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WIENER-HOPF C^*-ALGEBRAS OF STRONGLY PERFORATED SEMIGROUPS

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Abstract. If the Wiener-Hopf C^*-algebra $\mathcal{W}(G, M)$ for a discrete group G with a semigroup M has the uniqueness property, then the structure of it is to some extent independent of the choice of isometries on a Hilbert space. In this paper we show that if the Wiener-Hopf C^*-algebra $\mathcal{W}(G, M)$ of a partially ordered group G with the positive cone M has the uniqueness property, then (G, M) is weakly unperforated. We also prove that the Wiener-Hopf C^*-algebra $\mathcal{W}(\mathbb{Z}, M)$ of subsemigroup M generating the integer group $\mathbb{Z}$ is isomorphic to the Toeplitz algebra, but $\mathcal{W}(\mathbb{Z}, M)$ does not have the uniqueness property except the case $M = \mathbb{N}$.

1. Introduction and preliminaries

Let M be a countable discrete semigroup and $W : M \to \mathcal{B}$ be an isometric homomorphism for a unital C^*-algebra $\mathcal{B}$. We will consider a C^*-algebra generated by $\{W_x \mid \ x \in M\}$ and denote it by $C^*(W_M)$. Specially, the C^*-algebra generated by the left regular isometric representation has been much studied and called in the several names such as the Wiener-Hopf C^*-algebra [7, 12]. Besides the Wiener-Hopf C^*-algebra, we consider a semigroup C^*-algebra introduced by G. J. Murphy [10], which is obtained by enveloping all isometric representations of M and denoted by $C^*(M)$. From the definition of the semigroup C^*-algebra it has the following universal property: for any isometric homomorphism W of M and a C^*-algebra $C^*(W_M)$, there exists a unique homomorphism ϕ from the semigroup C^*-algebra $C^*(M)$ onto $C^*(W_M)$ sending a canonical isometry V_x to an isometry W_x for each $x \in M$, where V is the canonical isometric homomorphism of M to the semigroup C^*-algebra $C^*(M)$.

L. A. Coburn proved his well-known theorem [1], which asserts that C^*-algebras generated by a single non-unitary isometry on a Hilbert space do

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not depend on the particular choice of the isometry. That is, the C^*-algebra generated by a single non-unitary isometry on a Hilbert space is isomorphic to the Toeplitz algebra and the Toeplitz algebra is the universal algebra of C^*-algebras generated by a non-unitary isometry. So if M is the natural number semigroup $\mathbb{N}$, then the semigroup C^*-algebra $C^*(\mathbb{N})$ and the Wiener-Hopf C^*-algebra $\mathcal{W}(\mathbb{N})$ are generated by a non-unitary isometry and are isomorphic to the Toeplitz algebra.

Many authors have contributed to development of generalization of Coburn's result [1, 2, 4, 10], which is to find the condition when the Wiener-Hopf C^*-algebra $\mathcal{W}(M)$ and the semigroup C^*-algebra $C^*(M)$ are isomorphic or when the Wiener-Hopf C^*-algebra $\mathcal{W}(M)$ has a universal property for certain kinds of isometric representations of the semigroup M [2, 3, 4, 8, 10].

A. Nica called Coburn's result the uniqueness property of the C^*-algebras generated by isometric representations. Besides the Toeplitz algebra, the C^*-algebra generated by one-parameter semigroup of isometries and the Cuntz algebra are also remarkable examples of the C^*-algebras of isometries with the uniqueness property.

A. Nica introduced the quasi-lattice ordered group (G, M), the covariant isometric representations of semigroups and the amenability of quasi-lattice ordered groups in order to find the condition that the Wiener-Hopf C^*-algebra $\mathcal{W}(G, M)$ has a universal property for certain kinds of isometric representations of M [10]. The partially ordered group (G, M) is a *quasi-lattice ordered group* if every finite subset of G with an upper bound in M has a least upper bound in M. If a quasi-lattice ordered group (G, M) is amenable in the sense of Nica, the Wiener-Hopf C^*-algebra $\mathcal{W}(G, M)$ is the universal C^*-algebra of the C^*-algebras generated by covariant isometric representations. It seems that the quasi-lattice ordered group is an appropriate concept for the universal property of the Wiener-Hopf C^*-algebras. In [7] we had a very quite simple, non quasi-lattice ordered group $(\mathbb{Z}, P)$ with $P = \{0, 2, 3, \ldots\}$. Furthermore the semigroup P is a strongly perforated semigroup. We showed in [7] that $\mathcal{W}(\mathbb{Z}, P)$ is isomorphic to the classic Toeplitz algebra, but $\mathcal{W}(\mathbb{Z}, P)$ is not isomorphic to the semigroup C^*-algebra $C^*(P)$.

In Section 2, we show that if the Wiener Hopf C^*-algebra $\mathcal{W}(G, M)$ has the uniqueness property, (G, M) is weakly unperforated. In Section 3, we show that the Wiener-Hopf C^*-algebra $\mathcal{W}(\mathbb{Z}, M)$ of subsemigroup M generating the integer group $\mathbb{Z}$ is isomorphic to the Toeplitz algebra, but $\mathcal{W}(\mathbb{Z}, M)$ does not have the uniqueness property except the case $M = \mathbb{N}$. And we give examples which show that the quasi-lattice ordered group is a very appropriate concept for the universal property of the Wiener-Hopf C^*-algebras.

Let M be a countable, discrete semigroup with unit e, and B be a unital C^*-algebra. We say that a map $W : M \to B, x \to W_x$ is an *isometric homomorphism* from M to B if all of the elements W_x are isometries, if $W_e = 1$, and if $W_{xy} = W_x W_y$ $(x, y \in M)$. If H is a Hilbert space and $W : M \to B(H)$ is an isometric homomorphism, we call the pair (H, W) an isometric representation of M on H.

When M is a left-cancellative semigroup and H is an arbitrary non-zero Hilbert space, set $\widetilde{H} = l^2(M, H)$, the Hilbert space of all square-summable functions from M to H with the obvious operations and inner product. For $x \in M$, we define an isometry $\mathcal{L}_x$ on $\widetilde{H}$ by setting

$$\mathcal{L}_x f(z) = \begin{cases} f(y), & \text{if } z \in xM, \\ 0, & \text{if } z \notin xM. \end{cases}$$

We say that $(\widetilde{H}, \mathcal{L})$ is the *left regular isometric representation* of M on $\widetilde{H}$.

If x is invertible in M, then $\mathcal{L}_x f(z) = f(x^{-1}z)$ for each $z \in M$, so $\mathcal{L}_x$ is a left regular unitary on $\widetilde{H}$.

If the Hilbert space H is $\mathbb{C}$, the complex field, then $\widetilde{H}$ is just $l^2(M)$ and the left regular isometric representation $\mathcal{L}$ on $l^2(M)$ acts on as follows:

$$\mathcal{L}_x(\delta_y) = \delta_{xy} \ (x, y \in M),$$

where $\{\delta_x \mid x \in M\}$ is the canonical orthonormal basis of $l^2(M)$ defined by

$$\delta_x(y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

If M is the semigroup $\mathbb{N}$ of natural numbers, the left regular isometry $\mathcal{L}_1$ is the unilateral shift on $l^2(\mathbb{N})$ and $\mathcal{L}_n$ is the n-copy of the unilateral shift $\mathcal{L}_1$ with respect to the canonical orthonormal basis $\{\delta_n \mid n \in \mathbb{N}\}$ for each $n \in \mathbb{N}$.

The C^*-algebra generated by the left regular isometries $\{\mathcal{L}_x \mid x \in M\}$ on $l^2(M)$ is called the *Wiener-Hopf C^*-algebra* and denoted by $\mathcal{W}(M)$. And we can see that $\mathcal{W}(M)$ is the closed linear span of $\{\mathcal{L}_{x_{i_1}} \mathcal{L}^*_{x_{i_2}} \cdots \mathcal{L}^*_{x_{i_{2n_i}}} \mathcal{L}_{x_{i_{2n_i+1}}} \mid x_{ij} \in M\}$.

If M is the semigroup $\mathbb{N}$ of natural numbers, $\mathcal{W}(\mathbb{N})$ is generated by $\{\mathcal{L}_n \mid n \in \mathbb{N}\}$. Since $\mathcal{L}_n = \mathcal{L}_1^n$ for each $n \in \mathbb{N}$, $\mathcal{W}(\mathbb{N})$ is generated by a single non-unitary isometry $\mathcal{L}_1$ on $l^2(\mathbb{N})$. Therefore $\mathcal{W}(\mathbb{N})$ is the classical Toeplitz algebra. In this sense the Wiener Hopf C^*-algebra is a generalized Toeplitz algebra.

2. Unperforated semigroups and uniqueness property

Let M be a countable discrete semigroup. We can give an order on M as follows: if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote this by $y \leq x$. This relation makes M a pre-ordered semigroup.

Rørdam made a statement in [14, 15] what we call unperforated property of a unperforated property of a partially ordered group. If M is abelian, M can be equipped with the algebraic order $y \leq x$ if and only if $x = y + z$ for some $z \in M$. An element $x \in M$ is called *positive* if $y \leq y + x$ for all $y \in M$, and M is *positive* if all element in M are positive. If M has a zero element 0, then M is positive if and only if $0 \leq x$ for all $x \in M$.

A positive ordered abelian semigroup W is said to be *almost unperforated* if for all $x, y \in M$ and all $n, m \in M$, with $nx \leq my$ and $n > m$, one has $x \leq y$. A partially ordered abelian group G with the positive cone M is said to be almost unperforated if the statement that $x \in G$ and $n \in \mathbb{N}$ with $nx, (n+1)x \in M$ implies that $x \in M$. It is known that G is almost unperforated if and only if the positive semigroup M is almost unperforated for a partially ordered abelian group (G, M) [14, 15].

If the condition that $n \in \mathbb{N}$ and $x \in G$ with $nx \in M$ implies that $x \in M$, then the partially ordered abelian group (G, M) is *weakly unperforated.* Any weakly unperforated group is almost unperforated, but the converse is not true. The negation of almost unperforated property is strongly perforated.

If we put $P_x = \mathcal{L}_x \mathcal{L}_x^*$ for each $x \in M$, the range projection P_x of $\mathcal{L}_x$ is the orthogonal projection onto the closed linear span of $\{\delta_y \mid y \geq x,\ y \in m\}$. For each $x, y \in M$

$$P_x P_y = \begin{cases} 0, & \text{if } x \text{ and } y \text{ are not comparable,} \\ P_y, & \text{if } x \geq y \\ P_x & \text{if } y \geq y. \end{cases}$$

Hence $P_x P_y = P_y P_x$ for each $x, y \in M$, so the left regular isometries have the mutually commuting range projections.

Theorem 2.1. *Let (G, M) be a partially ordered abelian group with the positive cone M. If for any two isometric representations W and U of M there exists an isomorphism from the C^*-algebra $C^*(W_M)$ to the C^*-algebra $C^*(U_M)$ sending W_x to U_x, then (G, M) is weakly unperforated.*

Proof. Suppose that (G, M) is not weakly unperforated. Then there exists an element $x_0 \in G$ such that $nx_0 \in M$ for some $n \in \mathbb{N}$ and $x_0 \notin M$. Let n_0 = the smallest integer of $\{n \in \mathbb{N} \mid nx_0 \in M\}$. Put $M_0 = \{z \in M \mid z \leq n_0 x_0$ or z is not comparable with $n_0 x_0\}$ and $M_1 = M - M_0 \cup \{0\}$.

Let W be the isometric representation of M on $l^2(M_1)$ defined by $W_x(\delta_y) = \delta_{x+y}$ for $x \in M,\ y \in M_1$ where $\{\delta_y \mid y \in M\}$ is the canonical basis of $l^2(M_1)$. We can see that W is well-defined. Next, since $2n_0 x_0$ is contained in M, we can let n_1 the smallest integer of $\{n \mid n \neq n_0, nx_0 \in M\}$. Then the following relation holds;

$$W_{n_0 x_0}^* W_{n_1 x_0} (I - W_{n_0 x_0} W_{n_0 x_0}^*)(I - W_{n_1 x_0} W_{n_1 x_0}^*) = 0.$$

To define another isometric representation of M, we put $M' = M \cup \{nx_0 \mid n \in \mathbb{N}\}$. An isometric representation U of M on $l^2(M')$ can be defined by setting $U_x(\delta'_{y'}) = \delta'_{x+y'}$ for $x \in M,\ y' \in M'$ where $\{\delta'_{y'} \mid y' \in M'\}$ is the canonical orthonormal basis of $l^2(M')$. Then the above relation does not hold for the representation U. If $C^*(W_M)$ is the C^*-algebra generated by the isometric representation W and $C^*(U_M)$ is the C^*-algebra generated by the isometric representation U, then we don't have an isomorphism from the C^*-algebra $C^*(W_M)$ to the C^*-algebra $C^*(U_M)$ sending W_x to U_x. $\square$

When an ordered abelian group (G, M) is simple, it is almost unperforated if and only if it is weakly unperforated. Elliott considered in [5] a notion what he call (weak) unperforation of ordered abelian groups with torsion. In sense of Elliott's a torsion free group G is weakly unperforated if and only if it is unperforated: $g \geq 0$ whenever $ng \geq 0$ for some natural numbers $n \geq 2$.

Example 2.2. Let $M_1 = \{0, 3, 4, 6, 7, 8, 9, \ldots\}$ be the subsemigroup of $\mathbb{N}$ generated by 3 and 4. Then the left regular isometric representation $\mathcal{L}$ of $l^2(M_1)$ satisfies the relation

$$\mathcal{L}_3^* \mathcal{L}_4 (I - \mathcal{L}_3 \mathcal{L}_3^*)(I - \mathcal{L}_4 \mathcal{L}_4^*) = 0.$$

If we define another isometric representation U of M_1 as $U(n) = \mathcal{S}^n$ for $n = 0, 3, 4, 6, 7, \ldots$, where $\mathcal{S}$ is the unilateral shift on $l^2(\mathbb{N})$, this representation does not satisfy the above relation, i.e.,

$$\mathcal{S}^{*3} \mathcal{S}^4 (I - \mathcal{S}^3 \mathcal{S}^{*3})(I - \mathcal{S}^4 \mathcal{S}^{*4}) \neq 0.$$

Therefore we see that $\mathcal{W}(G, M_1)$ is not isomorphic to the semigroup C^*-algebra $C^*(M_1)$ since the semigroup C^*-algebra $C^*(M_1)$ is the universal C^*-algebra generated by isometries of M_1.

The semigroup $M_1 = \{0, 3, 4, 6, 7, 8, 9, 10, \ldots\}$ looks very simple, but it is strongly perforated.

3. Generalized Toeplitz algebras

One of the most beautiful property of the Toeplitz algebra is that it is the universal algebra of C^*-algebras generated by a single non-unitary isometry. In the following theorem we find semigroups of the integer group $\mathbb{Z}$ with more large extent whose the Wiener-Hopf C^*-algebras are isomorphic to the classical Toeplitz algebra.

Lemma 3.1. *Let M be a countable discrete semigroup. If the Wiener Hopf C^*-algebra $\mathcal{W}(M)$ acts irreducibly on $l^2(M)$ if and only if the unit of M is its only invertible element.*

Proof. Suppose that $\mathcal{W}(M)$ acts irreducibly on $l^2(M)$ and x is invertible in M. We define an bounded operator V_x on $l^2(M)$ by setting

$$V_x(f(y)) = f(yx) \ (y \in M).$$

Then V_x commutes with the operators $\mathcal{L}_y$ and $\mathcal{L}_y^*$ for all $y \in M$, so V_x commutes with $\mathcal{W}(M)$. Since $\mathcal{W}(M)$ acts irreducibly on $l^2(M)$, V_x should be λI. Hence x is the unit of M.

Conversely, let T be a bounded linear operator on $l^2(M)$ commuting with $\mathcal{W}(M)$. $[T_{[x,y]}]_{x,y \in M}$ is the matrix operator of T with respect to the canonical basis $\{\delta_x \mid x \in M\}$ of $l^2(M)$. Then we have

$$T_{[x,y]} = \langle T(\delta_y), \delta_x \rangle = \langle T(\delta_y), \mathcal{L}_x(\delta_e) \rangle = \langle T\mathcal{L}_x^*(\delta_y), \delta_e \rangle.$$

Hence the matrix element $T_{[x,y]} \neq 0$ only when $y \in xM$. Similarly,

$$T_{[x,y]} = \langle T\mathcal{L}_y(\delta_e), \delta_x\rangle = \langle T(\delta_e), \mathcal{L}_y^*(\delta_x)\rangle.$$

So $T_{[x,y]} \neq 0$ only when $x \in yM$ and $y \in xM$. Since the unit of M is the only invertible element, it follows that $T_{[x,y]} \neq 0$ only when $y = x$.

Since isometries $\mathcal{L}_x$'s are isometries, we have

$$T_{[x,x]} = \langle T\mathcal{L}_x(\delta_e), \mathcal{L}_x(\delta_e)\rangle = \langle \mathcal{L}_x^*\mathcal{L}_x T(\delta_e), (\delta_e)\rangle = T_{[e,e]}.$$

So T scalar operators and $\mathcal{W}(M)$ acts irreducibly on $l^2(M)$. □

Theorem 3.2. *If M is the subsemigroup of the integer group $\mathbb{Z}$ with $M + (-M) = \mathbb{Z}$ and $M \cap (-M) = 0$, then the Wiener Hopf C^*-algebra $\mathcal{W}(\mathbb{Z}, M)$ is isomorphic to the Toeplitz algebra.*

Proof. Case 1: If M contains 1, then $M = \mathbb{N}$, the set of natural numbers. And the result is known already.

Case 2: We consider the case of when M does not contain 1. Since M generates $\mathbb{Z}$, there exist elements $m_1, m_2 \in M$ such that $m_2 = m_1 + 1$. We put m_0= the smallest element of $\{n \in M \mid n+1 \in M\}$ in the usual order of $\mathbb{N}$. We renumber the elements of M by using the usual order of $\mathbb{N}$, i.e., $M = \{n_0 = 0, n_1, n_2, \dots, n_k, n_{k+1}, \dots\}$ and put $m_0 = n_k, m_0 + 1 = n_{k+1}$. We define a compact operator K_i for each $0 \leq i \leq k-1$, such as

$$K_i(\delta_n) = \begin{cases} \delta_{n_{i+1}}, & n = n_i, \\ 0, & \text{otherwise.} \end{cases}$$

If there exists an element $n_l \in M$ such that $n_l + 1 \notin M$, then we define a compact operator F_l,

$$F_l(\delta_n) = \begin{cases} \delta_{n_{l+1}}, & n = n_l, \\ 0, & \text{otherwise.} \end{cases}$$

Since $M + (-M) = \mathbb{Z}$, we have only finite n_l's with $n_l + 1 \notin M$.

Let $\mathcal{L}$ be the left regular isometric representation on $l^2(M)$ and put $U = \mathcal{L}_{m_0}^*\mathcal{L}_{m_0+1} + K_{n_1} + \cdots + K_{n_{k-1}} + \sum F_l$. The compact operator algebra $\mathcal{K}(l^2(M))$ is contained in $\mathcal{W}(M)$ because $\mathcal{W}(M)$ acts irreducibly on $l^2(M)$, and thus U is contained in $\mathcal{W}(M)$. We can see that the operator U translates the elements of the canonical orthonormal basis $\{\delta_n \mid n \in M\}$ of $l^2(M)$ to the left, one by one. If we put the C^*-subalgebra $\mathcal{U}$ of $W(M)$ generated by U, then $\mathcal{U}$ is isomorphic to the Toeplitz algebra.

Eventually, $\mathcal{L}_{m_0}$ and $\mathcal{L}_{m_0+1}$ generates $\mathcal{W}(M)$, so it is enough to show that $\mathcal{L}_{m_0}$ and $\mathcal{L}_{m_0+1}$ can be written as U + {suitable operators in $\mathcal{U}$} in order to say that U generates $\mathcal{W}(M)$. So we consider U^{m_0} and U^{m_0+1}. Since the terms of U^{m_0} containing K_i are removed,

$$U^{m_0} = (\mathcal{L}_{m_0}^*\mathcal{L}_{m_0+1})^{m_0} + \sum (\mathcal{L}_{m_0}^*\mathcal{L}_{m_0+1})^{s_1} F_{j_1}^{s_2} (\mathcal{L}_{m_0}^*\mathcal{L}_{m_0+1})^{s_3} \cdots F_{j_p}^{s_q},$$

where $s_1 + \cdots + s_q = m_0$ and s_i may be zero. In order to make up the gaps of $(\mathcal{L}_{m_0}^* \mathcal{L}_{m_0+1})^{m_0}$ we consider compact operators L_i's for $1 \le i \le k-1$, defined by

$$L_i(\delta_n) = \begin{cases} \mathcal{L}_{m_0}(\delta_n), & n = n_i, \\ 0, & \text{otherwise.} \end{cases}$$

Next, if there exist elements n_l between $m_0 + 1$ and $2m_0 - 2$ in the usual order of $\mathbb{N}$ such that $n_l + j \notin M$ for some $1 \le j \le m_0 - 2$, then we define compact operators Q_l as follows;

$$Q_l(\delta_n) = \begin{cases} \mathcal{L}_{m_0}(\delta_n), & n = n_l, \\ 0, & \text{otherwise.} \end{cases}$$

Due to the compact operators L_i and Q_l, we have

$$\mathcal{L}_{m_0} = U^{m_0} + \sum_i L_i + \sum_l Q_l - \sum (\mathcal{L}_{m_0}^* \mathcal{L}_{m_0+1})^{s_1} F_{j_1}^{s_2} (\mathcal{L}_{m_0}^* \mathcal{L}_{m_0+1})^{s_3} \cdots F_{j_p}^{s_q}.$$

Similarly, $\mathcal{L}_{m_0+1} = U^{m_0+1} + T$ for a suitable compact operator T. Therefore, U generates $\mathcal{W}(M)$ and $\mathcal{W}(M)$ is isomorphic to the Toeplitz algebra $\mathcal{T}$. □

In Theorem 2.1 we show that the uniqueness property implies the weakly unperforatedness. Even though we don't show that the converse of Theorem 2.1 is true, we can have many examples of strongly perforated group (G, M) whose the Wiener-Hopf algebra $\mathcal{W}(G, M)$ are not isomorphic to their semigroup C^*-algebras. The semigroups in the above theorem are one of these examples. The semigroups $(\mathbb{Z}, M)$'s in the above theorem except $M = \mathbb{N}$ are strongly perforated, We can see that their Wiener-Hopf C^*-algebras $\mathcal{W}(\mathbb{Z}, M)$ in the above theorem can not be isomorphic to their semigroup C^*-algebras. Even though $\mathcal{W}(\mathbb{Z}, M)$'s are isomorphic to the Toeplitz algebra, $\mathcal{W}(\mathbb{Z}, M)$'s are not isomorphic to $C^*(M)$ except $M = \mathbb{N}$. From the very reason it looks very strangely and interesting. We can give an example of a strongly perforated semigroup, which can show the above statement.

Example 3.3. If M_1 is the subsemigroup of $\mathbb{N}$ generated by 3 and 4 in the example 2.2, then by Theorem 3.2, $\mathcal{W}(M_1)$ is isomorphic to the Toeplitz algebra. We consider two operators of rank one K_0 and F_1 defined by

$$K_0(\delta_n) = \begin{cases} \delta_3, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$F_1(\delta_n) = \begin{cases} \delta_6, & \text{if } n = 4, \\ 0, & \text{otherwise.} \end{cases}$$

If we put $U = \mathcal{L}_3^* \mathcal{L}_4 + K_0 + F_1$, then U generates $\mathcal{W}(M_1)$. As we see in Example 2.2 $\mathcal{W}(M_1)$ is not isomorphic to the semigroup C^*-algebra $C^*(M_1)$ The semigroup $M_1 = \{0, 3, 4, 6, 7, 8, 9, 10, \dots\}$ is strongly perforated, and not quasi-lattice group.

The following examples show that there exist isometric representations of subsemigroups of free groups and abelian groups which can not be factored through the left regular isometric representations. Nica showed [12] that the quasi-lattice ordered group is amenable if and only if every covariant representation can be factored through the left regular isometric representation and also showed that the quasi-lattice abelian ordered group and the free group are amenable quasi-lattice groups. It seems that the quasi-lattice ordered group is a very appropriate concept for the universal property of the Wiener-Hopf C^*-algebras. So we can see that the Wiener-Hopf C^*-algebra $\mathcal{W}(M)$ and the semigroup C^*-algebra $C^*(M)$ can be isomorphic only for the particular semigroups.

Example 3.4. Let $\mathcal{F}_2$ be a free group with two generators z_1 and z_2 and $\mathcal{F}_2^+ = \{z_{i_1}^{\epsilon_{i_1}} z_{i_2}^{\epsilon_{i_2}} \dots z_{i_n}^{\epsilon_{i_n}} \mid i_j = 1 \text{ or } 2,\ \epsilon_{i_j} = 1\}$ be a subsemigroup of $\mathcal{F}_2$. Let $\mathcal{L}$ be the left regular isometric representation of $\mathcal{F}_2^+$ on $l^2(\mathcal{F}_2^+)$ and W be another isometric representation of $\mathcal{F}_2^+$ to $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ defined by $W_{z_1} = \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{smallmatrix}\right)$, $W_{z_2} = \left(\begin{smallmatrix} \mathcal{S} & 0 \\ 0 & I \end{smallmatrix}\right)$, where $\mathcal{S}$ is the unilateral shift on $l^2(\mathbb{N})$. The isometric representation W of $\mathcal{F}_2^+$ can't be factored through the left regular isometric representation of $\mathcal{F}_2^+$ because the $\mathcal{L}_{z_i}$'s are isometries with orthogonal ranges for $i = 1,\ 2$, but $W_{z_1} W_{z_1}^*$ is not orthogonal to $W_{z_2} W_{z_2}^*$.

Example 3.5. Let $G = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and M be the subsemigroup generated by $m_1 = (1, 0)$, $m_2 = (1, 1\ _{\text{mod } 2})$, and $(0, 0)$. If W is an isometric representation of M on a non-zero Hilbert space and $W(m_i) = W_i$ for $i = 1,\ 2$, then $W_1^2 = W_2^2$ and $W_1 W_2 = W_2 W_1$. If we look at the left regular isometric representation $\mathcal{L}$ of M on $l^2(M)$, we can see that $\mathcal{L}$ satisfies another relation:

$$(\mathcal{L}_1 \mathcal{L}_1^*)(\mathcal{L}_2 \mathcal{L}_2^*) = (\mathcal{L}_2 \mathcal{L}_2^*)(\mathcal{L}_1 \mathcal{L}_1^*).$$

This relation does not hold for arbitrary isometric representation of M. For example, if a Hilbert space K is $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$, $\mathcal{S}$ is the unilateral shift on $l^2(\mathbb{N})$, and V is an isometric representation of M on the Hilbert space K defined by

$$V_{m_1} = \begin{pmatrix} \mathcal{S} & 0 \\ 0 & \mathcal{S} \end{pmatrix}, \quad V_{m_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{S} & \mathcal{S}^2 \\ 1 & -\mathcal{S} \end{pmatrix},$$

then V does not satisfy the above relation.

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