Bull. Korean Math. Soc. **47** (2010), No. 6, pp. 1285–1296
DOI 10.4134/BKMS.2010.47.6.1285

CHARACTERIZATIONS OF SOME ISOMETRIC IMMERSIONS IN TERMS OF CERTAIN FRENET CURVES

Jin Ho Choi, Young Ho Kim*, and Hiromasa Tanabe

Abstract. We give criterions for a submanifold to be an extrinsic sphere and to be a totally geodesic submanifold by observing some Frenet curves of order 2 on the submanifold. We also characterize constant isotropic immersions into arbitrary Riemannian manifolds in terms of Frenet curves of proper order 2 on submanifolds. As an application we obtain a characterization of Veronese embeddings of complex projective spaces into complex projective spaces.

1. Introduction

Let $f : M^n \to \widetilde{M}^{n+p}$ be an isometric immersion of a Riemannian manifold M^n into an ambient Riemannian manifold $\widetilde{M}^{n+p}$. By examining the extrinsic shape of curves on the submanifold M^n, we can know the properties of the immersion f in some cases. The aim of this paper is to give characterizations of some isometric immersions by observing the extrinsic shape of Frenet curves of order 2 on the submanifold.

A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M^n parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot{\gamma}, V_2, \ldots, V_d\}$ along γ and positive smooth functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ satisfying the following system of ordinary differential equations

$$\nabla_{\dot{\gamma}} V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \ldots, d, \tag{1.1}$$

where $V_0 \equiv V_{d+1} \equiv 0$ and $\nabla_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M^n. The equation (1.1) is called the *Frenet formula* for the Frenet curve γ. The function $\kappa_j(s)$ $(j = 1, \ldots, d-1)$ and the orthonormal frame fields $\{V_1, \ldots, V_d\}$ are called the *j-th curvature* and the *Frenet frame* of γ, respectively. When $d = 1$, it is nothing but a geodesic. When $d = 2$ and its curvature $\kappa_1(s)$ is a positive constant k, namely when the

Received May 7, 2009.
2000 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C40.
Key words and phrases. Frenet curves of proper order 2, extrinsic spheres, totally geodesic submanifolds, constant isotropic immersions, Veronese embeddings.
* Supported by KOSEF R01-2007-000-20014-0(2007).

curve γ satisfies $\nabla_{\dot\gamma}V_1(s) = kV_2(s)$, $\nabla_{\dot\gamma}V_2(s) = -kV_1(s)$ and $V_1(s) = \dot\gamma(s)$, it is called a *circle* of curvature k. We regard a geodesic as a circle of null curvature for convenience. A Frenet curve is called a Frenet curve *of order d* if it is a Frenet curve of proper order $r(\leq d)$.

Motivated by the well-known result in K. Nomizu and K. Yano [5] which provides a characterization of *extrinsic spheres* (that is totally umbilic submanifolds with parallel mean curvature vector) in terms of circles, the third author showed the following ([8]): A submanifold M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$ if and only if there exists a positive smooth function κ such that every Frenet curve γ of proper order 2 with curvature κ on M^n is mapped to a Frenet curve of proper order 2 in the ambient space $\widetilde{M}^{n+p}$. Moreover, if the function κ is nonconstant, M^n is a totally geodesic submanifold of $\widetilde{M}^{n+p}$. However, in these characterizations we need a very large quantity of information in the meaning that "every" Frenet curve γ of proper order 2 with curvature κ on M^n must satisfy the condition. Therefore, in the first half of this paper, we give practical criterions for a submanifold to be an extrinsic sphere and to be a totally geodesic submanifold.

On the other hand, S. Maeda [3] characterized constant isotropic immersions by circles on submanifolds (for the definition of constant isotropic immersions, see Section 3). Namely, he proved that an isometric immersion is constant isotropic if and only if there exists a positive constant k satisfying that for each circle γ of curvature k on the submanifold M^n the curve $f \circ \gamma$ in the ambient space $\widetilde{M}^{n+p}$ has constant first curvature. The purpose of the latter half of this paper is to give a characterization of constant isotropic immersions into arbitrary Riemannian manifolds by using Frenet curves of proper order 2 on submanifolds (Theorem 1). As an application of Theorem 1 we obtain a characterization of Veronese embeddings of complex projective spaces into complex projective spaces by Frenet curves of proper order 2 (Theorem 3).

2. Characterizations of extrinsic spheres and totally geodesic submanifolds

We first review a few fundamental notions in submanifold theory. Let M^n, $\widetilde{M}^{n+p}$ be Riemannian manifolds and $f : M^n \to \widetilde{M}^{n+p}$ an isometric immersion. Throughout this paper we will identify a vector X of M^n with a vector $f_*(X)$ of $\widetilde{M}^{n+p}$. The Riemannian metrics on M^n, $\widetilde{M}^{n+p}$ are denoted by the same notation $\langle\ ,\ \rangle$. We denote by ∇ and $\widetilde\nabla$ the covariant differentiations of M^n and $\widetilde{M}^{n+p}$, respectively. Then the formulae of Gauss and Weingarten are

$$\widetilde\nabla_X Y = \nabla_X Y + \sigma(X,Y), \quad \widetilde\nabla_X \xi = -A_\xi X + D_X \xi,$$

where σ, A_ξ and D denote the second fundamental form of f, the shape operator in the direction of a normal vector ξ and the covariant differentiation in the normal bundle, respectively. We define the covariant differentiation $\bar\nabla$

of the second fundamental form σ with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows:

$$(\bar{\nabla}_X\sigma)(Y,Z) = D_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

If $\bar{\nabla}\sigma = 0$, the isometric immersion f is called *parallel*.

Now we shall prove the following proposition which is an improvement of the results in [2], [5] and [8].

Proposition 1. *Let M^n $(n \geq 2)$ be an n-dimensional connected Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. We denote by ∇ the covariant differentiation of M^n. Then the submanifold M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$ if and only if for some positive smooth function $\kappa = \kappa(s)$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ at each point p of M^n satisfying the following three conditions*:

(1) *For every Frenet curve $\gamma_{ij} = \gamma_{ij}(s)$ of proper order 2 with curvature κ on M^n with $\gamma_{ij}(0) = p$, $\dot{\gamma}_{ij}(0) = v_i$ and $(\nabla_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij})(0) = \kappa(0)v_j$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \gamma_{ij}$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^{n+p}$.*

(2) *For every Frenet curve $\tau_{ij} = \tau_{ij}(s)$ of proper order 2 with curvature κ on M^n with $\tau_{ij}(0) = p$, $\dot{\tau}_{ij}(0) = v_i$ and $(\nabla_{\dot{\tau}_{ij}}\dot{\tau}_{ij})(0) = -\kappa(0)v_j$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \tau_{ij}$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$.*

(3) *For every Frenet curve $\eta_{ij} = \eta_{ij}(s)$ of proper order 2 with curvature κ on M^n with $\eta_{ij}(0) = p$, $\dot{\eta}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ and $(\nabla_{\dot{\eta}_{ij}}\dot{\eta}_{ij})(0) = \kappa(0)(v_i - v_j)/\sqrt{2}$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \eta_{ij}$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$.*

Proof. The "only if" part is obvious from the result in K. Nomizu and K. Yano [5]. So we shall prove the "if" part. Let $\gamma_{ij} = \gamma_{ij}(s)$ be a Frenet curve of proper order 2 on M^n satisfying condition (1). Then there exists a field of unit vectors $V_{ij} = V_{ij}(s)$ along γ_{ij} which satisfies

$$\nabla_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = \kappa(s)V_{ij}(s), \quad \nabla_{\dot{\gamma}_{ij}}V_{ij}(s) = -\kappa(s)\dot{\gamma}_{ij}$$

with the initial condition $V_{ij}(0) = v_j$. Denote by $\widetilde{\nabla}$ the covariant differentiation of $\widetilde{M}^{n+p}$. Since the curve $f \circ \gamma_{ij}$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$ by assumption, there exist a function $\tilde{\kappa}_{ij} = \tilde{\kappa}_{ij}(s)$ and a field of unit vectors $\widetilde{V}_{ij} = \widetilde{V}_{ij}(s)$ along $f \circ \gamma_{ij}$ in $\widetilde{M}^{n+p}$ satisfying that

$$\widetilde{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = \tilde{\kappa}_{ij}(s)\widetilde{V}_{ij}(s), \quad \widetilde{\nabla}_{\dot{\gamma}_{ij}}\widetilde{V}_{ij}(s) = -\tilde{\kappa}_{ij}(s)\dot{\gamma}_{ij}.$$

Then by the equation of Gauss we have

$$\tilde{\kappa}_{ij}\widetilde{V}_{ij} = \kappa V_{ij} + \sigma(\dot{\gamma}_{ij}, \dot{\gamma}_{ij}), \tag{2.1}$$

hence

$$\tilde{\kappa}_{ij} = \tilde{\kappa}_{ij}(s) = \sqrt{\kappa(s)^2 + \|\sigma(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})\|^2}. \tag{2.2}$$

We here note that $\tilde{\kappa}_{ij}(s) > 0$ for any s, because $\kappa(s) > 0$ for any s.

Differentiating the left-hand side of (2.1), we see

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma_{ij}}(\tilde\kappa_{ij}\widetilde{V}_{ij}) &= \dot{\tilde\kappa}_{ij}\widetilde{V}_{ij} + \tilde\kappa_{ij}\widetilde{\nabla}_{\dot\gamma_{ij}}\widetilde{V}_{ij}\\ &= \frac{\dot{\tilde\kappa}_{ij}}{\tilde\kappa_{ij}}\{\kappa V + \sigma(\dot\gamma_{ij},\dot\gamma_{ij})\} - (\tilde\kappa_{ij})^2\dot\gamma_{ij}.\end{aligned} \tag{2.3}$$

On the other hand, differentiating the right-hand side of (2.1), by using the equations of Gauss and Weingarten, we obtain

$$\begin{aligned}&\widetilde{\nabla}_{\dot\gamma_{ij}}\{\kappa V_{ij} + \sigma(\dot\gamma_{ij},\dot\gamma_{ij})\}\\ &= \dot\kappa V_{ij} + \kappa\widetilde{\nabla}_{\dot\gamma_{ij}}V_{ij} - A_{\sigma(\dot\gamma_{ij},\dot\gamma_{ij})}\dot\gamma_{ij} + D_{\dot\gamma_{ij}}(\sigma(\dot\gamma_{ij},\dot\gamma_{ij}))\\ &= \dot\kappa V_{ij} + \kappa\{\nabla_{\dot\gamma_{ij}}V_{ij} + \sigma(\dot\gamma_{ij}, V_{ij})\}\\ &\quad - A_{\sigma(\dot\gamma_{ij},\dot\gamma_{ij})}\dot\gamma_{ij} + (\bar\nabla_{\dot\gamma_{ij}}\sigma)(\dot\gamma_{ij},\dot\gamma_{ij}) + 2\sigma(\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij},\dot\gamma_{ij})\\ &= \dot\kappa V_{ij} - \kappa^2\dot\gamma_{ij} + 3\kappa\sigma(\dot\gamma_{ij},V_{ij}) - A_{\sigma(\dot\gamma_{ij},\dot\gamma_{ij})}\dot\gamma_{ij} + (\bar\nabla_{\dot\gamma_{ij}}\sigma)(\dot\gamma_{ij},\dot\gamma_{ij}).\end{aligned} \tag{2.4}$$

We compare the tangential components and the normal components for the submanifold M^n in (2.3) and (2.4), respectively. Then we get

$$\dot{\tilde\kappa}_{ij}\kappa V_{ij} - (\tilde\kappa_{ij})^3\dot\gamma_{ij} = \tilde\kappa_{ij}\{\dot\kappa V_{ij} - \kappa^2\dot\gamma_{ij} - A_{\sigma(\dot\gamma_{ij},\dot\gamma_{ij})}\dot\gamma_{ij}\}, \tag{2.5}$$

$$\dot{\tilde\kappa}_{ij}\sigma(\dot\gamma_{ij},\dot\gamma_{ij}) = \tilde\kappa_{ij}\{3\kappa\sigma(\dot\gamma_{ij},V_{ij}) + (\bar\nabla_{\dot\gamma_{ij}}\sigma)(\dot\gamma_{ij},\dot\gamma_{ij})\}. \tag{2.6}$$

The equation (2.6) gives

$$\dot{\tilde\kappa}_{ij}\tilde\kappa_{ij}\sigma(\dot\gamma_{ij},\dot\gamma_{ij}) = (\tilde\kappa_{ij})^2\{3\kappa\sigma(\dot\gamma_{ij},V_{ij}) + (\bar\nabla_{\dot\gamma_{ij}}\sigma)(\dot\gamma_{ij},\dot\gamma_{ij})\}. \tag{2.7}$$

On the other hand, we see from (2.2)

$$\begin{aligned}\dot{\tilde\kappa}_{ij}\tilde\kappa_{ij} &= \frac{1}{2}\frac{d}{ds}(\tilde\kappa_{ij})^2\\ &= \dot\kappa\kappa + \frac{1}{2}\frac{d}{ds}\langle\sigma(\dot\gamma_{ij},\dot\gamma_{ij}),\sigma(\dot\gamma_{ij},\dot\gamma_{ij})\rangle\\ &= \dot\kappa\kappa + \langle D_{\dot\gamma_{ij}}(\sigma(\dot\gamma_{ij},\dot\gamma_{ij})),\sigma(\dot\gamma_{ij},\dot\gamma_{ij})\rangle\\ &= \dot\kappa\kappa + \langle(\bar\nabla_{\dot\gamma_{ij}}\sigma)(\dot\gamma_{ij},\dot\gamma_{ij}),\sigma(\dot\gamma_{ij},\dot\gamma_{ij})\rangle + 2\kappa\langle\sigma(V_{ij},\dot\gamma_{ij}),\sigma(\dot\gamma_{ij},\dot\gamma_{ij})\rangle.\end{aligned} \tag{2.8}$$

Substituting (2.2) and (2.8) into (2.7), at $s = 0$ we obtain

$$\begin{aligned}&\Big\{\kappa(0)\dot\kappa(0) + \langle(\bar\nabla_{v_i}\sigma)(v_i,v_i),\sigma(v_i,v_i)\rangle + 2\kappa(0)\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle\Big\}\sigma(v_i,v_i)\\ &= \Big\{\kappa(0)^2 + \|\sigma(v_i,v_i)\|^2\Big\}\Big\{3\kappa(0)\sigma(v_i,v_j) + (\bar\nabla_{v_i}\sigma)(v_i,v_i)\Big\}.\end{aligned} \tag{2.9}$$

Here we consider a Frenet curve $\tau_{ij} = \tau_{ij}(s)$ satisfying condition (2). Applying the above discussion to the curve τ_{ij}, we can see that

$$\begin{aligned}&\Big\{\kappa(0)\dot\kappa(0) + \langle(\bar\nabla_{v_i}\sigma)(v_i,v_i),\sigma(v_i,v_i)\rangle - 2\kappa(0)\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle\Big\}\sigma(v_i,v_i)\\ &= \Big\{\kappa(0)^2 + \|\sigma(v_i,v_i)\|^2\Big\}\Big\{-3\kappa(0)\sigma(v_i,v_j) + (\bar\nabla_{v_i}\sigma)(v_i,v_i)\Big\}.\end{aligned} \tag{2.9'}$$

Hence, from (2.9) and (2.9') we have

$$2\kappa(0)\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle\sigma(v_i,v_i) = 3\kappa(0)\Big\{\kappa(0)^2+\|\sigma(v_i,v_i)\|^2\Big\}\sigma(v_i,v_j)$$

so that

$$2\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle\sigma(v_i,v_i) = 3\Big\{\kappa(0)^2+\|\sigma(v_i,v_i)\|^2\Big\}\sigma(v_i,v_j). \tag{2.10}$$

Taking the inner product of both sides of the equation (2.10) with $\sigma(v_i,v_i)$, we get

$$2\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle||\sigma(v_i,v_i)||^2 = 3\Big\{\kappa(0)^2+\|\sigma(v_i,v_i)\|^2\Big\}\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle$$

hence

$$\Big\{3\kappa(0)^2+\|\sigma(v_i,v_i)\|^2\Big\}\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle = 0.$$

So we have

$$\langle\sigma(v_i,v_i),\sigma(v_i,v_j)\rangle = 0,$$

because $3\kappa(0)^2+\|\sigma(v_i,v_i)\|^2>0$. This, combined with (2.10), shows that

$$\sigma(v_i,v_j)=0 \quad (1\le i\ne j\le n). \tag{2.11}$$

Exchanging i for j in condition (3) in Proposition 1, we have similarly

$$\sigma\left(\frac{v_i+v_j}{\sqrt{2}},\frac{v_i-v_j}{\sqrt{2}}\right)=0$$

so that

$$\sigma(v_i,v_i)=\sigma(v_j,v_j) \quad (1\le i\ne j\le n). \tag{2.12}$$

Since p is an arbitrary point of M^n, it follows from (2.11) and (2.12) that our immersion $f: M^n\to\widetilde{M}^{n+p}$ is totally umbilic.

Then by taking the inner product of both sides of the equation (2.5) with V_{ij} we see that

$$\dot{\tilde{\kappa}}_{ij}(s)\kappa(s)=\tilde{\kappa}_{ij}(s)\dot{\kappa}(s) \tag{2.13}$$

for each s. Moreover the equation (2.6) reduces to

$$\dot{\tilde{\kappa}}_{ij}(s)\mathfrak{h}_{\gamma_{ij}(s)}=\tilde{\kappa}_{ij}(s)(D_{\dot{\gamma}_{ij}(s)}\mathfrak{h})_{\gamma_{ij}(s)} \tag{2.14}$$

for each s, where $\mathfrak{h}$ denotes the mean curvature vector field of our immersion. Therefore, the equations (2.13) and (2.14) imply

$$(D_{\dot{\gamma}_{ij}(s)}\mathfrak{h})_{\gamma_{ij}(s)}=\frac{\dot{\tilde{\kappa}}_{ij}(s)}{\tilde{\kappa}_{ij}(s)}\mathfrak{h}_{\gamma_{ij}(s)}=\frac{\dot{\kappa}(s)}{\kappa(s)}\mathfrak{h}_{\gamma_{ij}(s)}. \tag{2.15}$$

In particular, at $s=0$ we get

$$(D_{v_i}\mathfrak{h})_p=\frac{\dot{\kappa}(0)}{\kappa(0)}\mathfrak{h}_p.$$

Similarly, we apply the same process to condition (3). Then we get

$$(D_{(v_i+v_j)/\sqrt{2}}\,\mathfrak{h})_p = \frac{\dot\kappa(0)}{\kappa(0)}\mathfrak{h}_p$$

so that

$$(D_{(v_i+v_j)}\,\mathfrak{h})_p = \frac{\sqrt{2}\dot\kappa(0)}{\kappa(0)}\mathfrak{h}_p.$$

On the other hand, we have

$$\big((D_{v_i}+D_{v_j})\mathfrak{h}\big)_p = \frac{2\dot\kappa(0)}{\kappa(0)}\mathfrak{h}_p.$$

Thus we see

$$(D_{v_i}\mathfrak{h})_p = 0.$$

Since p is an arbitrary point, we have shown that the mean curvature vector field $\mathfrak{h}$ of f is parallel. Consequently, M^n is an extrinsic sphere of $\widetilde{M}^{n+p}$. □

As an immediate consequence of this proposition we obtain the following:

Corollary 1. *Let M^n $(n \geq 2)$ be an n-dimensional connected Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. We denote by ∇ the covariant differentiation of M^n. Then the submanifold M^n is totally geodesic in the ambient space $\widetilde{M}^{n+p}$ if and only if for some positive smooth non-constant function $\kappa = \kappa(s)$ there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ at each point p of M^n satisfying the following three conditions*:

(1) *For every Frenet curve $\gamma_{ij} = \gamma_{ij}(s)$ of proper order 2 with curvature κ in M^n with $\gamma_{ij}(0) = p$, $\dot\gamma_{ij}(0) = v_i$ and $(\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij})(0) = \kappa(0)v_j$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \gamma_{ij}$ is a Frenet curve of order 2 in the ambient space $\widetilde{M}^{n+p}$.*

(2) *For every Frenet curve $\tau_{ij} = \tau_{ij}(s)$ of proper order 2 with curvature κ in M^n with $\tau_{ij}(0) = p$, $\dot\tau_{ij}(0) = v_i$ and $(\nabla_{\dot\tau_{ij}}\dot\tau_{ij})(0) = -\kappa(0)v_j$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \tau_{ij}$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$.*

(3) *For every Frenet curve $\eta_{ij} = \eta_{ij}(s)$ of proper order 2 with curvature κ in M^n with $\eta_{ij}(0) = p$, $\dot\eta_{ij}(0) = (v_i+v_j)/\sqrt{2}$ and $(\nabla_{\dot\eta_{ij}}\dot\eta_{ij})(0) = \kappa(0)(v_i - v_j)/\sqrt{2}$ $(1 \leq i \neq j \leq n)$ the curve $f \circ \eta_{ij}$ is a Frenet curve of order 2 in $\widetilde{M}^{n+p}$.*

Proof. By assumption, as the curvature function κ is not constant, there exists some s_0 with $\dot\kappa(s_0) \neq 0$. This, together with (2.15), implies that $\mathfrak{h}$ vanishes at the point $\gamma_{ij}(s_0)(\in M)$. Hence, Proposition 1 gives our corollary. □

If M^n is a hypersurface of a Riemannian manifold $\widetilde{M}^{n+1}$, we have:

Proposition 2. *Let M^n be an n-dimensional Riemannian manifold isometrically immersed in an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}^{n+1}$. Then, the following are equivalent*:

(1) *M^n is totally umbilic in $\widetilde{M}^{n+1}$.*

(2) *For each point p in M^n, there exists an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for the tangent space T_pM^n of M^n at p satisfying that the geodesics γ_i, γ_{ij}^+, γ_{ij}^- $(1 \leq i < j \leq n)$ on M^n with initial conditions $\gamma_i(0) = \gamma_{ij}^+(0) = \gamma_{ij}^-(0) = p$ and $\dot\gamma_i = v_i$, $\dot\gamma_{ij}^+(0) = (v_i + v_j)/\sqrt{2}$, $\dot\gamma_{ij}^- = (v_i - v_j)/\sqrt{2}$ have the same normal curvature at the point p in the ambient space $\widetilde{M}^{n+1}$, that is, $\langle \widetilde\nabla_{\dot\gamma_i}\dot\gamma_i(0), N_p\rangle = \langle \widetilde\nabla_{\dot\gamma_{ij}^+}\dot\gamma_{ij}^+(0), N_p\rangle = \langle \widetilde\nabla_{\dot\gamma_{ij}^-}\dot\gamma_{ij}^-(0), N_p\rangle$, where N is a local normal unit vector field on M^n in the ambient space $\widetilde{M}^{n+1}$.*

Proof. (2) $\Rightarrow$ (1). It follows from the Gauss formula

$$\widetilde\nabla_X Y = \nabla_X Y + \sigma(X, Y) = \nabla_X Y + \langle AX, Y\rangle N$$

for any tangent vector fields X and Y, and the condition (2) that

$$\langle A\dot\gamma_i(0), \dot\gamma_i(0)\rangle = \langle A\dot\gamma_{ij}^+(0), \dot\gamma_{ij}^+(0)\rangle = \langle A\dot\gamma_{ij}^-(0), \dot\gamma_{ij}^-(0)\rangle,$$

so that

$$\langle Av_i, v_i\rangle = \frac{1}{2}\langle A(v_i + v_j), v_i + v_j\rangle = \frac{1}{2}\langle A(v_i - v_j), v_i - v_j\rangle, \tag{2.16}$$

where $1 \leq i < j \leq n$. Here, A is simply denoted by the shape operator A_N in the direction of the unit normal vector field N. By the second equation of (2.16) we have

$$\langle Av_i, v_j\rangle = 0 \tag{2.17}$$

for $1 \leq \forall i < \forall j \leq n$. Hence, from (2.17) and the first equation in (2.16) we find that

$$\langle Av_i, v_i\rangle = \langle Av_j, v_j\rangle \quad \text{for} \quad 1 \leq \forall i < \forall j \leq n. \tag{2.18}$$

In view of equations (2.17) and (2.18) we see that the point p is an umbilic point. Since the point p is arbitrary, the hypersurface M^n is totally umbilic in $\widetilde{M}^{n+1}$.

(1) $\Rightarrow$ (2). By hypothesis, there exists a function, say λ on M^n satisfying $AX = \lambda X$ for each tangent vector field X on M^n. Therefore, for each orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for T_pM^n at every point $p \in M^n$ we see that equations in (2.16) hold. Then, we derive the condition (2). □

3. Characterization of constant isotropic immersions

We recall the notion of *isotropic* immersions introduced by B. O'Neill [7]. An isometric immersion f of a Riemannian manifold M^n into an ambient Riemannian manifold $\widetilde{M}^{n+p}$ is said to be isotropic at $x \in M^n$ if $\|\sigma_x(v, v)\|/\|v\|^2$ does not depend on the choice of $v\,(\neq 0) \in T_xM^n$. If the immersion is isotropic at every point $x \in M^n$, then there exists a function λ on M^n defined by $x \mapsto \|\sigma_x(v, v)\|/\|v\|^2$ and the immersion is said to be λ-isotropic or, simply, isotropic. When the function λ is constant on M^n, we say that M^n is *constant* (λ-)*isotropic* in the ambient space $\widetilde{M}^{n+p}$. Note that a totally umbilic immersion is isotropic, but not *vice versa*.

The following is well-known([7]):

Lemma 1. *Let f be an isometric immersion of M^n into $\widetilde{M}^{n+p}$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ satisfies $\langle \sigma(u,u), \sigma(u,v)\rangle = 0$ for an arbitrary orthogonal pair $v, u \in T_xM$.*

The following gives a geometric meaning of constant isotropic immersions in terms of Frenet curves of proper order 2 on submanifolds which is an improvement of S. Maeda's result [3].

Theorem 1. *Let M^n ($n \geq 2$) be an n-dimensional connected Riemannian submanifold of an $(n+p)$-dimensional Riemannian manifold $\widetilde{M}^{n+p}$ through an isometric immersion f. Then the following are equivalent*:

(1) *M^n is constant (λ-)isotropic in $\widetilde{M}^{n+p}$.*

(2) *There exists a positive smooth function $\kappa = \kappa(s)$ satisfying that for each Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ on the submanifold M^n the curve $f \circ \gamma$ in $\widetilde{M}^{n+p}$ has constant first curvature $\tilde{\kappa}_1$ along this curve.*

Proof. (1) $\Rightarrow$ (2): It is obvious from the result in [3].

(2) $\Rightarrow$ (1): We denote by ∇ and $\widetilde{\nabla}$ the covariant differentiations of M^n and $\widetilde{M}^{n+p}$, respectively. Suppose that there exists a function $\kappa = \kappa(s)$ which satisfies the condition (2). Let x be an arbitrary point of M^n and u, v an arbitrary orthonormal pair of vectors in T_xM^n. Let $\gamma = \gamma(s)$ be a Frenet curve of proper order 2 with curvature κ on the submanifold M^n satisfying

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa(s)V(s), \quad \nabla_{\dot{\gamma}}V(s) = -\kappa(s)\dot{\gamma}$$

and the initial condition

$$\gamma(0) = x, \quad \dot{\gamma}(0) = u \quad \text{and} \quad V(0) = v.$$

Then by the formula of Gauss we have

$$\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \kappa V + \sigma(\dot{\gamma}, \dot{\gamma}),$$

so that

$$\|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\|^2 = \kappa^2 + \|\sigma(\dot{\gamma}, \dot{\gamma})\|^2.$$

As the first curvature $\tilde{\kappa}_1 = \|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\|$ of the curve $f \circ \gamma$ is constant by assumption, we see

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{ds}\|\widetilde{\nabla}_{\dot{\gamma}}\dot{\gamma}\|^2 \\ &= \dot{\kappa}\kappa + \langle D_{\dot{\gamma}}(\sigma(\dot{\gamma}, \dot{\gamma})), \sigma(\dot{\gamma}, \dot{\gamma})\rangle \\ &= \dot{\kappa}\kappa + \langle (\bar{\nabla}_{\dot{\gamma}}\sigma)(\dot{\gamma}, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle + 2\kappa\langle \sigma(V, \dot{\gamma}), \sigma(\dot{\gamma}, \dot{\gamma})\rangle. \end{aligned}$$

Evaluating this equation at $s = 0$, we obtain

$$\dot{\kappa}(0)\kappa(0) + \langle (\bar{\nabla}_u\sigma)(u, u), \sigma(u, u)\rangle + 2\kappa(0)\langle \sigma(v, u), \sigma(u, u)\rangle = 0. \tag{3.1}$$

For another Frenet curve $\tau = \tau(s)$ of proper order 2 with the same curvature κ on M^n satisfying the equations $\nabla_{\dot\tau}\dot\tau = \kappa(s)Z(s)$ and $\nabla_{\dot\tau}Z(s) = -\kappa(s)\dot\tau$ with initial condition $\tau(0) = x$, $\dot\tau(0) = u$ and $Z(0) = -v$, we can see

$$\dot\kappa(0)\kappa(0) + \langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle - 2\kappa(0)\langle\sigma(v,u), \sigma(u,u)\rangle = 0. \tag{3.1'}$$

Noting $\kappa(0) > 0$, from (3.1) and (3.1') we have

$$\langle\sigma(v,u), \sigma(u,u)\rangle = 0$$

for an arbitrary orthonormal pair of vectors u, v at each point x of M^n. Thus, by virtue of Lemma 1, the immersion f is (λ-)isotropic.

Now, we shall show that the function $\lambda : M^n \to \mathbb{R}$ is constant. It follows from (3.1) and (3.1') that

$$\langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle = -\dot\kappa(0)\kappa(0).$$

This equation shows that $\langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle$ is independent of the choice of a unit vector $u \in T_xM^n$. Changing u into $-u$ we find

$$\langle(\bar\nabla_u\sigma)(u,u), \sigma(u,u)\rangle = 0$$

for an arbitrary unit vector u at each point x of M^n. Then, for every geodesic $\rho = \rho(s)$ on the submanifold M^n we see $\lambda = \lambda(s)$ is constant along ρ. Hence, we conclude the function λ is constant on M^n and Theorem is proved. □

As an immediate consequence of Theorem 1 we obtain the following:

Theorem 2. *Let M^n $(n \geq 2)$ be an n-dimensional connected hypersurface of an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}^{n+1}$ through an isometric immersion f. Then the following are equivalent*:

(1) *M^n is an extrinsic sphere in $\widetilde{M}^{n+1}$.*

(2) *There exists a positive smooth function $\kappa = \kappa(s)$ satisfying that for each Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ on the hypersurface M^n the curve $f \circ \gamma$ in $\widetilde{M}^{n+1}$ has constant first curvature $\tilde\kappa_1$ along this curve.*

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1): Theorem 1 implies that the hypersurface M^n is constant isotropic in $\widetilde{M}^{n+1}$. Since the notion of isotropic is equivalent to that of totally umbilic in case that the codimension $p = 1$, the hypersurface M^n is totally umbilic with constant mean curvature $\|\mathfrak{h}\|$ in $\widetilde{M}^{n+1}$. Here, $\|\mathfrak{h}\|$ denotes the length of the mean curvature vector $\mathfrak{h}$ of M^n in $\widetilde{M}^{n+1}$. Moreover, in case that $p = 1$ it is well-known that "$\|\mathfrak{h}\|$ =constant" is equivalent to "$D_X\mathfrak{h} = 0$ for $\forall X \in TM^n$". Consequently, the hypersurface M^n is an extrinsic sphere in $\widetilde{M}^{n+1}$. □

Remark 1. (1) In the statements of Theorems 1 and 2, if we set $\kappa \equiv 0$, they are no longer true. For example, we take a Clifford hypersurface $M_{r,n-r} := S^r(c_1) \times S^{n-r}(c_2)$ (through the natural embedding f) in the ambient sphere $S^n(c)$ of constant sectional curvature c with $(1/c_1) + (1/c_2) = (1/c)$ and $1 \leq r \leq n-1$. It is known that this hypersurface $M_{r,n-r}$ has parallel shape operator (namely, parallel second fundamental form). Hence we can see that for every geodesic $\gamma = \gamma(s)$ on $M_{r,n-r}$ the curve $f \circ \gamma$ has constant first curvature along this curve. However, needless to say this hypersurface is not umbilic at its each point. The Clifford hypersurface $M_{r,n-r}$ has two constant principal curvatures $c_1/\sqrt{c_1+c_2}$ (with multiplicity r) and $-c_2/\sqrt{c_1+c_2}$ (with multiplicity $n-r$).

(2) Theorem 2 is an improvement of the result in [5] for codimension $p = 1$.

4. Characterization of Veronese embeddings

In this section we shall give a characterization of Veronese embeddings of complex projective spaces into complex projective spaces as an application of Theorem 1.

Let $\mathbb{C}P^n(c)$ denote an n-dimensional complex projective space of constant holomorphic sectional curvature c. Then we consider the Kähler isometric embedding $f_\nu : \mathbb{C}P^n(c/\nu) \to \mathbb{C}P^N(c)$ which is given by all homogeneous monomials of degree ν

$$[z_i]_{0\le i\le n} \mapsto \left[\sqrt{\frac{\nu!}{\nu_0!\cdots\nu_n!}}\, z_0^{\nu_0}\cdots z_n^{\nu_n}\right]_{\nu_0+\cdots+\nu_n=\nu},$$

where $[*]$ stands for the point of the projective space with homogeneous coordinates $*$ and $N = (n+\nu)!/(n!\nu!) - 1$. The embedding f_ν has various geometric properties. We usually call f_ν the ν-th *Veronese embedding*.

We denote by $M_N(c)$ a complex space form of constant holomorphic sectional curvature c, which is locally congruent to either a complex Euclidean space $\mathbb{C}^N$, a complex projective space $\mathbb{C}P^N(c)$ or a complex hyperbolic space $\mathbb{C}H^N(c)$ according as c is zero, positive or negative. The following theorem is due to E. Calabi [1] and H. Nakagawa and K. Ogiue [4].

Theorem A. *Let $f : M_n(c) \to M_N(\tilde{c})$ be a Kähler isometric immersion of a complex space form of constant holomorphic sectional curvature c into another complex space form of constant holomorphic sectional curvature $\tilde{c}$. If $\tilde{c} > 0$ and f is full, then $\tilde{c} = \nu c$, $N = (n+\nu)!/(n!\nu!) - 1$ and f is locally equivalent to the ν-th Veronese embedding f_ν for some positive integer ν.*

We now prove:

Theorem 3. *Let $f : M_n \to M_N(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold M_n into an N-dimensional complex space form $M_N(c)$ of constant holomorphic sectional curvature $c > 0$. Then the following two conditions are equivalent*:

(1) *For some positive integer ν, the submanifold M_n is locally congruent to $\mathbb{C}P^n(c/\nu)$, $N = (n+\nu)!/(n!\nu!) - 1$ and f is locally equivalent to the ν-th Veronese embedding f_ν.*

(2) *There exists a positive smooth function $\kappa = \kappa(s)$ satisfying that for each Frenet curve $\gamma = \gamma(s)$ of proper order 2 with curvature κ on the submanifold M_n the curve $f \circ \gamma$ in $M_N(c)$ has constant first curvature $\tilde{\kappa}_1$ along this curve.*

Proof. (1) $\Rightarrow$ (2): For each Veronese embedding $f_\nu : \mathbb{C}P^n(c/\nu) \to \mathbb{C}P^N(c)$ we see that $\|\sigma(u,u)\|^2 = c(\nu-1)/(2\nu)$ for any unit vector u at each point $x \in \mathbb{C}P^n(c/\nu)$ (see [6]). Let us consider an arbitrary circle γ of curvature k on $\mathbb{C}P^n(c/\nu)$. Then we find that the curve $f_\nu \circ \gamma$ has constant first curvature $\kappa_1 = \sqrt{k^2 + \frac{c(\nu-1)}{2\nu}}$ in the ambient manifold $\mathbb{C}P^N(c)$.

(2) $\Rightarrow$ (1): Let $f : M_n \to M_N(c)$ be a Kähler isometric full immersion satisfying the condition (2). Then, by virtue of Theorem 1, M_n is constant (λ-)isotropic in $M_N(c)$. Denoting by R, $\widetilde{R}$ the curvature tensors of M_n, $M_N(c)$ respectively, the equation of Gauss can be written as

$$\langle \widetilde{R}(u,v)w,\, z\rangle = \langle R(u,v)w,\, z\rangle + \langle \sigma(u,w), \sigma(z,v)\rangle - \langle \sigma(u,z), \sigma(v,w)\rangle.$$

Since M_n is a Kähler submanifold in $M_N(c)$, from this equation and

$$\widetilde{R}(u,v)w = \frac{c}{4}\{\langle v,w\rangle u - \langle u,w\rangle v + \langle Jv,w\rangle Ju - \langle Ju,w\rangle Jv + 2\langle u,Jv\rangle Jw\},$$

we find that the holomorphic sectional curvature $K(u,Ju)$ of M_n determined by a unit vector u is given by

$$K(u,Ju) = \langle R(u,Ju)Ju,\, u\rangle = c - 2\|\sigma(u,u)\|^2.$$

Thus we can see that the submanifold M_n is a complex space form. Therefore from Theorem A we obtain the statement (1). □

Remark 2. Theorem 3 is not true if we set $\kappa \equiv 0$. For example, we consider the Segre embedding $g : \mathbb{C}P^n(c) \times \mathbb{C}P^m(c) \to \mathbb{C}P^N(c)$, which is given by $g([z_i],[w_j]) = [z_i w_j]$, where $0 \le i \le n,\ 0 \le j \le m$ and $N = nm + n + m$. Since the Segre embedding g has parallel second fundamental form, we find that Theorem 3 is not true when $\kappa \equiv 0$.

Acknowledgments. The first author would like to express his sincere gratitude to Professor S. Maeda for his valuable suggestions when he visited Saga University during the period from July through August in 2007.

References

[1] E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23.

[2] U.-H. Kim and S. Maeda, *Notes on extrinsic spheres*, Bull. Korean Math. Soc. **35** (1998), no. 3, 433–439.

[3] S. Maeda, *A characterization of constant isotropic immersions by circles*, Arch. Math. (Basel) **81** (2003), no. 1, 90–95.

[4] H. Nakagawa and K. Ogiue, *Complex space forms immersed in complex space forms*, Trans. Amer. Math. Soc. **219** (1976), 289–297.

[5] K. Nomizu and K. Yano, *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163–170.

[6] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Advances in Math. **13** (1974), 73–114.

[7] B. O'Neill, *Isotropic and Kähler immersions*, Canad. J. Math. **17** (1965), 907–915.

[8] H. Tanabe, *Characterization of totally geodesic submanifolds in terms of Frenet curves*, Sci. Math. Jpn. **63** (2006), no. 1, 83–88.

Jin Ho Choi
Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea
E-mail address: ji_no@hanmail.net

Young Ho Kim
Department of Mathematics
College of Natural Sciences
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yhkim@knu.ac.kr

Hiromasa Tanabe
Yonago Higashi Highschool
Yonago, Tottori Pref. 683-0051, Japan
E-mail address: tanabe_hr1@mailk.torikyo.ed.jp