Bull. Korean Math. Soc. **47** (2010), No. 6, pp. 1297–1310
DOI 10.4134/BKMS.2010.47.6.1297

# TWISTED FACE-PAIRING 3-MANIFOLDS WHICH ARE HYPERBOLIC

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Abstract. We construct a family of 3-balls using cones which represent closed orientable 3-manifolds and study twisted face-pairing construction due to Cannon, Floyd and Parry to understand the structure of such manifolds. Moreover, we prove that those manifolds are hyperbolic.

## 1. Introduction

The goal of this paper is to study the topological properties of certain closed orientable 3-manifolds obtained by twisted face-pairing construction. In [2], Cannon, Floyd and Parry established the basic properties of twisted face-pairing manifolds. Besides, Cannon, Floyd and Parry investigated a special case of twisted face-pairing manifolds and the ample manifolds, and showed that these manifolds have Gromov hyperbolic fundamental groups in [3], they also showed how to construct Heegaard diagrams for twisted face-pairing 3-manifolds in [4]. From their construction, it is easy to give framed surgery descriptions for twisted face-pairing 3-manifolds. In [5], they showed that the class of twisted face-pairing manifolds contains all lens spaces, the Heisenberg manifold (Nil geometry), $S^2 \times S^1$, every orientable torus bundle over $S^1$ (Sol geometry), most closed, connected, orientable, Seifert fibered manifolds, and all connected sums of twisted face-pairing manifolds. It still seems unlikely that all closed, connected, orientable 3-manifolds can be obtained as face-pairing manifolds. It is shown that any twisted face-pairing 3-manifold that comes from an ample faceted 3-ball has a Gromove hyperbolic fundamental group in [4]. As a consequence they suggested the following question in [5].

**Question.** *Is every twisted face-pairing 3-manifold that comes from an ample faceted 3-ball hyperbolic?*

They considered twisted face-pairing manifolds which are hyperbolic and have small volume. Using SnapPea and the Dehn surgery description of twisted

Received May 7, 2009.
2000 *Mathematics Subject Classification.* 57M99.
*Key words and phrases.* twisted face-pairing, faceted 3-ball, twisted face-pairing 3-manifolds, Takahashi manifolds.

face-paring manifolds, they were able to construct hyperbolic manifolds with small volume. In Example 7.2 of [5], they introduced a hyperbolic manifold that comes from a model faceted 3-ball $P$ identified with the tetrahedron by twisted face-pairing construction. Also, they applied twisted face-pairing construction to other model faceted 3-ball $P$ such that $P$ is gotten from two tetrahedra by identifying a face of one with a face of the other ($P$ is a hexahedron). In this case, they proved that the twisted face-pairing manifold with special multiplier is hyperbolic manifold of small volume in the Example 7.5 of [3].

In this paper, we obtain a partial solution for the above question. We first construct that the new model faceted 3-ball (cone faceted 3-ball) obtained from several tetrahedra by identifying a face of one with a face of the other. Also we show that, for all but finite number of the twisted face-pairing manifolds induced as above, the new faceted 3-ball are hyperbolic.

The main results of this paper are the followings.

**Theorem 3.3**. *For any $n \geq 3$, let $M$ be a periodic Takahashi manifold given by $M_n(\frac{p}{q}, \frac{r}{s})$. If $p = s$ and $r = -q = 1$, then $M$ is a cone twisted face-pairing manifold $M_n(p)$.*

**Theorem 4.3**. *For any $n \geq 4$, the cone twisted face-pairing manifold $M_n(1)$ is hyperbolic.*

This paper is organized as follows. In Section 2, we analyze the twisted face-pairing constructions, which are defined in [2], [3], [4], and [5]. Also we construct the cone twisted face-pairing 3-manifolds. In Section 3, we explain the terminology for twist moves on framed links in $S^3$, which are used throughout the paper. Most of these are from [6] and [9]. Also we introduce the periodic Takahashi manifolds. Furthermore, the connection between the periodic Takahashi manifold and the cone twisted face-pairing manifold will be discussed. In Section 4, using the topological structure of the periodic Takahashi manifolds, we show that most of cone twisted face-pairing manifolds are hyperbolic.

## 2. Cone twisted face-paring construction

### 2.1. Twisted face-paring

The twisted face-pairing construction is modeled on a faceted 3-ball. A faceted 3-ball $P$ is an oriented CW complex such that $|P|$ is a closed 3-ball, the interior of $P$ is the one open 3-cell of $P$, and the cell structure of $\partial P$ does not consist of just one 0-cell and one 2-cell. The face $f$ is gotten from a closed disk by identifying some vertices and some pairs of edges if the interior of $f$ is homeomorphic to the interior of a closed disk. Hence the such face $f$ admits faces of $\partial P$.

Now a face-pairing $\varepsilon$ on a given faceted 3-ball $P$ consists of the following. First, the faces of $P$ are paired: for every face f of $P$ there exists a face $f^{-1} \neq f$ of $P$ such that $(f^{-1})^{-1} = f$. Second, for every face $f$ of $P$, there exists a cellular homeomorphism $\varepsilon_f : f \to f^{-1}$ called a face-pairing map such that $\varepsilon_{f^{-1}} = {\varepsilon_f}^{-1}$. We set $\varepsilon = \{\varepsilon_f^{\pm 1} \mid f \text{ is a face of } P\}$.

Let $P$ be a faceted 3-ball with orientation-reversing face-pairing $\varepsilon$ (A face pairing $\varepsilon$ is orientation reversing if every face-pairing map $\varepsilon_f$ reverses an orientation). Let $\sim$ be the equivalence relation on the set of edges of $P$ such that if $f$ is a face of $P$ and $e$ is an edge of $f$, then $e \sim \varepsilon_f(e)$. The equivalence classes of $\sim$ are called edge cycles.

$$E \ : \ e_1 \xrightarrow{\varepsilon_{f_1}^{\pm 1}} e_2 \xrightarrow{\varepsilon_{f_2}^{\pm 1}} \cdots \xrightarrow{\varepsilon_{f_{n-1}}^{\pm 1}} e_n \xrightarrow{\varepsilon_{f_n}^{\pm 1}} e_1.$$

Let $e_1, \ldots, e_n$ be the distinct edges of an edge cycle $E$ so that for every $i \in \{1, \ldots, n\}$ there exists a face $f_i$ of $P$ with $e_i \subseteq f_i$ and $e_{i+1} = \varepsilon_{f_i}(e_i)$, where the indices are taken modulo $j$.

For each edge cycle $E$, let $l_E$ be the number of edges in $E$ and we call $l_E$ the length of $E$. The function $m : \{\text{edge cycles}\} \to \mathbb{N}$ defined by $m(E) = m_E$ is called the multiplier function and a positive integer $m_E$ is called the multiplier of $E$.

Then we obtain a quotient space $P/\varepsilon$ consisting of orbits of points of $P$ under $\varepsilon$. We require that face-pairing maps satisfy the face-pairing compactibility condition from every edge cycle diagram.

$$\varepsilon_{f_n}\Big|_{e_n} \circ \cdots \circ \varepsilon_{f_2}\Big|_{e_2} \circ \varepsilon_{f_1}\Big|_{e_1} = \mathbf{1}_{e_1}.$$

Hence we ensure that $P/\varepsilon$ is a cell complex.

Now we define a twisted face-pairing manifold from $P$ and $\varepsilon$. Let $Q = Q(\varepsilon, m)$ be a faceted 3-ball from $P$ by just subdividing the edges of $P$ as follows. Let $e$ be an edge of $P$ and $E$ be the edge cycle of $e$. We subdivide $e$ into $l_E m_E$ subedges. The face pairing maps $\varepsilon_f$ on the faces on $P$ pairing maps on the faces of $Q$. For each face $f$ of $Q$, let $\tau$ be an orientation-preserving cellular homeomorphism of $f$ which takes each vertex of $f$ to its following vertex. Let $\delta$ be the face-pairing on $Q$ with the same pairing of faces as for the face-paring $\varepsilon$, and with $\delta_f = \varepsilon_f \circ \tau_f$ for each $f$ of $Q$. We set $\delta = \{\delta_f^{\pm 1} \mid f \text{ is a face of } Q\}$. Then $\delta$ is a face-paring on $Q$ called the twisted face-pairing.

Define $M(\varepsilon, m)$ to be the quotient complex $Q/\delta$. The fundamental theorem of twisting face-pairing is that $M$ is a 3-manifold. This was proved in Theorem 3.1 of [2]. We call $M(\varepsilon, m)$ a twisted face-pairing 3-manifold. For more details on these definitions see [2], [3], and [4].

### 2.2. Cone twisted face-paring

We will consider a special kind of face-pairing manifolds. We construct a model faceted 3-ball $P_n$ representing the closed orientable 3-manifold as follows:
**Step 1 : The cone faceted 3-ball**.

Let $P_n$ be a bi-pyramid with $n$-gon in $\mathbb{R}^3$. It is the union of two cones over a regular $n$-gon along a common face. We assume that $P_n$ is centered at the origin and its $n$ verties of valence 4 are in the $xy$-plane. The other two vertices are $n$-valence in $\mathbb{R}^3$. The vertices of the $p$-gon are denoted by $V_1,\ V_2,\ldots,V_n$, the vertices of the two cones by $N$ and $S$. We call $P_n$ a cone faceted 3-ball. See the Figure 1.

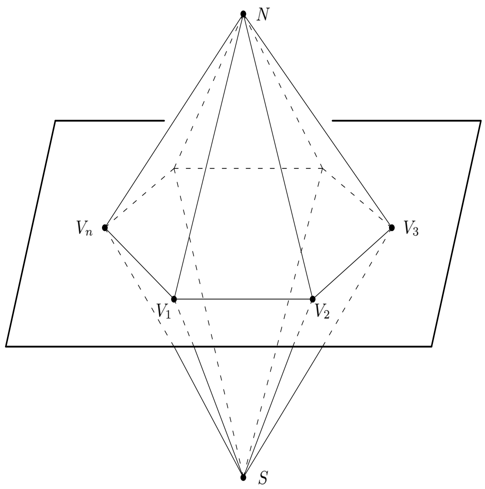


FIGURE 1. The cone faceted 3-ball $P_n$

**Step 2 : The reflected face-pairing**.

Let $\varepsilon = \{\varepsilon_i^{\pm 1} \mid i = 1, 2, \ldots, n\}$ be the orientation-reversing face-pairing map for which each face is paired with its image under reflection in the $xy$-plane and each face-pairing map is reflection in the $xy$-plane: For each $i$, we identify the face $f_i = \triangle V_i V_{i+1} N$ with the face ${f_i}^{-1} = \varepsilon_i(f_i) = \triangle V_i V_{i+1} S$, where the indices are taken modulo by $n$. We write using permutation notation in the following way:

$$\boldsymbol{\varepsilon_i} = \begin{pmatrix} V_i & V_{i+1} & N \\ V_i & V_{i+1} & S \end{pmatrix}, i \in \{1, \ldots, n\}.$$

In this case we call $\varepsilon$ reflected face-pairing. The edge cycles for $\varepsilon$ have $2n$ diagrams as follows.

$$E_i \ : \ V_i V_{i+1} \xrightarrow{\varepsilon_i} V_i V_{i+1} \ (1 \leq i \leq n : \text{indices mod } n).$$

$$E_j \ : \ V_{j-n} N \xrightarrow{\varepsilon_{j-1}} V_{j-n} S \xrightarrow{\varepsilon_{j-n}^{-1}} V_{j-n} N \ (n+1 \leq j \leq 2n : \text{indices mod } n).$$

Then $n$ edges in the $xy$-plane are in edge cycles $E_i$ of length 1, $l_{E_i} = 1$ $(1 \leq i \leq n)$, and each of the other $n$ edge cycles $E_j$ has length $l_{E_j} = 2$ $(n+1 \leq j \leq 2n)$.

**Step 3 : Edge subdivision**.

Let $m_E$ be a multiplier function for a reflected face-pairing $\varepsilon$. We choose a multiplier function $m_E$ for each edge cycle so that positive integer $m = m_{E_i} = m_{E_j}$ for $i \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, 2n\}$. Now, for each edge of edge cycle $E_i$, we subdivide edge into $l_{E_i}m_{E_i}$ subedges. So $n$ edges of $P_n$ in the $xy$-plane is subdivide into $l_{E_i}m_{E_i} = m$ $(1 \leq i \leq n)$ edges in $Q_n$, and each of the other $2n$ edges of $P_n$ is subdivided into $l_{E_j}m_{E_j} = 2m,$ $(n+1 \leq j \leq 2n)$ edges in $Q_n$. In our case each of the $2n$ faces of $Q_n$ is a $5m$-gon. See the Figure 2.

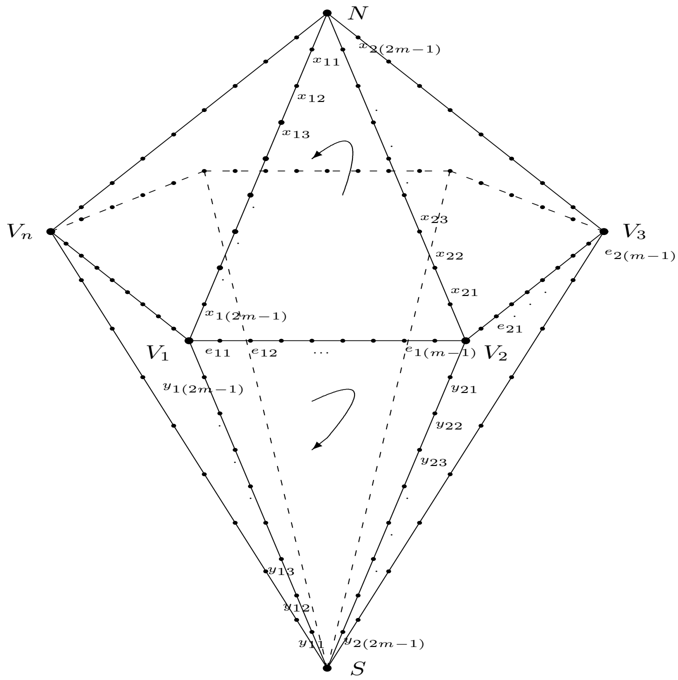


FIGURE 2. The complex $O_n$ for $P_n$

For example, $x_{11}$ is a new vertex in edge $V_1N$, and since the face-pairing map $\varepsilon_1$ takes $V_1N$ to $V_1S$, we have $\varepsilon_1(x_{11}) = y_{11}$. We now obtain

$$\varepsilon_i = \begin{pmatrix} V_i \ e_{i1}\cdots e_{i(2m-1)} & V_{i+1} \ x_{(i+1)1}\cdots x_{(i+1)(2m-1)} & N \ x_{i1}\cdots x_{i(2m-1)} \\ V_i \ e_{i1}\cdots e_{i(2m-1)} & V_{i+1} \ y_{(i+1)1}\cdots y_{(i+1)(2m-1)} & S \ y_{i1}\cdots y_{i(2m-1)} \end{pmatrix}, \quad (\text{mod } n).$$

**Step 4 : Cone twisted face-pairing**.

For our model we orient the boundary of every face of cone faceted 3-ball $P_n$ in the counter-clockwise direction. Before applying the induced face-pairing maps, we twist each face one subedge in the direction of the orientation of the boundary. Let $\tau_i$ be an orientation-preserving twisted cellular homeomorphism of each face of $P_n$ which takes each vertex of each face to its following vertex.

Then, we can define $\delta_i = \varepsilon_i \circ \tau_i$. So we have

$$\delta_i = \begin{pmatrix} V_i & e_{i1} & \cdots & e_{i(2m-1)} & V_{i+1} & x_{(i+1)1} & \cdots & x_{(i+1)(2m-1)} & N & x_{i1} & \cdots & x_{i(2m-1)} \\ e_{i1} & e_{i2} & \cdots & V_{i+1} & y_{(i+1)1} & y_{(i+1)2} & \cdots & S & y_{i1} & y_{i2} & \cdots & V_i \end{pmatrix}, \pmod n.$$

Define $M_n(\varepsilon, m)$ be the quotient complex $Q_n/\delta$. We will denote $M_n(\varepsilon, m)$ by $M_n(m)$. One can shows that the cell complex $M_n(\varepsilon, m)$ is completely determined by $n$ and $m$ since, in our case, we fixed the face-pairing $\varepsilon$ by the reflection in the $xy$-plane.

Now we have the following theorem.

**Theorem 2.1.** *Let $P_n$ be a cone faceted 3-ball with reflected face-pairing $\varepsilon$. Suppose given a multiplier function $m_{E_i} = m_{E_j} = m$ for $i \in \{1, \ldots, n\}$, $j \in \{n+1, \ldots, 2n\}$. Then the cell complex $M_n(m) = Q_n/\delta$ is an orientable closed 3-manifold.*

*Proof.* Using the twisted face-paring construction, it is natural to construct the cell complex $M_n(m)$ from the cone faceted 3-ball $P_n$. Thus, by Theorem 3.1 in [2], the cell complex $M_n(m)$ is an orientable closed 3-manifold. □

We call $M_n(m)$ a cone twisted face-pairing 3-manifold.

Now we compute the fundamental group of the cone twisted face-pairing manifold $M_n(m)$. As note above, we obtain the cycles of equivalent edges from cone twisted face-pairing $\delta$, as follows:

$$V_i e_{i1} \xrightarrow{\delta_i} e_{i1}e_{i2} \xrightarrow{\delta_i} e_{i2}e_{i3} \xrightarrow{\delta_i} \cdots \xrightarrow{\delta_i} e_{i(m-2)}e_{i(m-1)} \xrightarrow{\delta_i} e_{i(m-1)}V_{i+1} \xrightarrow{\delta_i}$$
$$V_{i+1}y_{(i+1)1} \xrightarrow{\delta_{i+1}^{-1}} x_{(i+1)1}x_{(i+1)2} \xrightarrow{\delta_i} y_{(i+1)2}\ y_{(i+1)3} \xrightarrow{\delta_{i+1}^{-1}} x_{(i+1)3}x_{(i+1)4} \xrightarrow{\delta_i} \cdots$$
$$\xrightarrow{\delta_{i+1}^{-1}} x_{(i+1)(2m-3)}x_{(i+1)(2m-2)} \xrightarrow{\delta_i} y_{(i+1)(2m-2)}\ y_{(i+1)(2m-1)} \xrightarrow{\delta_{i+1}^{-1}} x_{(i+1)(2m-1)}N$$
$$\xrightarrow{\delta_i} Sy_{i1} \xrightarrow{\delta_{i-1}^{-1}} x_{i1}x_{i2} \xrightarrow{\delta_i} y_{i2}\ y_{i3} \xrightarrow{\delta_{i-1}^{-1}} x_{i3}x_{i4} \xrightarrow{\delta_i} \cdots \xrightarrow{\delta_{i-1}^{-1}} x_{i(2m-3)}x_{i(2m-2)}$$
$$\xrightarrow{\delta_i} y_{i(2m-2)}\ y_{i(2m-1)} \xrightarrow{\delta_{i-1}^{-1}} x_{i(2m-1)}V_i \xrightarrow{\delta_i} V_i e_{i1}.$$

Let $\{x_i\}$ form a basis of a free group. We replace the $\delta_i$ atop the arrow in our edge cycle diagram by $x_i$. From our edge cycle diagram we give relations $x_i^m(x_i x_{i+1}^{-1})^m(x_i x_{i-1}^{-1})^m = 1$ for any $i = 1, 2, \ldots, n \pmod n$. Thus we deduce the following theorem.

**Theorem 2.2.** *For any $n \geq 3$, let $M_n(m)$ be a cone twisted face-pairing 3-manifold. Then the fundamental group of the 3-manifold $M_n(m)$ admits the finite presentation*

(1) $\pi_1(M_n(m)) = \langle x_1, x_2, \ldots, x_n \mid x_i^m(x_i x_{i+1}^{-1})^m(x_i x_{i-1}^{-1})^m = 1 : i = 1, \ldots, n\rangle,$

*where the subscripts are taken up to* mod $n$.

## 3. The periodic Takahashi manifolds

### 3.1. Twists

In this section we first review some well-known facts about the Dehn surgery. We focus on twist moves. This appears in Section 5.3 of [6] as Rolfsen twists and Section 9.H of [9]. Let $L = L_1 \cup L_2 \cup \cdots \cup L_k$ be a link $S^3$ framed by elements of $Q \cup \{\infty\}$. Suppose one component $L_1$ is unknotted. Then the complement $S^3 - L_1$ of regular neighborhood of $L_1$ is a solid torus $T$. Let $t$ be a right hand Dehn twist of $T$ and $n \in \mathbb{Z}$. Let $L' = L'_1 \cup L'_2 \cup \cdots \cup L'_k$ be a new link obtained from $L$ by applying $t^n$ to $L - L_1$.

We frame $L'$ as follows. If the first framing of $L_1$ is $r_1 = \frac{b_1}{a_1}$, then the new framing of $L'_1$ is $r'_1 = (n + \frac{1}{r_1})^{-1} = \frac{b_1}{a_1+nb_1}$. If the first framing of $L_2$ is $r_2$, then the new framing of $L'_2$ is $r'_2 = r_2 + n[lk(L_2, L_1)]^2$, where $lk(L_2, L_1)$ is the linking number of $L_1$ and $L_2$, and similarly for $r'_3$, etc. In [6] it is known that the manifold obtained by Dehn surgery on $L'$ is homeomorphic to the manifold obtained by Dehn surgery on $L$.

### 3.2. Periodic Takahashi manifolds

In this section we prove that most of periodic Takahashi manifolds are cone twisted face-pairing manifolds.

We first define the Takahashi manifold. Let us denoted by $\mathcal{L}_{2n}$ the $2n$-component link of the Figure 3, which is a closed chain of $2n$ unknotted components, with surgery coefficients $\frac{p_1}{q_1}, \frac{r_1}{s_1}, \ldots, \frac{p_n}{q_n}, \frac{r_n}{s_n} \in \mathbb{Q} \cup \{\infty\} = \tilde{\mathbb{Q}}$ cyclically associated to the components of $\mathcal{L}_{2n}$, respectively. Takahashi manifold is closed orientable 3-manifold obtained by Dehn surgery along the above link $\mathcal{L}_{2n}$ (see [11] for more details). We denote $M(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ by the Takahashi manifold determined by rational coefficients $\frac{p_i}{q_i}$, $\frac{r_i}{s_i}$ $(i = 1, \ldots, n)$ of $\mathcal{L}_{2n}$. And we call $\mathcal{L}_{2n}$ the Takahashi link. A Takahashi manifold is periodic when $\frac{p_i}{q_i} = \frac{p}{q}$ and $\frac{r_i}{s_i} = \frac{r}{s}$ for every $i$, $(i = 1, \ldots, n)$. Denote by $M_n(\frac{p}{q}, \frac{r}{s})$ the periodic Takahashi manifold. For details, see [8], [10], and [11]. From now on, without loss of generality, we assume that: $\gcd(p, q) = 1$, $\gcd(r, s) = 1$ and $p, r \geq 0$.

In [8], [10], and [11] they gave finite presentations of the fundamental group of the periodic Takahashi manifold $M_n(\frac{p}{q}, \frac{r}{s})$. As a consequence, $\pi_1(M_n(\frac{p}{q}, \frac{r}{s}))$ admits the following presentation with $2n$ generators and $2n$ relators:

$$\pi_1\left(M_n\left(\frac{p}{q}, \frac{r}{s}\right)\right) = \langle x_1, \ldots, x_{2n} \mid x_{2i-1}^q x_{2i}^{-r} x_{2i+1}^{-q} = 1, x_{2i}^s x_{2i+1}^p x_{2i+2}^{-s} = 1 \ ; \ i = 1, \ldots, n\rangle,$$

where the subscripts are taken up to mod $2n$. When $r = 1$, we can easily get a cyclic presentation with $n$ generators:

$$(2) \quad \pi_1\left(M_n\left(\frac{p}{q}, \frac{1}{s}\right)\right) = \langle z_1, \ldots, z_n \mid z_i^p (z_i^{-q} z_{i+1}^q)^s (z_i^{-q} z_{i-1}^q)^s = 1; i = 1, \ldots, n\rangle,$$

where the subscripts are taken up to mod $n$.

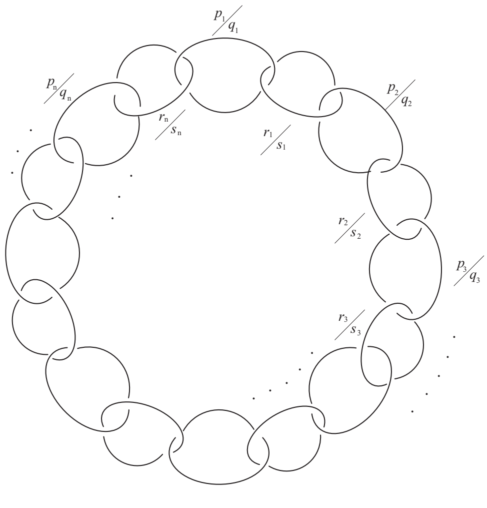


FIGURE 3. The Takahashi link $\mathcal{L}_{2n}$

**Lemma 3.1.** *For any $n \geq 3$, the fundamental group of the cone twisted face-pairing manifold $M_n(m)$ is isomorphic to the fundamental group of the periodic Takahashi manifold $M_n(-m, \frac{1}{m})$.*

*Proof.* Using the equation (2), this follows easily from the equation (1) of Theorem 2.2. That is, if $p = s = m$ and $r = -q = 1$, then we obtain the required presentation for the periodic Takahashi manifold. □

The following lemma relates the link $L_n$ and the periodic Takahashi link, where $L_n$ is obtained by the corridor construction.

**Lemma 3.2.** *For any $n \geq 3$, let $P_n$ be a cone faceted 3-ball with a multiplier function $m$. Let $D_n$ be a corridor complex link diagram for $P_n$. Let $L_n$ be a link in $S^3$ with diagram $D_n$. Define a framing of $L_n$ as follows. Every face component of $L_n$ has framing 0. The edge components of $L_n$ corresponding to $E_i$ $(i = 1, \ldots, 2n)$ has framing $\frac{1}{m}$. Then the link $L_n$ is isotopic to the periodic Takahashi link $\mathcal{L}_{2n}$.*

*Proof.* First of all, we construct $L_n$ as follows. As in Theorem 6.2.2 in [4], we can obtain the corridor complex link in Figure 4. A corridor complex for $P_n$ appears in Figure 4, drawn with thin arcs. A framed link diagram is drawn with thick arcs (circle).

A link component $L_f$ with framing 0 is presented for each triangular face in the northern cone. We call those components of $L_n$ as face components. Every edge $e$ of $P_n$ also gives a link component $L_e$ of $L_n$, called an edge component, as follows. Let $e$ be edge in the $xy$-plane of $P_n$. The edge cycle of $e$ is just $E_i$ $(1 \leq i \leq n)$. From the argument in Section 2.2, we assign $L_e$ to be an

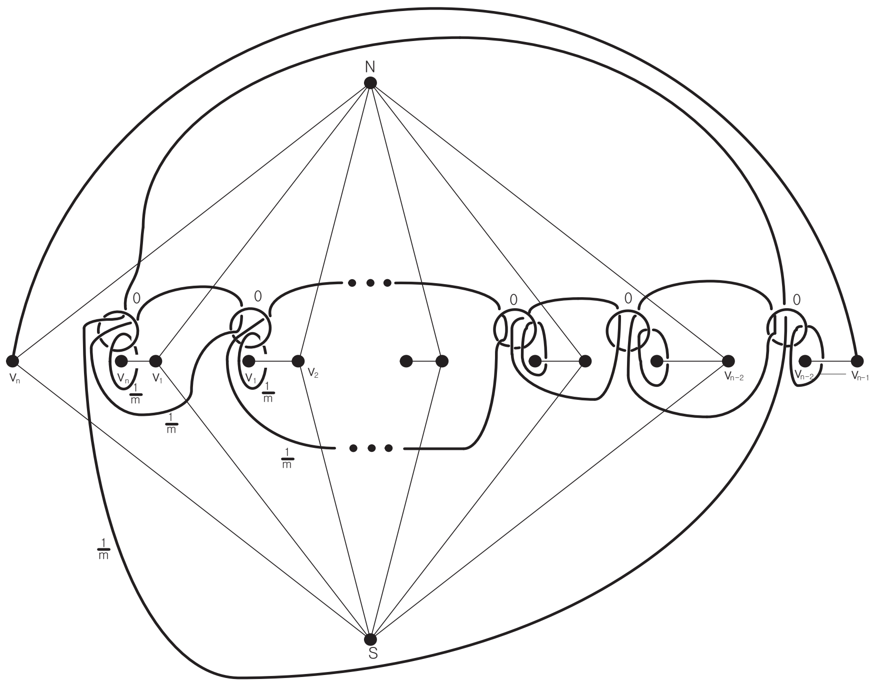


FIGURE 4. A corridor complex and framed link diagram for $P_n$

unknot with framing $\frac{1}{m_{E_i}}$, where $m_{E_i} = m$ is the multiplier of length 1 edge cycle. Now let $e$ be an edge of the northern cone not contained in the $xy$-plane. The edge cycle of $e$ is just $E_j$ $(n+1 \leq j \leq 2n)$. So we give $L_e$ to be an unknot with framing $\frac{1}{m_{E_j}}$, where $m_{E_j} = m$ is the multiplier of length 2 edge cycle. It is easy to see that the framed link in Figure 4 is isotopic to the framed link in Figure 5.

We simplify the component of the link $L_n$ in Figure 5 with framing $\frac{1}{m}$ by performing twist moves. Those components are made of the length 1 edge cycle. By twisting $-m$ times along those components with framing 0, those components with framing $\frac{1}{m}$ change to components with $\infty$ framing. The same procedure changes components with 0 framing to components with $-m$ framing. By doing so, $L_n$ is isotopic to the link in Figure 6.

The link $L_n$ in Figure 6 is the periodic Takahashi link $\mathcal{L}_{2n}$ with $2n$-components. □

**Theorem 3.3.** *For any $n \geq 3$, let $M$ be a periodic Takahashi manifold given by $M_n(\frac{p}{q}, \frac{r}{s})$. If $p = s$ and $r = -q = 1$, then $M$ is a cone twisted face-pairing manifold $M_n(p)$.*

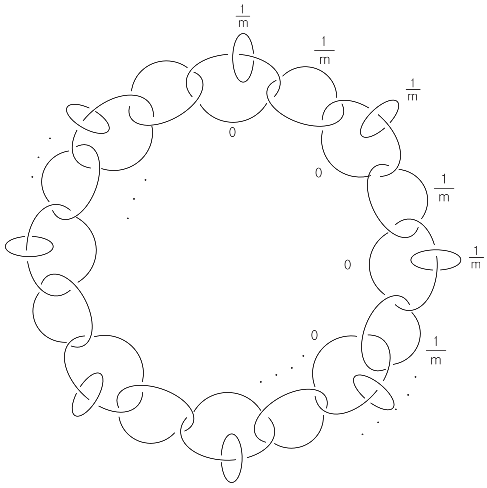


FIGURE 5. A simpler framed link $L_n$

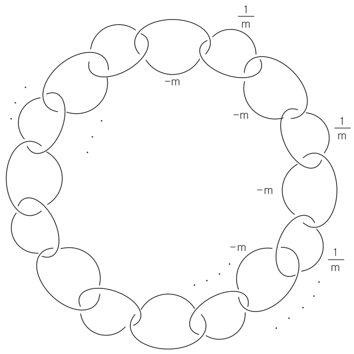


FIGURE 6. A simpler framed link $L_n$

*Proof.* Let $m$ be a positive integer such that $p = s = m$. The assumption implies that $\frac{p}{q} = -m$ and $\frac{r}{s} = \frac{1}{m}$. From the definition of the periodic Takahashi manifold, $M_n(-m, \frac{1}{m})$ is obtained by Dehn surgery on $S^3$ along the periodic Takahashi link $\mathcal{L}_{2n}$ in Figure 3. And Theorem 6.2.2 in [4] states that the associated cone twisted face-pairing manifold $M_n(m)$ is obtained by Dehn surgery on the framed link in Figure 6. Therefore Lemma 3.2 implies $M_n(m)$ is homeomorphic to $M_n(-m, \frac{1}{m})$. □

From Theorem 3.3 we can deduce the following result, which has already been proved by Theorem 3 in [8].

**Corollary 3.4.** *For any $n \geq 3$, the cone twisted face-pairing manifold $M_n(m)$ is the $n$-fold cyclic covering of the connected sum of lens space $L(m, 1)$, branched over a knot $K$, which does not depend on $n$.*

## 4. Hyperbolicity

In this section we state some results about the hyperbolicity of the cone twisted face-pairing manifold $M_n(m)$. Also we give the main theorems.

For arbitrary faceted 3-ball $P$, we need to define an ample faceted 3-ball. This appear on p. 1 of [3].

**Definition.** A faceted 3-ball $P$ is ample if it satisfies the following conditions:

(i) every two distinct faces of faceted 3-ball are either disjoint or meet in a vertex or meet in an edge.

(ii) three distinct faces of a faceted 3-ball which meet each other pairwise have exactly one vertex in common.

(iii) no face of a face 3-ball is a triangle.

Then these conditions are called the ampleness conditions.

We consider that a twisted face-pairing 3-manifold $M(\varepsilon, m)$ which is obtained by an ample faceted 3-ball $P$; such a 3-manifold $M(\varepsilon, m)$ is called an ample twisted face-pairing manifold on $P$.

**Theorem 4.1.** *Let $P_n$ be a cone faceted 3-ball, and let $m$ be a multiplier function. Let $M_n(m)$ be the associated cone twisted face-pairing manifold. Then $\pi_1(M_n(m))$ is Gromov hyperbolic.*

*Proof.* According to Theorem 6.1 of [3], it suffices to show that $M_n(m)$ is an ample twisted face-pairing manifold. We consider an ample faceted 3-ball since the ample twisted face-pairing manifold can be obtained by the ample faceted 3-ball. We can construct an ample faceted 3-ball from our cone faceted 3-ball $P_n$ as follows. We perform the dual cap subdivision of each faces of $\partial P_n$, which means that every triangle face is subdivided as in Figure 7. We denote by $P_n^*$ the subdivision 3-ball for $P_n$. Then $P_n^*$ has ampleness conditions (i), (ii) and (iii).

Actually $P_n^*$ can be obtained from $P_n$ so that it is $\varepsilon$-invariant. So a reflected face-pairing $\varepsilon$ on $P_n$ naturally determines a face-pairing $\varepsilon^*$ on $P_n^*$. This completes the proof of Theorem 4.1. □

The following is the main theorems.

**Theorem 4.2.** *For any $n \geq 4$, the cone twisted face-pairing manifold $M_n(1)$ is hyperbolic.*

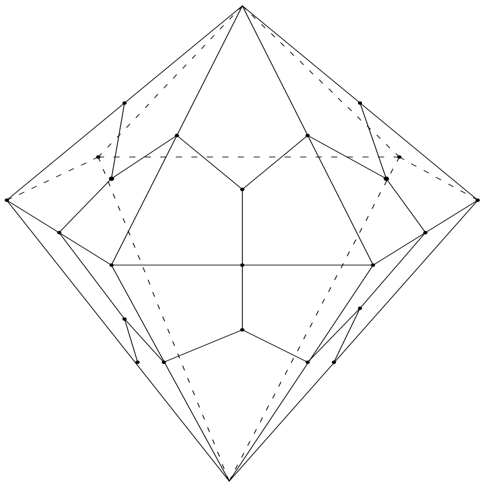

FIGURE 7. The ample faceted 3-ball $P_n^*$

*Proof.* By Theorem 3.3, we observe that, for $m = 1$, cone twisted face-pairing manifold $M_n(1)$ is homeomorphic to $M_n(-1, 1)$. On the other hand, the periodic Takahashi manifold $M_n(-1, 1)$ is the Fibonacci manifold $M_n(1, -1)$ by the symmetry of the periodic Takahashi link $\mathcal{L}_{2n}$ of Figure 3 (see p. 3 in [8] for more details). Then, for any $n \geq 4$, it was proved that these manifolds $M_n(1, -1)$ is hyperbolic in [7]. This completes the proof. □

**Proposition 4.3.** *For any $n \geq 3$, let $P_n$ be a cone faceted 3-ball with a multiplier function $m$. Let $D_n$ be a corrider complex link diagram for $P_n$. If $L_n$ is a link in $S^3$ with diagram $D_n$ (see Figure 5). Then the link $L_n$ is hyperbolic.*

*Proof.* Lemma 3.2 implies that the corridor complex link $L_n$ (see Figure 5) is isotopic to the Takahashi link $\mathcal{L}_{2n}$ of Figure 3. We simplify the Takahashi link $\mathcal{L}_{2n}$ in Figure 3 using twist moves. Twisting $-m$ times along the each components with framing $\frac{1}{m}$, and deleting resulting components with framing $\infty$ yields the link of $n$-components of Figure 8. The same procedure changes components with $-m$ framing to components with $-3m$ framing. After these simplifications, one can deduce that the simplified Takahashi link $\mathcal{L}_{2n}$ of Figure 8 is hyperbolic by using the result in [1] (see p. 222) and [12] (see p. 144). So the link $L_n$ is hyperbolic. □

As a result of the above proposition, we can deduce the hyperbolicity for the cone twisted face-pairing manifold $M_n(m)$. By using the Thurston-Jorgensen's theorem for hyperbolic surgery which states that all but finitely many Dehn

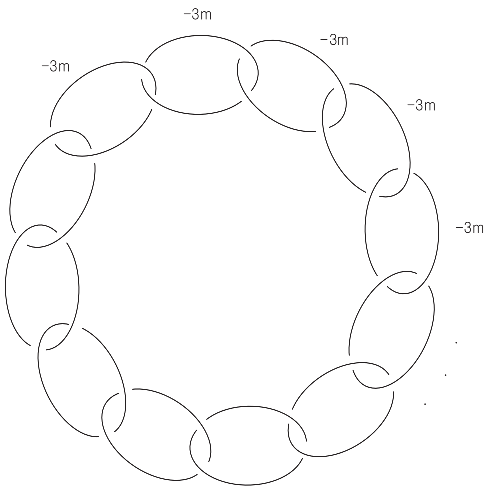


FIGURE 8. The simplified Takahashi link $\mathcal{L}_{2n}$

surgery on hyperbolic link complements yield hyperbolic structure (see [12]), we obtain the following theorem.

**Corollary 4.4.** *For any $n \geq 3$, and for all but finite number of multiplier $m = \{m_i \mid i = 1, \ldots, n\}$, the cone twisted face-pairing manifolds $M_n(m)$ are hyperbolic.*

## References

[1] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.
[2] J. W. Cannon, W. J. Floyd, and W. R. Parry, *Twisted face-pairing 3-manifolds*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2369–2397.
[3] ———, *Ample twisted face-pairing 3-manifolds*, preprint.
[4] ———, *Heegaard diagrams and surgery descriptions for twisted face-pairing 3-manifolds*, Algebr. Geom. Topol. **3** (2003), 235–285.
[5] ———, *A survey of twisted face-pairing 3-manifolds*, preprint.
[6] R. E. Gompf and A. I. Stipsicz, 4-*manifolds and Kirby Calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999.
[7] H. Helling, A. C. Kim, and J. Mennicke, *On Fibonacci groups*, preprint.
[8] M. Mulazzani, *On periodic Takahashi manifolds*, Tsukuba J. Math. **25** (2001), no. 2, 229–237.
[9] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976.
[10] B. Ruini and F. Spaggiari, *On the structure of Takahashi manifolds*, Tsukuba J. Math. **22** (1998), no. 3, 723–739.
[11] M. Takahashi, *On the presentations of the fundamental groups of* 3-*manifolds*, Tsukuba J. Math. **13** (1989), no. 1, 175–189.
[12] W. P. Thurston, *The geometry and topology of 3-manifold*, Lect. Notes, Princeton University, N. J., 1980.

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