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A COMPLEX SURFACE OF GENERAL TYPE WITH $p_g = 0$, $K^2 = 3$ AND $H_1 = \mathbb{Z}/2\mathbb{Z}$

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Abstract. As the sequel to our previous work [4], we construct a minimal complex surface of general type with $p_g = 0, K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ by using a rational blow-down surgery and $\mathbb{Q}$-Gorenstein smoothing theory.

1. Introduction

This paper is a continuation of our previous work [4], in which the authors constructed a simply connected minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Motivated by Y. Lee and the second author's recent construction [3] of a surface of general type with $p_g = 0$, $K^2 = 2$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$, we extend the result to the $K^2 = 3$ case in this paper. That is, we construct a new non-simply connected minimal surface of general type with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$ using a rational blow-down surgery and a $\mathbb{Q}$-Gorenstein smoothing theory.

The key ingredient of this paper is to find a right rational surface Z which makes it possible to get such a complex surface. Once we have a right candidate Z for $K^2 = 3$, the remaining argument is similar to that of $K^2 = 3$ case appeared in our previous work [4]. That is, by applying a rational blow-down surgery and a $\mathbb{Q}$-Gorenstein smoothing theory developed in Lee and Park [2] to Z, we obtain a minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Then we show that the surface has $H_1 = \mathbb{Z}/2\mathbb{Z}$. Since almost all the proofs are parallel to the case of the main construction in the our previous work [4, §3], we only explain how to construct such a minimal complex surface. The main result of this paper is the following

Theorem 1. *There exists a minimal complex surface of general type with $p_g = 0$, $K^2 = 3$ and $H_1 = \mathbb{Z}/2\mathbb{Z}$.*

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Remark. D. Cartwright and T. Steger [1] also constructed minimal surfaces of general type with $p_g = 0$, $K^2 = 3$ and $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ using a completely different method.

2. Main construction

We start with a special elliptic fibration $Y := \mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$ which is used in the main construction of this paper. Let L_1, L_2, L_3 and A be lines in $\mathbb{CP}^2$ and let B be a smooth conic in $\mathbb{CP}^2$ intersecting as in Figure 1. We consider a pencil of cubics $\{\lambda(L_1+L_2+L_3)+\mu(A+B) \mid [\lambda:\mu] \in \mathbb{CP}^1\}$ in $\mathbb{CP}^2$ generated by two cubic curves $L_1+L_2+L_3$ and $A+B$, which has 5 base points, say, p, q, r, s and t. In order to obtain an elliptic fibration over $\mathbb{CP}^1$ from the pencil, we blow up three times at q and twice at s and t, respectively, including infinitely near base-points at each point. We perform two further blowing-ups at the base points p and r. By blowing-up nine times, we resolve all base points (including infinitely near base-points) of the pencil and we then get an elliptic fibration $Y = \mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$ over $\mathbb{CP}^1$ (Figure 2). We denote by E_i (or $\widetilde{E_i}$), $i = 1, \ldots, 9$, the

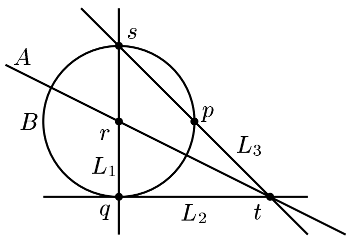

FIGURE 1. A pencil of cubics

exceptional divisors (or their proper transforms in Y, respectively) induced by the nine blowing-ups. Note that there are five sections of the elliptic fibration Y corresponding to the five base points p, q, r, s, and t, which are denoted by $E_5, E_6, \ldots, E_9$, respectively. Furthermore, the elliptic fibration Y has an I_7-singular fiber consisting of the proper transforms $\widetilde{L_i}$ of L_i ($i = 1, 2, 3$), $\widetilde{E_1}$, $\widetilde{E_2}$, $\widetilde{E_3}$ and $\widetilde{E_4}$. Also Y has an I_2-singular fiber consisting of the proper transforms $\widetilde{A}$ and $\widetilde{B}$ of A and B, respectively. According to the list of Persson [5], we may assume that Y has three more nodal singular fibers by choosing generally L_i's, A and B. Among the three nodal singular fibers, we use only two nodal singular fibers, say F_1 and F_2, for the main construction (Figure 2). Next, by blowing-up several times on Y, we construct a rational surface Z which contains special configurations of linear chains of $\mathbb{CP}^1$'s. At first we blow up five times at the marked point $\odot$ on $F_2 \cap E_5$. We also blow up two times at

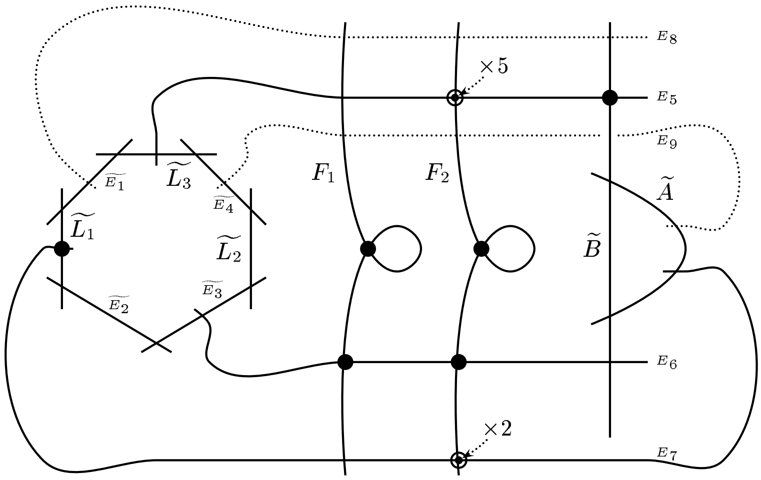

FIGURE 2. An elliptic fibration Y

the marked point $\odot$ on $F_2 \cap E_7$. Finally we blow up at the six marked points $\bullet$ on each fiber. We then get a rational surface $Z = Y \sharp 13\overline{\mathbb{CP}}^2$. We denote by e_i (or $\widetilde{e_i}$), $i = 1, \ldots, 13$, the exceptional divisors (or their proper transforms in Z, respectively) induced by the 13 blow-ups and we also denote by $\widetilde{F_i}$ $(i = 1, 2)$ the proper transforms of F_i. Then there exist two disjoint linear chains of $\mathbb{CP}^1$'s in Z: $C_{110,67} = \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-5}{\circ} - \overset{-7}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-3}{\circ} - \overset{-3}{\circ}$ (which consists of $\widetilde{e_{12}}$, $\widetilde{E_7}$, $\widetilde{F_1}$, $\widetilde{E_5}$, $\widetilde{L_3}$, $\widetilde{E_1}$, $\widetilde{L_1}$, $\widetilde{E_2}$, $\widetilde{E_3}$, $\widetilde{E_6}$, $\widetilde{B}$) and $C_{6,1} = \overset{-8}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ} - \overset{-2}{\circ}$ (which consists of $\widetilde{F_2}$, $\widetilde{e_8}$, $\widetilde{e_7}$, $\widetilde{e_6}$, $\widetilde{e_5}$) (Figure 3). Here $C_{p,q} = \overset{-b_k}{\circ} - \overset{-b_{k-1}}{\circ} - \cdots - \overset{-b_1}{\circ}$ is a small neighborhood linear chains of $\mathbb{CP}^1$ such that $p > q$, $\gcd(p, q) = 1$, $b_i \geq 2$, and $[b_k, \ldots, b_1]$ forms a continued fraction with

$$\frac{p^2}{pq-1} = b_k - \cfrac{1}{b_{k-1} - \cfrac{1}{\ddots - \cfrac{1}{b_1}}}$$

Next, by applying $\mathbb{Q}$-Gorenstein smoothing theory as in our previous work [4], we construct the minimal complex surface appeared in the main theorem. That is, we first contract two disjoint chains $C_{110,67}$ and $C_{6,1}$ of $\mathbb{CP}^1$'s from Z so that it produces a normal projective surface X with two permissible singular points. And then, by using a similar technique in our previous work [4], we can conclude that X has a $\mathbb{Q}$-Gorenstein smoothing and a general fiber X_t of

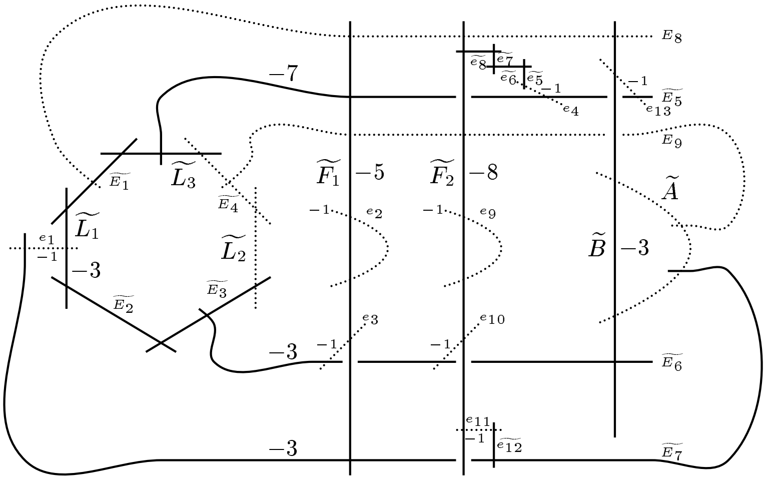

FIGURE 3. A rational surface $Z = Y\sharp 13\overline{\mathbb{CP}^2}$

the $\mathbb{Q}$-Gorenstein smoothing of X is a minimal complex surface of general type with $p_g = 0$ and $K^2 = 3$. Let us denote a general fiber of the $\mathbb{Q}$-Gorenstein smoothing of X by X_t. Finally it remains to show that $H_1(X_t;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

2.1. Proof of $H_1(X_t;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$

Let $Z_{110,6}$ be a rational blow-down 4-manifold obtained from Z by replacing two disjoint configurations $C_{110,67}$ and $C_{6,1}$ with the corresponding rational balls $B_{110,67}$ and $B_{6,1}$, respectively. Then, since a general fiber X_t of the $\mathbb{Q}$-Gorenstein smoothing of X is diffeomorphic to the rational blow-down 4-manifold $Z_{110,6}$, we have $H_1(X_t;\mathbb{Z}) = H_1(Z_{110,6};\mathbb{Z})$. Hence it suffices to show that $H_1(Z_{110,6};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Proposition 2. *$H_1(Z_{110,6};\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. One can prove this proposition using the similar technique in Section 2 of Lee and Park [3]. Here we present another way to prove it as follows: First note that the rational surface $Z = Y\sharp 13\overline{\mathbb{CP}^2}$ can be decomposed into $Z = Z_0 \cup \{C_{110,67} \cup C_{6,1}\}$ and the rational blow-down 4-manifold $Z_{110,6}$ can be decomposed into $Z_{110,6} = Z_0 \cup \{B_{110,67} \cup B_{6,1}\}$. Let $W = Z_0 \cup B_{110,67}$ and consider the following exact homology sequence for a pair $(W, \partial W)$:

$$\cdots \to H_2(W,\partial W;\mathbb{Z}) \xrightarrow{\partial_*} H_1(\partial W;\mathbb{Z}) \xrightarrow{i_*} H_1(W;\mathbb{Z}) \to H_1(W,\partial W;\mathbb{Z}) = 0.$$

Here the last term is zero because the punctured exceptional curve e_{11} lies in $Z_0 \cup C_{6,1}$ (refer to Figure 3) and $Z_0 \cup C_{6,1}$ is simply connected. So $Z_0 \cup C_{6,1} \cup$

$B_{110,67}$ is also simply connected by Van Kampen Theorem and $H_1(W, \partial W; \mathbb{Z}) \cong H_1(Z_0 \cup C_{6,1} \cup B_{110,67}, C_{6,1}; \mathbb{Z}) = 0$. Note that $\partial W = \partial B_{6,1} = L(36, -5)$ and a generator of $H_1(\partial W; \mathbb{Z}) = \mathbb{Z}/36\mathbb{Z}$ can be represented by a normal circle, say α, of a disk bundle $C_{6,1}$ over the (-8)-curve $\widetilde{F_2}$. Then we have

$$\partial_*([e_9|_W]) = 2\alpha \in H_1(\partial W; \mathbb{Z}) = \mathbb{Z}/36\mathbb{Z}.$$

Furthermore, by choosing a suitable basis $\mathcal{B}$ of $H_2(W, \partial W; \mathbb{Z})$ and by evaluating $\mathcal{B}$ under ∂_*, we can conclude that the generator $\alpha \in H_1(\partial W; \mathbb{Z})$ is not in the image of ∂_*. Hence it follows from the exact sequence above that we have $H_1(W; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H_1(W; \mathbb{Z})$ is generated by the element $i_*(\alpha)$.

Next, we consider the Mayer-Vietoris sequence for a triple $(Z_{110,6}; W, B_{6,1})$:

$$\begin{aligned} H_2(Z_{110,6}; \mathbb{Z}) \xrightarrow{\partial_*} H_1(L(36,-5); \mathbb{Z}) &\xrightarrow{i_* \oplus j_*} H_1(W; \mathbb{Z}) \oplus H_1(B_{6,1}; \mathbb{Z}) \\ &\to H_1(Z_{110,6}; \mathbb{Z}) \to 0. \end{aligned}$$

Since the map $i_* \oplus j_*$ sends the generator α to (a generator, a generator), we finally have $H_1(Z_{110,6}; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. □

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