover, if $i_n : S^{n-1} \rightarrow B^n$ and $i_A : A \rightarrow X$ are the inclusions, then it can be shown easily that a pair map $f : (B^n, S^{n-1}) \rightarrow (X, A)$ is relative cyclic if and only if $(f|_{S^{n-1}}, f) : i_n \rightarrow i_A$ is cyclic morphism [7]. So the concept of the cyclic morphism with respect to a morphism is a generalization of the cyclic morphism with respect to a map and the relative cyclic map.

Definition. We define the subset $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ of $\Pi(\alpha, \beta)$ as the set of homotopy classes of all cyclic morphisms with respect to $(h_1, h_2) : \eta \rightarrow \beta$. That is,

$$\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = \{[f_1, f_2] \in \Pi(\alpha, \beta) | (f_1, f_2) : \text{cyclic morphism w.r.t } (h_1, h_2)\}.$$

$\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ is called *Gottlieb subset of $\Pi(\alpha, \beta)$ with respect to (h_1, h_2) in the category of pairs*.

For this set, we have three kinds of questions:

- (1) Is it the most general concept in the class of all concepts of generalized Gottlieb subsets introduced by many authors until now?
- (2) Is it homotopy invariant in the category of pairs?
- (3) When does it have a group structure?

In this paper, we shall give answers to these questions.

By above definition, if $[g_1, g_2] \in \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$, then there exists a morphism $(f_1, f_2) : \alpha \rightarrow \beta$ which is a cyclic morphism with respect to (h_1, h_2) and homotopic to (g_1, g_2) in the category of pairs. Thus there is not a problem in the definition of Gottlieb subset with respect to a morphism in the category of pairs. But we are interested in the statement that the homotopic relation preserves the cyclicity with respect to a morphism in the category of pairs, that is, if two morphisms (f_1, f_2) and (g_1, g_2) from α to β are homotopic and (f_1, f_2) is cyclic with respect to a morphism (h_1, h_2) , then (g_1, g_2) is cyclic with respect to (h_1, h_2) . Is it true or false? If it is not true in general, when is it true? In the following lemma, we shall concentrate on solving this problem.

Lemma 3.1. *Let $\eta : X \rightarrow Y$ be a homeomorphism and $(h_1, h_2) : \eta \rightarrow \beta$ be a morphism for a map $\beta : B_1 \rightarrow B_2$. If $\alpha : A_1 \rightarrow A_2$ is an inclusion, two morphisms (f_1, f_2) and (g_1, g_2) from α to β are homotopic and (f_1, f_2) is cyclic with respect to a morphism (h_1, h_2) , then (g_1, g_2) is cyclic with respect to (h_1, h_2) .*

Proof. Suppose (f_1, f_2) and (g_1, g_2) are homotopic and (f_1, f_2) is cyclic with respect to (h_1, h_2) . Then there is a homotopy $(F_1, F_2) : \alpha \times id_I \rightarrow \beta$ between (f_1, f_2) and (g_1, g_2) , where id_I is the identity map on unit interval I . Moreover, there is an (h_1, h_2) -affiliated map (H_1, H_2) from $\alpha \times \eta$ to β of (f_1, f_2) . That is, we have commutative diagrams

$$\begin{array}{ccc}
 A_1 \vee X & \xrightarrow{f_1 \vee h_1} & B_1 \vee B_1 \\
 \downarrow j & & \downarrow \nabla \\
 A_1 \times X & \xrightarrow{H_1} & B_1 \\
 \downarrow \alpha \times \eta & & \downarrow \beta \\
 A_2 \times Y & \xrightarrow{H_2} & B_2 \\
 \uparrow j & & \uparrow \nabla \\
 A_2 \vee Y & \xrightarrow{f_2 \vee h_2} & B_2 \vee B_2
 \end{array}$$

and

$$\begin{array}{ccc} A_1 \times I & \xrightarrow{F_1} & B_1 \\ \downarrow \alpha \times id_I & & \downarrow \beta \\ A_2 \times I & \xrightarrow{F_2} & B_2. \end{array}$$

Define $\widehat{H}_1 : A_1 \times * \times I \sqcup * \times X \times I \sqcup A_1 \times X \times 0 \rightarrow B_1$ by

$$\widehat{H}_1(a, x, t) = \begin{cases} F_1(a, t) & \text{for } (a, x, t) \in A_1 \times * \times I, \\ h_1(x) & \text{for } (a, x, t) \in * \times X \times I, \\ H_1(a, x) & \text{for } (a, x, t) \in A_1 \times X \times 0. \end{cases}$$

Then \widehat{H}_1 is well-defined since $F_1(*, t) = * = h_1(*)$, $F_1(a, 0) = f_1(a) = H_1(a, *)$ and $H_1(*, x) = h_1(x)$. But since $A_1 \times * \cup * \times X$ is a sub-complex of $A_1 \times X$, there is an extension $\bar{H}_1 : A_1 \times X \times I \rightarrow B_1$ of \widehat{H}_1 . Similarly, we obtain another extension.

Define $\widehat{H}_2 : A_2 \times * \times I \sqcup A_1 \times Y \times I \sqcup A_2 \times Y \times 0 \rightarrow B_2$ by

$$\widehat{H}_2(a, y, t) = \begin{cases} F_2(a, t) & \text{for } (a, y, t) \in A_2 \times * \times I, \\ \beta \bar{H}_1(a, \eta^{-1}(y), t) & \text{for } (a, y, t) \in A_1 \times Y \times I, \\ H_2(a, y) & \text{for } (a, y, t) \in A_2 \times Y \times 0, \end{cases}$$

where $\eta^{-1} : Y \rightarrow X$ is the inverse of η . Since $\beta \bar{H}_1(a, \eta^{-1}(y), t) = F_2(a, t)$ for $a \in A_1$, $\beta \bar{H}_1(a, \eta^{-1}(y), 0) = H_2(a, y)$ and $H_2(a, *) = f_2(a) = F_2(a, 0)$, \widehat{H}_2 is well-defined. Since $A_1 \times Y \cup A_2 \times *$ is a sub-complex of $A_2 \times Y$, there is an extension $\bar{H}_2 : A_2 \times Y \times I \rightarrow B_2$ of \widehat{H}_2 . Let

$$G_1 = \bar{H}_1|_{A_1 \times X \times 1} \text{ and } G_2 = \bar{H}_2|_{A_2 \times Y \times 1}.$$

Since for $(a, x) \in A_1 \times X$,

$$\begin{aligned} G_2(\alpha \times \eta)(a, x) &= G_2(a, \eta(x)) = \bar{H}_2(a, \eta(x), 1) \\ &= \widehat{H}_2(a, \eta(x), 1) = \beta \bar{H}_1(a, \eta^{-1}(\eta(x)), 1) \\ &= \beta \bar{H}_1(a, x, 1) = \beta G_1(a, x), \end{aligned}$$

(G_1, G_2) is a morphism from $\alpha \times \eta$ to β . Moreover,

$$\begin{aligned} G_1(a, *) &= \bar{H}_1(a, *, 1) = \widehat{H}_1(a, *, 1) = F_1(a, 1) = g_1(a), \\ G_2(a, *) &= \bar{H}_2(a, *, 1) = \widehat{H}_2(a, *, 1) = F_2(a, 1) = g_2(a), \\ G_1(*, x) &= \bar{H}_1(*, x, 1) = \widehat{H}_1(*, x, 1) = h_1(x), \end{aligned}$$

and

$$\begin{aligned} G_2(*, y) &= \bar{H}_2(*, y, 1) = \widehat{H}_2(*, y, 1) = \beta \bar{H}_1(*, \eta^{-1}(y), 1) \\ &= \beta h_1(\eta^{-1}(y)) = h_2(\eta(\eta^{-1}(y))) = h_2(y). \end{aligned}$$

Therefore, (G_1, G_2) is an (h_1, h_2) -affiliated morphism of (g_1, g_2) . \square

By Remark 1, the subset $\mathcal{G}^{(h_1, \beta h_1)}(\alpha, \beta)$ is just the subset $\mathcal{G}^{h_1}(\alpha, \beta)$. Moreover, $\mathcal{G}^{(h_1, *)}(\alpha, *) = \mathcal{G}^{h_1}(A_1, B_1)$, where $\alpha : A_1 \rightarrow B_1$ is a map. Here, we denote $\mathcal{G}^{(h_1, h_2)}(\Sigma^n \alpha, \beta)$ by $\mathcal{G}_n^{(h_1, h_2)}(\alpha, \beta)$, where $\Sigma^n \alpha : \Sigma^n A_1 \rightarrow \Sigma^n A_2$ is the n -fold suspension map of α . In particular, if $i_n : \Sigma^n A \rightarrow C\Sigma^n A$ is the natural inclusion, then we denote $\mathcal{G}^{(h_1, h_2)}(i_n, \beta)$ by $\mathcal{G}_n^{(h_1, h_2)}(A, \beta)$. Thus, we conclude that if $\eta = id_X$ and $h_2 : X \rightarrow *$, then $\mathcal{G}_n^{(h_1, h_2)}(A, \beta)$ is just $\mathcal{G}_n^{h_1}(A, \beta)$. Therefore, the Gottlieb subset of $\Pi(\alpha, \beta)$ with respect to a morphism (h_1, h_2) in the category of pairs is the most general concept in the class of all generalized Gottlieb subsets introduced by many authors until now.

4. Properties of cyclic morphisms with respect to a morphism

Let $\alpha : A_1 \rightarrow A_2, \beta : B_1 \rightarrow B_2$ and $\gamma : C_1 \rightarrow C_2$ be maps. Then we have the following lemma.

Lemma 4.1. *Let $\eta : X \rightarrow Y$ be a map and $(h_1, h_2) : \eta \rightarrow \beta$ and $(g_1, g_2) : \gamma \rightarrow \alpha$ be morphisms. If $(f_1, f_2) : \alpha \rightarrow \beta$ is a cyclic morphism with respect to $(h_1, h_2) : \eta \rightarrow \beta$, then $(f_1, f_2) \circ (g_1, g_2) : \gamma \rightarrow \beta$ is a cyclic morphism with respect to (h_1, h_2) .*

Proof. Let $(H_1, H_2) : \alpha \times \eta \rightarrow \beta$ be an (h_1, h_2) -affiliated morphism of (f_1, f_2) . Then $(H_1(g_1 \times 1_X), H_2(g_2 \times 1_Y)) : \gamma \times \eta \rightarrow \beta$ is an (h_1, h_2) -affiliated morphism of $(f_1, f_2) \circ (g_1, g_2)$. It is clear by the following commutative diagram;

$$\begin{array}{ccccc}
 C_1 \vee X & \xrightarrow{g_1 \vee 1_X} & A_1 \vee X & \xrightarrow{f_1 \vee h_1} & B_1 \vee B_1 \\
 \downarrow j & & \downarrow j & & \downarrow \nabla \\
 C_1 \times X & \xrightarrow{g_1 \times 1_X} & A_1 \times X & \xrightarrow{H_1} & B_1 \\
 \downarrow \gamma \times \eta & & \downarrow \alpha \times \eta & & \downarrow \beta \\
 C_2 \times Y & \xrightarrow{g_2 \times 1_Y} & A_2 \times Y & \xrightarrow{H_2} & B_2 \\
 \uparrow j & & \uparrow j & & \uparrow \nabla \\
 C_2 \vee Y & \xrightarrow{g_2 \vee 1_Y} & A_2 \vee Y & \xrightarrow{f_2 \vee h_2} & B_2 \vee B_2.
 \end{array}$$

□

Corollary 1. *If $(g_1, g_2) : \gamma \rightarrow \alpha$ is a morphism, then the induced map $(g_1, g_2)^* : \Pi(\alpha, \beta) \rightarrow \Pi(\gamma, \beta)$ carries $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ into $\mathcal{G}^{(h_1, h_2)}(\gamma, \beta)$.*

Corollary 1 shows that for fixed $\beta, \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ is a homotopy invariant in the category of pairs. That is, we have the following theorem.

Theorem 4.2. *Let $(g_1, g_2) : \gamma \rightarrow \alpha$ be a homotopy equivalence. Then $(g_1, g_2)^* : \mathcal{G}^{(h_1, h_2)}(\alpha, \beta) \rightarrow \mathcal{G}^{(h_1, h_2)}(\gamma, \beta)$ is an one-to-one correspondence. Especially, $(g_1, g_2)^*$ is an isomorphism provided that $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ and $\mathcal{G}^{(h_1, h_2)}(\gamma, \beta)$ are groups.*

Lemma 4.3. *If $(g_1, g_2) : \beta \rightarrow \gamma$ is a morphism and $(f_1, f_2) : \alpha \rightarrow \beta$ is a cyclic morphism with respect to $(h_1, h_2) : \eta \rightarrow \beta$, then the composition $(g_1, g_2) \circ (f_1, f_2)$ is a cyclic morphism with respect to $(g_1 h_1, g_2 h_2) : \eta \rightarrow \gamma$.*

Proof. Let $(H_1, H_2) : \alpha \times \eta \rightarrow \beta$ be an (h_1, h_2) -affiliated morphism of (f_1, f_2) . Then $(g_1 H_1, g_2 H_2) : \alpha \times \eta \rightarrow \gamma$ is an $(g_1 h_1, g_2 h_2)$ -affiliated morphism of $(g_1 f_1, g_2 f_2)$. \square

Corollary 2. *If $(g_1, g_2) : \beta \rightarrow \gamma$ is a morphism, then the induced map $(g_1, g_2)_* : \Pi(\alpha, \beta) \rightarrow \Pi(\alpha, \gamma)$ carries $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ into $\mathcal{G}^{(g_1 h_1, g_2 h_2)}(\alpha, \gamma)$.*

Is $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ a homotopy invariant in the category of pairs for fixed α ? It may not be true, since it does not follow from Lemma 4.3 or Corollary 2 as Theorem 4.2. But we can give a partial solution of this question, by proving the following lemma.

Lemma 4.4. *Suppose $\alpha : A_1 \rightarrow A_2$ is an inclusion and $\eta : X \rightarrow Y$ is a homeomorphism. Then if two morphisms (h_1, h_2) and (h'_1, h'_2) from η to a map $\beta : B_1 \rightarrow B_2$ are homotopic, $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = \mathcal{G}^{(h'_1, h'_2)}(\alpha, \beta)$.*

Proof. It is sufficient to show that one of them contains the other. Let us show that $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) \subset \mathcal{G}^{(h'_1, h'_2)}(\alpha, \beta)$. By Lemma 3.1, it is sufficient to show that there is an (h'_1, h'_2) -affiliated morphism of (f_1, f_2) for each $[f_1, f_2] \in \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$. Suppose $[f_1, f_2] \in \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$. Then there is an (h_1, h_2) -affiliated morphism of (f_1, f_2) , that is, there is a morphism $(H_1, H_2) : \alpha \times \eta \rightarrow \beta$ such that $H_i|_{A_i \times * } = f_i$ for $i = 1, 2$, $H_1|_{* \times X} = h_1$ and $H_2|_{* \times Y} = h_2$. Moreover, there is a homotopy $(F_1, F_2) : \eta \times id_I \rightarrow \beta$ between (h_1, h_2) and (h'_1, h'_2) . Define $\widehat{H}_1 : A_1 \times * \times I \sqcup * \times X \times I \sqcup A_1 \times X \times 0 \rightarrow B_1$ by

$$\widehat{H}_1(a, x, t) = \begin{cases} f_1(a) & \text{for } (a, x, t) \in A_1 \times * \times I, \\ F_1(x, t) & \text{for } (a, x, t) \in * \times X \times I, \\ H_1(a, x) & \text{for } (a, x, t) \in A_1 \times X \times 0. \end{cases}$$

Then \widehat{H}_1 is well-defined. Since $A_1 \times * \cup * \times X$ is a sub-complex of $A_1 \times X$, there is an extension $\bar{H}_1 : A_1 \times X \times I \rightarrow B_1$ of \widehat{H}_1 . Similarly, we obtain another extension.

Define $\widehat{H}_2 : A_2 \times * \times I \sqcup A_1 \times Y \times I \sqcup A_2 \times Y \times 0 \rightarrow B_2$ by

$$\widehat{H}_2(a, y, t) = \begin{cases} f_2(a) & \text{for } (a, y, t) \in A_2 \times * \times I, \\ \beta \bar{H}_1(a, \eta^{-1}(y), t) & \text{for } (a, y, t) \in A_1 \times Y \times I, \\ H_2(a, y) & \text{for } (a, y, t) \in A_2 \times Y \times 0, \end{cases}$$

where $\eta^{-1} : Y \rightarrow X$ is the inverse of η . Since $\beta \bar{H}_1(a, \eta^{-1}(y), t) = \beta f_1(a) = f_2(a)$ for $a \in A_1$, $\beta \bar{H}_1(a, \eta^{-1}(y), 0) = \beta H_1(a, \eta^{-1}(y)) = H_2(a, y)$ for $a \in A_1$ and $H_2(a, *) = f_2(a)$ for $a \in A_2$, \widehat{H}_2 is well-defined. Since $A_1 \times Y \cup A_2 \times *$ is a sub-complex of $A_2 \times Y$, there is an extension $\bar{H}_2 : A_2 \times Y \times I \rightarrow B_2$ of \widehat{H}_2 . Let

$$G_1 = \bar{H}_1|_{A_1 \times X \times 1} \text{ and } G_2 = \bar{H}_2|_{A_2 \times Y \times 1}.$$

Since for $(a, x) \in A_1 \times X$

$$\begin{aligned} G_2(\alpha \times \eta)(a, x) &= G_2(a, \eta(x)) = \bar{H}_2(a, \eta(x), 1) \\ &= \hat{H}_2(a, \eta(x), 1) = \beta \bar{H}_1(a, \eta^{-1}(\eta(x)), 1) \\ &= \beta \bar{H}_1(a, x, 1) = \beta G_1(a, x), \end{aligned}$$

(G_1, G_2) is a morphism from $\alpha \times \eta$ to β . Moreover,

$$G_1(a, *) = \bar{H}_1(a, *, 1) = \hat{H}_1(a, *, 1) = f_1(a) \text{ for } a \in A_1,$$

$$G_2(a, *) = \bar{H}_2(a, *, 1) = \hat{H}_2(a, *, 1) = f_2(a) \text{ for } a \in A_2,$$

$$G_1(*, x) = \bar{H}_1(*, x, 1) = \hat{H}_1(*, x, 1) = F(x, 1) = h'_1(x) \text{ for } x \in X,$$

and for $y \in Y$

$$\begin{aligned} G_2(*, y) &= \bar{H}_2(*, y, 1) = \hat{H}_2(*, y, 1) = \beta \bar{H}_1(*, \eta^{-1}(y), 1) \\ &= \beta F_1(\eta^{-1}(y), 1) = \beta h'_1(\eta^{-1}(y), 1) = h'_2(\eta(\eta^{-1}(y))) = h'_2(y). \end{aligned}$$

Therefore, (G_1, G_2) is an (h'_1, h'_2) -affiliated morphism of (f_1, f_2) . □

Theorem 4.5. *Suppose $\alpha : A_1 \rightarrow A_2$ is an inclusion, $\eta : X \rightarrow Y$ is a homeomorphism and $(g_1, g_2) : \beta \rightarrow \gamma$ is a homotopy equivalence. Then $(g_1, g_2)_* : \mathcal{G}^{(h_1, h_2)}(\alpha, \beta) \rightarrow \mathcal{G}^{(g_1 h_1, g_2 h_2)}(\alpha, \gamma)$ is an one-to-one correspondence. Especially, $(g_1, g_2)_*$ is an isomorphism provided that $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ and $\mathcal{G}^{(g_1 h_1, g_2 h_2)}(\alpha, \gamma)$ are groups.*

Proof. Let $(g'_1, g'_2) : \gamma \rightarrow \beta$ be a homotopy inverse of (g_1, g_2) . Then

$$(g'_1, g'_2)_*(\mathcal{G}^{(g_1 h_1, g_2 h_2)}(\alpha, \gamma)) \subset \mathcal{G}^{(g'_1 g_1 h_1, g'_2 g_2 h_2)}(\alpha, \beta) = \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$$

by Corollary 2 and Lemma 4.4. Thus

$$(g'_1, g'_2)_* : \mathcal{G}^{(g_1 h_1, g_2 h_2)}(\alpha, \gamma) \rightarrow \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$$

is the inverse function of $(g_1, g_2)_*$. □

Lemma 4.1 also gives us an information which characterizes the cyclic morphism as follows.

Theorem 4.6. *Let $\beta : B_1 \rightarrow B_2$ and $\eta : X \rightarrow Y$ be maps. If the identity morphism $(id_{B_1}, id_{B_2}) : \beta \rightarrow \beta$ is cyclic with respect to a morphism $(h_1, h_2) : \eta \rightarrow \beta$, then $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = \Pi(\alpha, \beta)$ for any map α .*

Proof. Suppose $(id_{B_1}, id_{B_2}) : \beta \rightarrow \beta$ is cyclic with respect to (h_1, h_2) and $[f_1, f_2] \in \Pi(\alpha, \beta)$. Since $(f_1, f_2) = (id_{B_1}, id_{B_2})(f_1, f_2)$, (f_1, f_2) is cyclic with respect to (h_1, h_2) by Lemma 4.1. Thus $[f_1, f_2] \in \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$. □

The reverse implication may not be true because the homotopic relation does not preserve the cyclicity in the category of pairs in general. But under the hypothesis that $\beta : B_1 \rightarrow B_2$ is an inclusion and $\eta : X \rightarrow Y$ is a homeomorphism, it is true and thus we have the following theorem.

Theorem 4.7. *Let $\beta : B_1 \rightarrow B_2$ be an inclusion. Then the identity morphism $(id_{B_1}, id_{B_2}) : \beta \rightarrow \beta$ is cyclic with respect to a morphism $(h_1, h_2) : \eta \rightarrow \beta$ if and only if $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = \Pi(\alpha, \beta)$ for each map α .*

Proof. By Theorem 4.6, it is sufficient to prove the ‘only if’ part. Suppose $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = \Pi(\alpha, \beta)$ for each map α . Especially, $\mathcal{G}^{(h_1, h_2)}(\beta, \beta) = \Pi(\beta, \beta)$. Thus

$$[id_{B_1}, id_{B_2}] \in \Pi(\beta, \beta) = \mathcal{G}^{(h_1, h_2)}(\beta, \beta).$$

Therefore, there is a morphism $(f_1, f_2) : \beta \rightarrow \beta$ which is cyclic with respect to (h_1, h_2) and homotopic to (id_{B_1}, id_{B_2}) in the category of pairs. By Lemma 3.1, (id_{B_1}, id_{B_2}) is cyclic with respect to (h_1, h_2) . \square

5. Group structures of Gottlieb subsets with respect to a morphism

In this section, we shall investigate group structures of $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$. First, we shall check the possibility that the Gottlieb subset with respect a morphism is an image of the induced map from an evaluation morphism in the category of pairs.

Let $\omega_1 : B_1^X \rightarrow B_1$ and $\omega_2 : B_2^Y \rightarrow B_2$ be evaluation maps at a base point $*$ $\in X$ and $*$ $\in Y$, respectively. Let $\bar{\beta} : B_1^X \rightarrow B_2^Y$ be an object and $(\omega_1, \omega_2) : \bar{\beta} \rightarrow \beta$ a morphism. Then $(\omega_1, \omega_2)_* : \Pi(\alpha, \bar{\beta}) \rightarrow \Pi(\alpha, \beta)$ is a homomorphism.

Definition. An object $\eta : X \rightarrow Y$ is called a *right invertible object* if it has a right inverse, that is, there is an object $\theta : Y \rightarrow X$ such that $\eta \circ \theta = id_Y$.

Example 1. (a) The identity map $id_X : X \rightarrow X$ is a right invertible object.

(b) The natural projection $p : X \times Y \rightarrow Y$ is a right invertible object.

(c) Every homeomorphism is a right invertible object.

(d) Every Hurewicz fibration with a cross-section is a right invertible object.

Lemma 5.1. *Let $\beta : B_1 \rightarrow B_2$ and $\eta : X \rightarrow Y$ be objects and $(h_1, h_2) : \eta \rightarrow \beta$ be a morphism. If η is a right invertible object with the inverse θ , then there exists an object $\bar{\beta} : B_1^X \rightarrow B_2^Y$ such that $\bar{\beta}(h_1) = h_2$, where B_1^X and B_2^Y are function spaces with compact open topology with base points h_1, h_2 , respectively.*

Proof. Define $\bar{\beta} : B_1^X \rightarrow B_2^Y$ by $\bar{\beta}(g) = \beta \circ g \circ \theta$ for each $g \in B_1^X$. Then $\bar{\beta}$ is continuous since the adjoint $\widehat{\bar{\beta}}$ of $\bar{\beta}$ is the composition $\widehat{\bar{\beta}} = \beta \circ ev \circ (1_{B_1^X} \times \theta)$ and it is continuous, where ev is the evaluation map from $B_1^X \times X$ to B_1 given by $ev(f, x) = f(x)$. Moreover,

$$\bar{\beta}(h_1) = \beta h_1 \theta = h_2 \eta \theta = h_2. \quad \square$$

Lemma 5.2. *Let $\alpha : A_1 \rightarrow A_2$ and $\beta : B_1 \rightarrow B_2$ be objects, $\eta : X \rightarrow Y$ be a right invertible map with the inverse θ and $(h_1, h_2) : \eta \rightarrow \beta$ be a morphism. Then $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$ is contained in the image $(\omega_1, \omega_2)_* \Pi(\alpha, \bar{\beta})$.*

Proof. Let $[f_1, f_2] \in \mathcal{G}^{(h_1, h_2)}(\alpha, \beta)$. Then there exists a morphism $(H_1, H_2) : \alpha \times \eta \rightarrow \beta$ such that the following diagram is commutative:

$$\begin{array}{ccc}
 A_1 \vee X & \xrightarrow{f_1 \vee h_1} & B_1 \vee B_1 \\
 \downarrow j & & \downarrow \nabla \\
 A_1 \times X & \xrightarrow{H_1} & B_1 \\
 \downarrow \alpha \times \eta & & \downarrow \beta \\
 A_2 \times Y & \xrightarrow{H_2} & B_2 \\
 \uparrow j & & \uparrow \nabla \\
 A_2 \vee Y & \xrightarrow{f_2 \vee h_2} & B_2 \vee B_2
 \end{array}$$

Consider the adjoint maps \widetilde{H}_1 and \widetilde{H}_2 of H_1 and H_2 , respectively. Define $\overline{\beta} : B_1^X \rightarrow B_2^Y$ by $\overline{\beta}(g) = \beta \circ g \circ \theta$. Then $(\widetilde{H}_1, \widetilde{H}_2) : \alpha \rightarrow \overline{\beta}$ is a morphism. In fact,

$$\begin{aligned}
 \overline{\beta} \widetilde{H}_1(a)(y) &= (\beta \circ \widetilde{H}_1(a) \circ \theta)(y) = \beta H_1(a, \theta(y)) = H_2(\alpha(a), \eta \theta(y)) \\
 &= \widetilde{H}_2(\alpha(a))(\eta \theta(y)) = \widetilde{H}_2(\alpha(a))(y).
 \end{aligned}$$

Moreover,

$$\omega_1 \widetilde{H}_1(a) = \widetilde{H}_1(a)(*) = H_1(a, *) = f_1(a), \quad \omega_2 \widetilde{H}_2(a) = \widetilde{H}_2(a)(*) = f_2(a).$$

Thus $[\widetilde{H}_1, \widetilde{H}_2] \in \Pi(\alpha, \overline{\beta})$ and $(\omega_1, \omega_2)_*[\widetilde{H}_1, \widetilde{H}_2] = [\omega_1 \widetilde{H}_1, \omega_2 \widetilde{H}_2] = [f_1, f_2]$. Therefore $\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) \subset (\omega_1, \omega_2)_* \Pi(\alpha, \overline{\beta})$. \square

Lemma 5.3. *Under the same hypothesis in Lemma 5.2, we have*

$$(\omega_1, \omega_2)_* \Pi(\alpha, \overline{\beta}) \subset \mathcal{G}^{(h_1 \theta \eta, h_2)}(\alpha, \beta)$$

for the object $\overline{\beta} : B_1^X \rightarrow B_2^Y$ given by $\overline{\beta}(g) = \beta g \theta$, where θ is a right inverse of η .

Proof. Let $[f_1, f_2] \in (\omega_1, \omega_2)_* \Pi(\alpha, \overline{\beta})$. Then there exists a morphism $(\overline{f}_1, \overline{f}_2) : \alpha \rightarrow \overline{\beta}$ such that $(\omega_1, \omega_2)_*[\overline{f}_1, \overline{f}_2] = [f_1, f_2]$. By the definition of $\mathcal{G}^{(h_1 \theta \eta, h_2)}(\alpha, \beta)$, we may assume $(\omega_1, \omega_2)_*(\overline{f}_1, \overline{f}_2) = (f_1, f_2)$. Consider the adjoint maps $\widetilde{\overline{f}}_1 : A_1 \times X \rightarrow B_1$ and $\widetilde{\overline{f}}_2 : A_2 \times Y \rightarrow B_2$ of \overline{f}_1 and \overline{f}_2 respectively. Define $\widehat{f}_1 = \widetilde{\overline{f}}_1(1_{A_1} \times \theta \eta)$ and $\widehat{f}_2 = \widetilde{\overline{f}}_2$. Then $(\widehat{f}_1, \widehat{f}_2)$ is a morphism from $\alpha \times \eta$ to β . In fact,

$$\begin{aligned}
 \widehat{f}_2(\alpha \times \eta)(a, x) &= \widehat{f}_2(\alpha(a), \eta(x)) = \overline{f}_2 \alpha(x)(\eta(x)) \\
 &= \overline{\beta} \overline{f}_1(a)(\eta(x)) = \beta \overline{f}_1(a) \theta \eta(x) = \beta \widetilde{\overline{f}}_1(a, \theta \eta(x)) \\
 &= \beta \widetilde{\overline{f}}_1(1_{A_1} \times \theta \eta)(a, x) = \beta \widehat{f}_1(a, x).
 \end{aligned}$$

Furthermore,

$$\begin{aligned} \widehat{f}_1(a, *) &= \widetilde{f}_1(a, \theta\eta(*)) = \overline{f}_1(a)(*) = \omega_1 \overline{f}_1(a) = f_1(a), \\ \widehat{f}_1(*, x) &= \widetilde{f}_1(*, \theta\eta(x)) = \overline{f}_1(*)\theta\eta(x) = h_1\theta\eta(x), \\ \widehat{f}_2(a, *) &= \overline{f}_2(a)(*) = \omega_2 \overline{f}_2(a) = f_2(a) \end{aligned}$$

and

$$\widehat{f}_2(*, x) = \overline{f}_2(*)\theta\eta(x) = h_2.$$

Thus $[f_1, f_2] \in \mathcal{G}^{(h_1\theta\eta, h_2)}(\alpha, \beta)$. □

By Lemma 5.2 and Lemma 5.3 , we have the following theorem.

Theorem 5.4. *Let $\alpha : A_1 \rightarrow A_2$ and $\beta : B_1 \rightarrow B_2$ be maps, $\eta : X \rightarrow Y$ be a homeomorphism and $(h_1, h_2) : \eta \rightarrow \beta$ be a morphism. Then we have*

$$\mathcal{G}^{(h_1, h_2)}(\alpha, \beta) = (\omega_1, \omega_2)_* \Pi(\alpha, \overline{\beta}),$$

where $\overline{\beta} : B_1^X \rightarrow B_2^Y$ is defined by $\overline{\beta}(g) = \beta g \theta$ for $g \in B_1^X$ and the inverse θ of η .

Definition. A map $\alpha : A_1 \rightarrow A_2$ is said to be *co-H-object* if there exists a morphism $(\phi_1, \phi_2) : \alpha \rightarrow \alpha \vee \alpha$ such that $j_i \circ \phi_i : A_i \rightarrow A_i \times A_i$ is homotopic to Δ_i for $i = 1, 2$, where $(j_1, j_2) : \alpha \vee \alpha \rightarrow \alpha \times \alpha$ is the inclusion and $(\Delta_1, \Delta_2) : \alpha \rightarrow \alpha \times \alpha$ is the diagonal morphism. In this case, (ϕ_1, ϕ_2) is called *co-H-structure morphism of α* .

Example 2. If (A_1, μ_1) and (A_2, μ_2) are co-H-spaces and $(\mu_1, \mu_2) : \alpha \rightarrow \alpha \vee \alpha$ is a morphism, then $\alpha : A_1 \rightarrow A_2$ is a co-H-object. Thus if $f : C_1 \rightarrow C_2$ is a map, then the suspension map $\Sigma f : \Sigma C_1 \rightarrow \Sigma C_2$ is a co-H-object. Let $\phi_i : \Sigma C_i \rightarrow \Sigma C_i \vee \Sigma C_i$ be a co-H-structure given by

$$\phi_i \langle c, t \rangle = \begin{cases} (\langle c, 2t \rangle, *) & \text{if } 0 \leq t \leq 1/2 \\ (*, \langle c, 2t - 1 \rangle) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

for $i = 1, 2$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \Sigma C_1 & \xrightarrow{\phi_1} & \Sigma C_1 \vee \Sigma C_1 \\ \downarrow \Sigma f \downarrow & & \downarrow \Sigma f \vee \Sigma f \\ \Sigma C_2 & \xrightarrow{\phi_2} & \Sigma C_2 \vee \Sigma C_2 \end{array}$$

In fact,

$$\phi_2 \Sigma f \langle a, t \rangle = \phi_2 (\langle f(a), t \rangle) = \begin{cases} (\langle f(a), 2t \rangle, *) & \text{for } 0 \leq t \leq 1/2 \\ (*, \langle f(a), 2t - 1 \rangle) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

$$\begin{aligned}
 (\Sigma f \vee \Sigma f)\phi_1\langle a, t \rangle &= \begin{cases} (\Sigma f \vee \Sigma f)(\langle a, 2t \rangle, *) & \text{for } 0 \leq t \leq 1/2 \\ (\Sigma f \vee \Sigma f)(*, \langle a, 2t - 1 \rangle) & \text{for } 1/2 \leq t \leq 1 \end{cases} \\
 &= \begin{cases} (\langle f(a), 2t \rangle, *) & \text{for } 0 \leq t \leq 1/2 \\ (*, \langle f(a), 2t - 1 \rangle) & \text{for } 1/2 \leq t \leq 1. \end{cases}
 \end{aligned}$$

It is well-known fact if α is a suspension map, then $\Pi(\alpha, \beta)$ has a group structure for each object β .

Theorem 5.5. *Let $\Sigma\alpha : \Sigma A_1 \rightarrow \Sigma A_2$ be the suspension map of $\alpha : A_1 \rightarrow A_2$, $\eta : X \rightarrow Y$ be a right invertible map with the inverse θ and $(h_1, h_2) : \eta \rightarrow \Sigma\alpha$ be a morphism. Then $\mathcal{G}^{(h_1, h_2)}(\Sigma\alpha, \beta)$ is a subgroup of $\Pi(\Sigma\alpha, \beta)$ for each map $\beta : B_1 \rightarrow B_2$.*

Proof. By Lemma 4.1, it is sufficient to show that the group operation of $\Pi(\Sigma\alpha, \beta)$ is closed in $\mathcal{G}^{(h_1, h_2)}(\Sigma\alpha, \beta)$. Let $[f_1, f_2]$ and $[g_1, g_2]$ be elements in $\mathcal{G}^{(h_1, h_2)}(\Sigma\alpha, \beta)$. Then there are two (h_1, h_2) -affiliated morphisms $(H_1, H_2) : \Sigma\alpha \times \eta \rightarrow \beta$ and $(G_1, G_2) : \Sigma\alpha \times \eta \rightarrow \beta$ of (f_1, f_2) and (g_1, g_2) , respectively. Consider the adjoint maps $\widetilde{H}_1, \widetilde{G}_1 : A_1 \rightarrow B_1^X$ of H_1 and G_1 respectively and $\widetilde{H}_2, \widetilde{G}_2 : A_2 \rightarrow B_2^Y$ of H_2 and G_2 respectively. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
 \Sigma A_1 & \xrightarrow{\phi_1} & \Sigma A_1 \vee \Sigma A_1 & \xrightarrow{\widetilde{H}_1 \vee \widetilde{G}_1} & B_1^X \vee B_1^X & \xrightarrow{\nabla} & B_1^X & \xrightarrow{\omega_1} & B_1 \\
 \downarrow \Sigma\alpha & & \downarrow \Sigma\alpha \vee \Sigma\alpha & & \downarrow \overline{\beta} \vee \overline{\beta} & & \downarrow \overline{\beta} & & \downarrow \beta \\
 \Sigma A_2 & \xrightarrow{\phi_2} & \Sigma A_2 \vee \Sigma A_2 & \xrightarrow{\widetilde{H}_2 \vee \widetilde{G}_2} & B_2^Y \vee B_2^Y & \xrightarrow{\nabla} & B_2^Y & \xrightarrow{\omega_2} & B_2
 \end{array}$$

where $(\phi_1, \phi_2) : \Sigma\alpha \rightarrow \Sigma\alpha \vee \Sigma\alpha$ is a co-H-structure morphism and $\overline{\beta} : B_1^X \rightarrow B_2^Y$ is given by $\overline{\beta}(g) = \beta g \theta$ for each $g \in B_1^X$. In order to show that the diagram is commutative, it is sufficient to check the commutativity of the second rectangle. Since $\beta(\widetilde{H}_1(a)(x)) = \beta(H_1(a, x)) = H_2(\Sigma\alpha(a), \eta(x)) = \widetilde{H}_2(\Sigma\alpha(a))(\eta(x))$, we have

$$\begin{aligned}
 (\overline{\beta} \vee \overline{\beta})(\widetilde{H}_1 \vee \widetilde{G}_1)(a, *) &= (\overline{\beta}\widetilde{H}_1(a), \overline{\beta}\widetilde{G}_1(*)) \\
 &= (\beta\widetilde{H}_1(a)\theta, \overline{\beta}(h_1)) \\
 &= (\widetilde{H}_2\Sigma\alpha(a)\eta\theta, h_2),
 \end{aligned}$$

$$(\widetilde{H}_2 \vee \widetilde{G}_2)(\Sigma\alpha \vee \Sigma\alpha)(a, *) = (\widetilde{H}_2\Sigma\alpha(a), \widetilde{G}_2\Sigma\alpha(*)) = (\widetilde{H}_2\Sigma\alpha(a), h_2)$$

for each $(a, *) \in \Sigma A_1 \vee \Sigma A_1$. Similarly, we can prove the equality for each $(*, a') \in \Sigma A_1 \vee \Sigma A_1$. Consider the adjoint maps $\nabla(\widetilde{H}_i \vee \widetilde{G}_i)$ of $\nabla(\widetilde{H}_i \vee \widetilde{G}_i)$ for

$i = 1, 2$. Then we have the following diagram:

$$\begin{array}{ccccc}
 \Sigma A_1 \times X & \xrightarrow{\psi_1} & (\Sigma A_1 \vee \Sigma A_1) \times X & \xrightarrow{\psi_2} & B_1 \\
 \downarrow \Sigma\alpha \times \eta & & \downarrow (\Sigma\alpha \vee \Sigma\alpha) \times \eta & & \downarrow \beta \\
 \Sigma A_2 \times Y & \xrightarrow{\psi_3} & (\Sigma A_2 \vee \Sigma A_2) \times Y & \xrightarrow{\psi_4} & B_2
 \end{array}$$

where $\psi_1 = \phi_1 \times 1_X$, $\psi_2 = \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)$, $\psi_3 = \phi_2 \times 1_Y$ and $\psi_4 = \nabla(\widetilde{H}_2 \vee \widetilde{G}_2)$. Let us represent each element in ΣA_1 (or ΣA_2) as $\langle a, t \rangle$. Then we have

$$\begin{aligned}
 \overline{\beta \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1 \times 1_X)(\langle a, t \rangle, x) &= \overline{\beta \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1(\langle a, t \rangle), x) \\
 &= \beta \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)(\phi_1(\langle a, t \rangle))(x) \\
 &= \beta \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)(\langle a, 2t \rangle, *) (x) \\
 &= \beta \widetilde{H}_1(\langle a, 2t \rangle)(x) \\
 &= \beta H_1(\langle a, 2t \rangle, x) \\
 &= H_2(\Sigma\alpha \times \eta)(\langle a, 2t \rangle, x) \\
 &= \widetilde{H}_2(\langle \alpha(a), 2t \rangle)(\eta(x)) \\
 &= \nabla(\widetilde{H}_2 \vee \widetilde{G}_2)(\langle \alpha(a), 2t \rangle, *) (\eta(x)) \\
 &= \nabla(\widetilde{H}_2 \vee \widetilde{G}_2)(\phi_2(\alpha(a), t))(\eta(x)) \\
 &= \overline{\nabla(\widetilde{H}_2 \vee \widetilde{G}_2)}(\phi_2(\alpha(a), t), \eta(x)) \\
 &= \overline{\nabla(\widetilde{H}_2 \vee \widetilde{G}_2)}\phi_2(\Sigma\alpha \times \eta)(\langle a, t \rangle, x)
 \end{aligned}$$

for $0 \leq t \leq 1/2$. By the same way we can prove the equality for the case $1/2 \leq t \leq 1$. Furthermore,

$$\begin{aligned}
 \overline{\nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1 \times 1_X)(\langle a, t \rangle, *) &= \overline{\nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1(\langle a, t \rangle), *) \\
 &= \nabla(\widetilde{H}_1 \vee \widetilde{G}_1)(\phi_1(\langle a, t \rangle))(*) \\
 &= \omega_1(\nabla(\widetilde{H}_1 \vee \widetilde{G}_1)\phi_1)(\langle a, t \rangle) \\
 &= \omega_1(\widetilde{H}_1 + \widetilde{G}_1)(\langle a, t \rangle),
 \end{aligned}$$

where ‘+’ is the group operation in $\Pi(\Sigma A_1, B_1^X)$. Similarly,

$$\overline{\nabla(\widetilde{H}_2 \vee \widetilde{G}_2)}(\phi_2 \times 1_Y)(\langle a, t \rangle, *) = \omega_2(\widetilde{H}_2 + \widetilde{G}_2)(\langle a, t \rangle).$$

Since

$$\begin{aligned}
 \overline{\nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1 \times 1_X)(*, x) &= \overline{\nabla(\widetilde{H}_1 \vee \widetilde{G}_1)}(\phi_1(*), x) \\
 &= (\nabla(\widetilde{H}_1(*), \widetilde{G}_1(*)))(x) = h_1(x)
 \end{aligned}$$

and $\overline{\nabla(\widetilde{H}_2 \vee \widetilde{G}_2)}(\phi_2 \times 1_Y)(*, y) = h_2(y)$,

$$[\omega_1(\widetilde{H}_1 + \widetilde{G}_1), \omega_2(\widetilde{H}_2 + \widetilde{G}_2)] \in \mathcal{G}^{(h_1, h_2)}(\Sigma\alpha, \beta).$$

But

$$\begin{aligned}
 (\omega_1, \omega_2)_*([\widetilde{H}_1, \widetilde{H}_2] + [\widetilde{G}_1, \widetilde{G}_2]) &= [\omega_1 \widetilde{H}_1, \omega_2 \widetilde{H}_2] + [\omega_1 \widetilde{G}_1, \omega_2 \widetilde{G}_2] \\
 &= [f_1, f_2] + [g_1, g_2] \\
 &= [f_1 + g_1, f_2 + g_2] \in \mathcal{G}^{(h_1, h_2)}(\Sigma\alpha, \beta). \quad \square
 \end{aligned}$$

References

- [1] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.
- [2] ———, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756.
- [3] ———, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432–437.
- [4] P. J. Hilton, *Homotopy Theory and Duality*, Mimeographed Notes, Cornell Univ. Ithaca, NY, 1959.
- [5] G. E. Lang, *Evaluation subgroups of factor spaces*, Pacific J. Math. **42** (1972), 701–709.
- [6] K. Y. Lee and M. H. Woo, *The G-sequence and the ω -homology of a CW-pair*, Topology Appl. **52** (1993), no. 3, 221–236.
- [7] ———, *Cyclic morphisms in the category of pairs and generalized G-sequences*, J. Math. Kyoto Univ. **38** (1998), no. 2, 271–285.
- [8] K. Y. Lee, M. H. Woo, and X. Zhao, *Certain generalizations of G-sequences and their exactness*, Bull. Korean Math. Soc. **45** (2008), no. 1, 119–131.
- [9] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc. Ser. A **32** (1982), no. 3, 349–357.
- [10] N. Oda, *The homotopy set of the axes of pairings*, Canad. J. Math. **42** (1990), no. 5, 856–868.
- [11] J. Pan, X. Shen, and M. Woo, *The G-sequence of a map and its exactness*, J. Korean Math. Soc. **35** (1998), no. 2, 281–294.
- [12] J. Siegel, *G-spaces, W-spaces and H-spaces*, Pacific J. Math. **31** (1969), 209–214.
- [13] K. Varadarajan, *Generalised Gottlieb groups*, J. Indian Math. Soc. (N.S.) **33** (1969), 141–164.
- [14] M. Woo and J. Kim, *Certain subgroups of homotopy groups*, J. Korean Math. Soc. **21** (1984), no. 2, 109–120.
- [15] M. H. Woo and K. Y. Lee, *On the relative evaluation subgroups of a CW-pair*, J. Korean Math. Soc. **25** (1988), no. 1, 149–160.

JIYEAN KIM
 DEPARTMENT OF MATHEMATICS
 KOREA UNIVERSITY
 SEOUL 136-701, KOREA
E-mail address: eskim@korea.ac.kr

KEE YOUNG LEE
 DEPARTMENT OF INFORMATION AND MATHEMATICS
 KOREA UNIVERSITY
 CHOCHIEON 339-700, KOREA
E-mail address: keyolee@korea.ac.kr