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ON FILLING DISCS IN THE STRONG BOREL DIRECTION OF ALGEBROID FUNCTION WITH FINITE ORDER

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Abstract. Using Ahlfors' covering surface method, some properties on the strong Borel direction of algebroid function of finite order are obtained. The main object of this paper is to prove existence theorem of a strong Borel direction and the existence of filling discs in such direction, which briefly extends some results of meromorphic function.

1. Introduction and main results

Suppose that $A_v(z), \ldots, A_0(z)$ are analytic functions without common zeros on the whole complex plane $\mathbf{C}$ and the indecomposable equation

$$A_v(z)w^v + A_{v-1}(z)w^{v-1} + \cdots + A_0(z) = 0 \tag{1}$$

defines a v-valued algebroid function $w(z)$ on the complex plane (if $A_v(z)$= 1, then $w(z)$ is called a v-valued integral algebroid function), where $A_0(z) \not\equiv 0$, otherwise (1) is an reducible algebriod function; where $A_v(z) \not\equiv 0$, otherwise it is $v-1$ valued or less. In particular, when $v = 1$, $w(z)$ is exactly a meromorphic function.

The value distribution theory of meromorphic functions due to Nevanlinna (see [2] and [14]) such as the growth and the singular direction can be extended to the algebroid function in despite of its multivaluedness and the complexity of its branch points (see [8], [12], [3] and [10]). For instance, G. Valiron [13] conjectured that there exists at least a Borel direction for any v-valued algebroid function of order $0 < \rho < +\infty$. Lü and Gu [6] proved that there exists a direction such that the number of Borel exceptional values is $2v$ at most. Later,

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Sun [10] investigated the growth relationship of algebroid function and its coefficients, and obtained a basic in-equation between the maximum modulus function and Nevanlinna characteristic function.

Given an algebroid function defined on the z-plane, the existence of its filling discs was proved by Sun in [9]. Gao proved in [1] that for an algebroid function of positive order ρ, there exists a Borel direction of the largest type and a sequence of filling discs in this direction, which extends some results of A. Rauch [7] for meromorphic function. In this paper we will prove existence theorem of a strong Borel direction and the existence of filling discs in such direction by using a Type-function from [5], which improve the results of [6], [9], [1], [7] and some similar results of [4]. To arrive at our main results, we introduce some definitions and notations, which can be found in [3] and [10].

The single-valued domain $\widetilde{R_z}$ of definition of $w(z)$ is a v-valued covering of the z-plane and it is a Riemann surface. A point in $\widetilde{R_z}$ is denoted by $\widetilde{z}$ provided its projection on the z-plane is z. The open set which lies over $|z| < r$ is denoted by $|\widetilde{z}| < r$. Let $n(r, a)$ be the number of roots, with due count of multiplicity, of the equation $w(z) = a$ in $|\widetilde{z}| \leq r$, $\overline{n}(r, a)$ be the number of distinct roots of the equation $w(z) = a$ in $|\widetilde{z}| \leq r$. Specially, $\overline{n}(r, \Delta(\theta, \varepsilon), a)$ denotes the number of distinct roots of the equation $w(z) = a$ in the angular domain $\{|\widetilde{z}| < r\} \cap \Delta(\theta, \varepsilon)$, where $\Delta(\theta, \varepsilon) = \{\theta | \theta - \varepsilon < \arg \widetilde{z} < \theta + \varepsilon\}$. Let

$$
\begin{aligned}
m(r, w) &= \frac{1}{2v\pi} \int_0^{2\pi} \ln^+ |w(re^{i\theta})| \mathrm{d}\theta, \\
N(r, w) &= \frac{1}{v} \left[\int_0^r \frac{n(r, w) - n(0, w)}{r} \mathrm{d}r + n(0, w) \ln r \right], \\
N_\chi(r, w) &= \frac{1}{v} \left[\int_0^r \frac{n_\chi(r, w) - n_\chi(0, w)}{r} \mathrm{d}r + n_\chi(0, w) \ln r \right], \\
S(r, w) &= \frac{1}{\pi} \sum_{n=1}^{v} \iint_{|\widetilde{z}| \leq r} \left(\frac{|w_j'(z)|}{1 + |w_j(z)|^2} \right)^2 \mathrm{d}w \quad z = re^{i\theta}, \\
T_0(r, w) &= \frac{1}{v} \int_0^r \frac{S(r, w)}{r} \mathrm{d}r, \quad T(r, w) = N(r, w) + m(r, w),
\end{aligned}
$$

where $n_\chi(r, w)$ is the number of the branch points of $\widetilde{R_z}$ in $|\widetilde{z}| \leq r$, counted with the order of branch. Moreover, $S(r, w)$ is a conformal invariant and is called the mean covering number of $|\widetilde{z}| \leq r$ into w-sphere. We call $T(r, w)$ the characteristic function of $w(z)$. It is known from [3] or [11] that

$$
(2) \qquad T(r, w) = T_0(r, w) + O(1), \quad N_\chi(r, w) \leq 2(v - 1)T(r, w) + O(1).
$$

Definition 1. Let $w(z)$ be the v-valued algebroid function, the order of $w(z)$ is defined by

$$
\rho = \overline{\lim_{r \to +\infty}} \frac{\ln T(r, w)}{\ln r}.
$$

When $\rho \in (0, +\infty)$, we say that $w(z)$ is a v-valued algebroid function of finite order.

Definition 2. Let $w(z)$ be the v-valued algebroid function. If $\forall \varepsilon > 0$, $\forall a \in \overline{\mathbf{C}} := \mathbf{C} \cup \infty$, we have

$$\overline{\lim_{r\to+\infty}} \frac{\ln^+ n(r, \Delta(\theta_0, \varepsilon), a)}{\ln r} = \rho,$$

with at most $2v$ possible exceptional values of a, then ray $L(\theta_0) = \{z : \arg z = \theta_0\}$ is called a Borel direction.

By [5], we can introduce a Type-function $V(\ln r) = r^{\rho(r)}$ from Valiron's. If $w(z)$ is a v-valued algebroid function of finite order $\rho \in (0, +\infty)$, where $\rho(r)$ is a non-decreasing continuous function and satisfies

$$\lim_{r\to+\infty} \rho(r) = \rho, \qquad \lim_{r\to+\infty} \rho'(r) r \ln r = 0. \tag{3}$$

We can see that $V(\ln r)$ is a non-decreasing continuous function and Type-function satisfies

$$\overline{\lim_{r\to+\infty}} \frac{T(r, w)}{V(\ln r)} = 1, \quad \lim_{r\to+\infty} \frac{V(\ln r)}{\ln^2 r} = \infty. \tag{4}$$

$V(\ln r)$ is called Type-function of $w(z)$ with finite order and $\rho(r)$ the proximate order of $w(z)$.

Definition 3. Let $w = w(z)$ be the v-valued algebroid function of finite order and $V(\ln r) = r^{\rho(r)}$ be its Type-function. If $\forall \varepsilon \in (0, \pi/2), \forall a \in \overline{\mathbf{C}}$, we have

$$\overline{\lim_{r\to+\infty}} \frac{n(r, \Delta(\theta_0, \varepsilon), a)}{V'(\ln r)} = C_\varepsilon > 0,$$

with at most $2v$ possible exceptional values of a. The ray $L(\theta_0) = \{z : \arg z = \theta_0\}$ is called a strong Borel direction of $w(z)$ based on $V(\ln r)$, where C_ε is a positive constant defined by ε.

Theorem 1. *Suppose that $w(z)$ is a v-valued algebroid function of finite order ρ on the whole complex plane, then there exists a strong Borel direction $L(\theta_0) = \{z : \arg z = \theta_0\}$ of $w(z)$ on Type-function $V(\ln r)$.*

Theorem 2. *Suppose that $w(z)$ is a v-valued algebroid function of finite order, then the strong Borel direction $L(\theta_0) = \{z : \arg z = \theta_0\}$ of Theorem 1 is also a Borel direction.*

Theorem 3. *Suppose that $w = w(z)$ is a v-valued algebroid function of finite order ρ on the whole complex plane. If $L(\theta_0) = \{z : \arg z = \theta_0\}$ $(0 \le \theta_0 < 2\pi)$ is a strong Borel direction of $w(z)$ on $V(\ln r)$, then there exists a sequence of filling discs*

$$D_n = \{z : |z - z_n| < \sigma_n |z_n|\}, n = 1, 2, \ldots$$

such that

$$\overline{n}(D_n, w = b) \ge V^{1-\varepsilon_n}(|z_n|) \quad \textit{and} \quad \varepsilon_n \to 0 \ (n \to \infty),$$

holds for any complex value $b \in \overline{\mathbf{C}}$ except at most some possible exceptions which can be enclosed by spherical circles of radius $V^{-\frac{1}{40}}(\ln|z_n|)$, where $z_n = r_n e^{i\theta_n}$, $\lim_{n\to\infty} r_n = \infty$, $\lim\limits_{n\to\infty} \sigma_n = 0$ $(\sigma_n > 0)$.

Remark. For any complex number $b \in \overline{\mathbf{C}}$, with some possible exceptions, we can find a point $z \in D_n$ such that $w(z) = b$ at least $V^{1-\varepsilon_n}(|z_n|)$ times, that is to say the complex plane $\overline{\mathbf{C}}$ can be almost covered by the certain disc D_n through $w(z)$ at least $V^{1-\varepsilon_n}(|z_n|)$ times, so D_n is called a filling disc in a strong Borel direction of $w(z)$.

2. Lemmas and their proofs

Lemma 1. *Let $V(\ln r) = r^{\rho(r)}$ be a Type-function of $w(z)$ of finite order and $\rho(r)$ be the proximate order of $w(z)$. Then $V'(\ln r)$ is a non-decreasing continuous function on r and satisfies*

$$V'(\ln r) = V(\ln r)(\rho + o(1)).$$

Moreover

$$\lim_{r\to+\infty} \frac{V(\ln rh)}{V(\ln r)} = h^\rho, \quad \lim_{r\to+\infty} \frac{V'(\ln rh)}{V'(\ln r)} = h^\rho, \quad \lim_{r\to+\infty} \frac{V(\ln rh)}{V'(\ln r)} = \frac{h^\rho}{\rho}, \tag{5}$$

holds for any $h > 1$.

Proof. As $\rho(r)$ is a nondecreasing continuous function,

$$\ln V(\ln r) = \rho(r)\ln r \Rightarrow V'(\ln r) = V(\ln r)[r\rho(r)\ln r + \rho(r)],$$

and then $V'(\ln r)$ is also a nondecreasing continuous function on r. By (3), it follows that

$$V'(\ln r) = V(\ln r)(\rho + o(1)) \quad (r \to +\infty).$$

For any $h > 1$, we have

$$\lim_{r\to+\infty} \frac{V(\ln rh)}{V(\ln r)} = \lim_{r\to+\infty} [r^{\rho(rh)-\rho(r)} \cdot h^{\rho(rh)}] = h^\rho,$$

then

$$\lim_{r\to+\infty} \frac{V'(\ln rh)}{V'(\ln r)} = \lim_{r\to+\infty} \frac{V(\ln rh)[rh\rho(rh)\ln rh + \rho(rh)]}{V(\ln r)[r\rho(r)\ln r + \rho(r)]} = h^\rho,$$

and

$$\lim_{r\to+\infty} \frac{V(\ln rh)}{V'(\ln r)} = \lim_{r\to+\infty} \frac{V(\ln rh)}{V(\ln r)[r\rho(r)\ln r + \rho(r)]} = \frac{h^\rho}{\rho}.$$

Lemma 1 is proved. □

From [6], we have Lemma 2.

Lemma 2. *Let $w = w(z)$ $(z \in \boldsymbol{C})$ be a v-valued algebroid function defined by* (1). *For $0 < \delta < \delta_0 \le \frac{\pi}{2}, 0 \le \theta_0 < 2\pi$, let*

$$\Delta_0 = \{|\arg z - \theta_0| \le \delta_0\}, \quad \overline{\Delta} = \{z \mid |\arg z - \theta_0| \le \delta\},$$

$$S(r, \overline{\Delta}, w) = \frac{1}{\pi} \iint_{\widetilde{\overline{\Delta}}} \left[\frac{w'(z)}{1 + |w(z)|^2} \right]^2 dw \quad z = re^{i\theta}.$$

The part of $\widetilde{R_z}$ which lies over $\overline{\Delta} \cap \{|z| < r\}$ is denoted by $\widetilde{\overline{\Delta}}$. Then $\forall \lambda > 1, \forall \alpha \in \mathbb{N}_+$ and q $(q \ge 3)$ distinct points $a_1, a_2, \ldots, a_q$ on the sphere (the spherical distance of any two of which is no less than K), we have

$$(q-2)S(r, \overline{\Delta}, w) \le 2\sum_{j=1}^{q} n(\lambda^{2\alpha} r, \Delta_0, a_j) + (1 + \frac{1}{\alpha}) n_\chi(\lambda^{2\alpha} r, \Delta_0, w)$$
$$+ (q-2)S(\lambda^\alpha, \overline{\Delta}, w) + \frac{2A \ln r}{(1-K)\ln \lambda},$$

where A is a constant depends only on $a_1, a_2, \ldots, a_q$; and K is a constant and satisfies $0 < K < 1$.

Lemma 3 ([9, Lemma 1])**.** *Suppose that $w(z)$ is a v-valued algebroid function in $|z| < R$, and $a_1, a_2, \ldots, a_q$ $(q \ge 3)$ are distinct points on the sphere, the spherical distance of any two of which is no less than $\delta \in (0, \frac{1}{2})$. Then $\forall r \in (0, R)$,*

$$(q-2)S(r, w) \le \sum_{j=1}^{q} \overline{n}(R, a_j) + n_\chi(R, w) + \frac{2^{51}\pi^{40} v R}{(R-r)\delta^{38}}.$$

Lemma 4. *Let $L(\theta_0) = \{z : \arg z = \theta_0\}$ be a strong Borel direction mentioned in Theorem 1. Then $\forall \varepsilon \in (0, \frac{\pi}{2})$, it follows that*

$$\overline{\lim_{r\to+\infty}} \frac{S(r, \Delta(\theta_0, \varepsilon), w)}{V'(\ln r)} \ge C_\varepsilon > 0.$$

Moreover,

$$\overline{\lim_{r\to+\infty}} \frac{S(r, \Delta(\theta_0, \varepsilon), w)}{V(\ln r)} \ge C_\varepsilon \rho > 0.$$

Proof. For any $\varepsilon \in (0, \frac{\pi}{2})$, there exists $C_\varepsilon > 0$ such that $\forall a \in \overline{\mathbf{C}}$ (with $2v$ exceptions at most),

$$\overline{\lim_{n\to+\infty}} \frac{n(r, \Delta(\theta_0, \varepsilon), a)}{V'(\ln r)} > C_\varepsilon.$$

Then $\forall 0 < \eta < C_\varepsilon$, there exists $\{r_n\}$ such that

$$n(r_n, \Delta(\theta_0, \varepsilon), a) > (C_\varepsilon - \eta)V'(\ln r_n).$$

Let $E_n = \{a \in V \mid n(r_n, \Delta(\theta_0, \varepsilon), a) > (C_\varepsilon - \eta)V'(\ln r_n)\}$. Then

$$\pi S(r, \Delta(\theta_0, \varepsilon), w) > (C_\varepsilon - \eta)V'(\ln r_n) \cdot \mathrm{mes} E_n,$$

where mesE_n is the Lebesgue measure of E_n. Since $\#\{S - E_n\} \leq 2v$, where S is a complex sphere, mes$E_n = \pi$, i.e.,

$$\frac{S(r, \Delta(\theta_0, \varepsilon), w)}{V'(\ln r)} \geq C_\varepsilon - \eta.$$

Then

$$\overline{\lim_{r\to+\infty}} \frac{S(r, \Delta(\theta_0, \varepsilon), w)}{V'(\ln r)} \geq C_\varepsilon > 0.$$

By Lemma 1, $V'(\ln r) = V(\ln r)(\rho + o(1))$, we have

$$\overline{\lim_{r\to+\infty}} \frac{S(r, \Delta(\theta_0, \varepsilon), w)}{V(\ln r)} = \overline{\lim_{r\to+\infty}} \frac{S(r, \Delta(\theta_0, \varepsilon), w)(\rho + o(1))}{V'(\ln r)} \geq C_\varepsilon \rho. \qquad \square$$

3. Proofs of theorems

Proof of Theorem 1. We prove Theorem 1 in two steps.

Step i). First, let $V(\ln r)$ be a Type-function of $w(z)$ in Lemma 1. Then for $m(m \in \mathbf{N}_+)$ domains $\Delta\theta_i = \{z \mid |\arg z - \frac{2i\pi}{m}| < \frac{2\pi}{m}\}$ $(i = 0, \ldots, m-1)$, there is at least a domain $\Delta\theta_{i_0}$ such that

$$\overline{\lim_{r\to+\infty}} \frac{n(r, \Delta\theta_{i_0}, a)}{V'(\ln r)} \geq \frac{1}{(4v+3)m} > 0$$

holds for any $a \in \overline{\mathbf{C}}$, with at most $2v$ exceptions.

Otherwise, for every $\Delta\theta_i$ $(i = 0, \ldots, m-1)$, there will be $q = 2v+1$ complex numbers a_i^j $(j = 1, \ldots, 2v+1)$ such that

$$\overline{\lim_{r\to+\infty}} \frac{n(r, \Delta\theta_i, a_i^j)}{V'(\ln r)} < \frac{1}{(4v+3)m}.$$

Then when r is sufficiently close to ∞, we have

$$n(r, \Delta\theta_i, a_i^j) < \frac{V'(\ln r)}{(4v+3)m}. \tag{6}$$

Let α be any positive integer and $\lambda(>1)$ be any positive real number. Put

$$\theta_{i,k} = \frac{2\pi i}{m} + \frac{2\pi k}{\alpha m}, \qquad \Delta_{i,k} = \{z \mid |z| < \lambda^{2\alpha} r, \theta_{i,k} \leq \arg z < \theta_{i,k+1}\},$$

where $0 \leq i \leq m-1, 0 \leq k \leq \alpha - 1$ and $r > 1$. Since

$$\{|z| < \lambda^{2\alpha} r\} = \bigcup_{k=0}^{\alpha-1} \bigcup_{i=0}^{m-1} \Delta_{i,k},$$

there exists a $k_0 \in [0, \alpha-1]$, without loss of generality, we may assume that $k_0 = 0$, such that

$$\sum_{i=0}^{m-1} n_\chi(\Delta_{i,0}, w) \leq \frac{1}{\alpha} n_\chi(\lambda^{2\alpha} r, w).$$

Put

$$\overline{\Delta}_i = \left\{ z \mid \frac{\theta_{i,0}+\theta_{i,1}}{2} \le \arg z \le \frac{\theta_{i+1,0}+\theta_{i+1,1}}{2} \right\},$$

$$\Delta_i^0 = \{z \mid \theta_{i,0} < \arg z < \theta_{i+1,1}\}, 0 \le i \le m-1.$$

Since Δ_i^0 overlaps only with one Δ_j^0 , namely Δ_{i+1}^0 and $\Delta_i^0 \bigcap \Delta_{i+1}^0 = \Delta_{i+1,0}$, then

$$\sum_{i=0}^{m-1} n_\chi(\lambda^{2\alpha} r, \Delta_i^0, w) \le \left(1 + \frac{1}{\alpha}\right) n_\chi(\lambda^{2\alpha} r, w).$$

For $\overline{\Delta}_i \subset \Delta_i^0$, by Lemma 2, we have

$$\begin{aligned}(q-2)S(r, \overline{\Delta}_i, w) \le 2\sum_{j=1}^{q} n(\lambda^{2\alpha} r, \Delta_i^0, w = a_i^j) + \left(1 + \frac{1}{\alpha}\right) n_\chi(\lambda^{2\alpha} r, \Delta_i^0, w) \\ + (q-2)S(\lambda^\alpha, \overline{\Delta}_i, w) + \frac{2A_i \ln r}{(1-K)\ln \lambda} \quad (q = 2v+1).\end{aligned}$$

Adding the above inequality from $i = 0$ to $m-1$ and by (6), it follows that

$$(q-2)S(r, w) \le \frac{2q}{4v+3} V'(\ln \lambda^{2\alpha} r) + \left(1 + \frac{1}{\alpha}\right)^2 n_\chi(\lambda^{2\alpha} r, w) + O(\ln r).$$

Dividing both sides of this inequality by r, and then integrating both sides from 1 to r, by (2), we obtain

$$\begin{aligned}(q-2)T(r, w) &\le \frac{2q}{4v+3} \int_1^r \frac{V'(\ln \lambda^{2\alpha} t)}{t} \mathrm{d}t + \left(1 + \frac{1}{\alpha}\right)^2 \lambda^{2\alpha} N_\chi(\lambda^{2\alpha} r, w) \\ &\quad + (q-2)T(1, w) + o(\ln^2 r) \\ &\le \frac{2q}{4v+3} V(\ln \lambda^{2\alpha} r) + 2(v-1)\left(1 + \frac{1}{\alpha}\right)^2 \lambda^{2\alpha} T(\lambda^{2\alpha} r, w) \\ &\quad + (q-2)T(1, w) + o(\ln^2 r).\end{aligned}$$

Dividing both sides by $V(\ln r)$ and letting $r \to +\infty$. Then from (4) and (5), we have

$$2v - 1 = q - 2 \le \lambda^{2\alpha\rho} \frac{2q}{4v+3} + 2(v-1)\left(1 + \frac{1}{\alpha}\right)^2 \lambda^{2\alpha\rho + 2\alpha}.$$

Let $\lambda \to 1$ and $\alpha \to \infty$, it follows that

$$1 \le \frac{2q}{4v+3} = \frac{4v+2}{4v+3},$$

this is a contradiction. Therefore Step i) holds.

Step ii). By Step i), for any given positive integer $m \ge 4$, there always exists an angular domain

$$\Delta_m = \left\{ z : |\arg z - \theta_m| < \frac{2\pi}{m} \right\}$$

such that
$$\overline{\lim_{r\to+\infty}}\frac{n(r,\Delta_m,a)}{V'(\ln r)}>0$$
for any value of a with at most $2v$ possible exceptions. By choosing a subsequence (we still note it by $\{\theta_m\}$), we can assume that $\theta_m\to\theta_0$, when $m\to\infty$. Then the ray $L(\theta_0)=\{z:\arg z=\theta_0\}$ is called a strong Borel direction of $w(z)$ on $V(\ln r)$. □

Proof of Theorem 2. Let $w(z)$ be a v-valued algebroid function of finite order. Then $\forall\varepsilon\in(0,\pi/2),\forall a\in\overline{\mathbf{C}}$ (with $2v$ exceptional values at most), we have
$$\overline{\lim_{r\to+\infty}}\frac{n(r,\Delta(\theta_0,\varepsilon),a)}{V'(\ln r)}=C_\varepsilon>0.$$
Then $\forall\delta>0$, when r is sufficiently large, we have
$$n(r,\Delta(\theta_0,\varepsilon),a)\le(C_\varepsilon+\delta)V'(\ln r).$$
And there exists $\{r_n\}$ verifying
$$n(r_n,\Delta(\theta_0,\varepsilon),a)\le(C_\varepsilon-\delta)V'(\ln r_n).$$
By Lemma 1, we have
$$n(r,\Delta(\theta_0,\varepsilon),a)\le(C_\varepsilon+\delta))r^{\rho(r)}(\rho(r)+o(1)),$$
$$n(r_n,\Delta(\theta_0,\varepsilon),a)\le(C_\varepsilon-\delta)r_n^{\rho(r_n)}(\rho(r_n)+o(1)).$$
Let $r\to+\infty$ and by $\rho(r)\to\rho$, we have
$$\overline{\lim_{r\to+\infty}}\frac{\ln n^+(r,\Delta(\theta_0,\varepsilon),a)}{\ln r}=\rho.$$
□

Proof of Theorem 3. We prove this theorem in two parts.

Case i). Put
$$B_p=\{a^{p-1}\le|z|<a^{p+2}\}\bigcap\left\{z:|\arg z-\theta_0|<\frac{a-1}{a}\right\},\quad p=1,2,\ldots.$$
For arbitrarily constant $\varepsilon\in(0,\rho)$ and $R>1$, there exists an $a_0\in(1,2)$ (which defined by ε and R) such that for any $a\in(1,a_0)$, the following assertion is true:

If $L(\theta_0)=\{z:\arg z=\theta_0\}$ be a strong Borel direction of $w(z)$ on $V(\ln r)$, then there exists at least a p_0 with $a^{p_0}>R$ so that
$$\overline{n}(B_p,w=b)\ge V^{1-\varepsilon}(\ln a^{p_0})$$
holds for any complex value b except at most $[\frac{(2v-2)(2a^3)^\rho a^3}{\rho C_a\ln 2}]+2$ possible exceptions enclosed by spherical circles of radius $\delta=V^{-1/40}(\ln a^{p_0})$ on the Riemann sphere, where $[x]$ denotes the maximal integer not more than x.

Otherwise, $\exists\varepsilon_0>0,R>1,\forall a_0\in(1,2),\exists a\in(1,a_0),\forall p>\frac{\ln R}{\ln a}$, there are $q=[\frac{(2v-2)(2a^3)^\rho a^3}{\rho C_a\ln 2}]+3$ distinct complex numbers $\{a_j=a_j(p)\}_{j=1}^q$ with spherical distance of any two of them is equal to or larger than $\delta = V^{-1/40}(p \ln a)$, such that

$$\overline{n}(B_p, w = a_j) < V^{1-\varepsilon_0}(p \ln a). \tag{7}$$

Taking $r > R$ arbitrarily and setting $N = [\ln r / \ln a]$, then $a^N < r < a^{N+1}$. For any positive integer M, put

$$r_{p,t} = a^{p+\frac{t}{M}}, \quad t = 0, 1, \ldots, M-1,$$

$$\Delta\left(\theta_0, \frac{a-1}{a}\right) = \left\{z : |\arg z - \theta_0| < \frac{a-1}{a}\right\},$$

$$\Omega_{p,t} = \{r_{p,t} \leq |z| < r_{p,t+1}\} \bigcap \Delta\left(\theta_0, \frac{a-1}{a}\right).$$

Since

$$\{a^{-1} \leq r \leq a^{N+3}\} \bigcap \Delta\left(\theta_0, \frac{a-1}{a}\right) = \bigcup_{t=0}^{M-1} \bigcup_{p=1}^{N+2} \Omega_{p,t},$$

then there is a t_0 $(0 \leq t_0 \leq M-1)$ depending on N, without loss of generality, we may assume that $t_0 = 0$ such that

$$\bigcup_{p=1}^{N+2} n_\chi(\Omega_{p,t}, w) \leq \frac{1}{M} n_\chi(a^{N+3}, w).$$

Put

$$B_p^0 = \left\{\frac{r_{p,0} + r_{p,1}}{2} \leq |z| < \frac{r_{p+1,0} + r_{p+1,1}}{2}\right\} \bigcap \Delta\left(\theta_0, \frac{a-1}{a}\right),$$

$$\overline{B}_p = \{r_{p,0} \leq |z| < r_{p+1,1}\} \bigcap \Delta\left(\theta_0, \frac{a-1}{a}\right).$$

Then $B_p^0 \subset \overline{B}_p \subset B_p$.

Since $\overline{B}_p$ overlaps only with one $\overline{B}_{p'}$, namely $\overline{B}_{p+1}$, and it holds

$$\overline{B}_p \bigcap \overline{B_{p+1}} = \Omega_{p+1,0},$$

thus

$$\sum_{p=1}^{N+1} n_\chi(\overline{B}_p, w) \leq \left(1 + \frac{1}{M}\right) n_\chi(a^{N+3}, w).$$

Obviously, there is a conformal mapping which maps $\{\overline{B}_p\}$ into the unit disc $|\zeta| < 1$ and its center to $\zeta = 0$. Correspondingly, it maps B_p^0 into $|\zeta| < K < 1$. Since, for each $p > \frac{\ln R}{\ln a}$, $\{\overline{B}_p\}$ and $\{B_p^0\}$ are similar. Then K is a positive constant independent of p. From Lemma 3, $\forall a_j$ $(1 \leq j \leq q)$, we have

$$(q-2)S(B_p^0, w) \leq \sum_{j=1}^{q} \overline{n}(\overline{B}_p, a_j) + n_\chi(\overline{B}_p, w) + \frac{2^{51}\pi^{40}v}{(1-K)\delta_p^{38}}.$$

Thus

$$(q-2)\sum_{p=1}^{N+1} S(B_p^0,w) \le \sum_{p=1}^{N+1}\sum_{j=1}^{q}\overline{n}(\overline{B}_p,a_j)+\sum_{p=1}^{N+1} n_\chi(\overline{B}_p,w)+\sum_{p=1}^{N+1}\frac{2^{51}\pi^{40}v}{(1-K)\delta_p^{38}}.$$

By (7), we have

$$\begin{aligned}
&(q-2)\left[S\left(a^{N+1},\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)-S\left(a^2,\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)\right]\\
&\le q(N+1)V^{1-\varepsilon}(\ln a^{N+1})+\left(1+\frac{1}{M}\right)n_\chi(a^{N+3},w)\\
&\quad+\frac{2^{51}\pi^{40}v(N+1)}{(1-K)}V^{\frac{38}{40}}(\ln a^{N+1}).
\end{aligned}\tag{8}$$

Moreover

$$\frac{q(N+1)}{V^{\frac{\varepsilon}{2}}(\ln a^{N+1})}<\frac{q\ln a^{N+1}}{V^{\frac{\varepsilon}{2}}(\ln a^{N+1})}<\frac{q\ln a^{N+1}}{a^{\frac{\varepsilon}{2}(N+1)\rho(a^{N+1})}}\to 0 \quad (N\to+\infty).$$

Taking $N(=[\ln r/\ln a])$ sufficiently large, then r is sufficiently large too and $r\in(a^N,a^{N+1})$. By (8), we have

$$\begin{aligned}
&(q-2)\left[S\left(r,\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)-S\left(a^2,\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)\right]\\
&<(q-2)\left[S\left(a^{N+1},\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)-S\left(a^2,\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)\right]\\
&<V^{1-\frac{\varepsilon}{2}}(\ln a^{N+1})+\left(1+\frac{1}{M}\right)n_\chi(a^{N+3},w)+V^{\frac{39}{40}}(\ln a^{N+1})\\
&\le V^{1-\frac{\varepsilon}{2}}(\ln ar)+\left(1+\frac{1}{M}\right)n_\chi(a^3r,w)+V^{\frac{39}{40}}(\ln ar).
\end{aligned}$$

Dividing both sides of the above inequality by r, and integrating both sides from r to $2r$, we obtain

$$\begin{aligned}
&(q-2)S\left(r,\Delta\left(\theta_0,\frac{a-1}{a}\right),w\right)\ln 2+\text{constant}\\
&\le V^{1-\frac{\varepsilon}{2}}(\ln 2ar)\ln 2+(2v-2)a^3\left(1+\frac{1}{M}\right)T(2a^3r,w)+V^{\frac{39}{40}}(\ln 2ar)\ln 2.
\end{aligned}$$

Dividing them by $V(\ln r)$ and letting $r\to+\infty$, we have

$$\begin{aligned}
&(q-2)\ln 2\cdot\overline{\lim_{r\to+\infty}}\frac{S(r,\Delta(\theta_0,\frac{a-1}{a}),w)}{V(\ln r)}\\
&\le \ln 2\cdot\overline{\lim_{r\to+\infty}}\frac{V^{1-\frac{\varepsilon}{2}}(\ln 2ar)}{V(\ln r)}+(2v-2)a^3\left(1+\frac{1}{M}\right)\overline{\lim_{r\to+\infty}}\frac{T(2a^3r,w)}{V(\ln r)}\\
&\quad+\ln 2\cdot\overline{\lim_{r\to+\infty}}\frac{V^{\frac{39}{40}}(\ln 2ar)}{V(\ln r)}.
\end{aligned}$$

Applying (4), (5) and Lemma 4, then

$$\rho C_a(q-2)\ln 2 \leq (2v-2)\left(1+\frac{1}{M}\right)(2a^3)^\rho a^3.$$

Let $M \to \infty$, then

$$q \leq \frac{(2v-2)(2a^3)^\rho a^3}{\rho C_a \ln 2} + 2,$$

which is contrary to $q = [\frac{(2v-2)(2a^3)^\rho a^3}{\rho C_a \ln 2}] + 3$.

Case ii). Let $\varepsilon_n = \frac{1}{2n}, R_n = 2^n$. From Case i), there are $a_n \in (1, 1+\frac{1}{n})$, p_n and

$$B_{p_n} = \{a_n^{p_n-1} \leq |z| < a_n^{p_n+2}\} \bigcap \left\{|\arg z - \theta_0| < \frac{a_n - 1}{a_n}\right\}, \quad n = 1, 2, \ldots.$$

Let $z_n = a_n^{p_n} e^{i\theta_0}$. Then $|z_n| = a_n^{p_n} > R_n = 2^n \to \infty$ $(n \to \infty)$. Since $(a_n^{p_n+2} - a_n^{p_n+1}) + [a_n^{p_n+1}(a_n - 1)] = 2a_n^{p_n+1}(a_n - 1) < 4a_n^{p_n}(a_n - 1) = r_n 4(a_n - 1)$, we can choose $\sigma_n = 4(a_n - 1) < \frac{4}{n}$, then $\sigma_n \to 0$. Put $D_n = \{z : |z - z_n| < \sigma_n r_n\}$ then $B_{p_n} \subset D_n$, then from Case i)

$$\overline{n}(D_n, w(z) = b) \geq V^{1-\varepsilon_n}(p_n \ln a_n), \quad n = 1, 2, \ldots$$

holds for any complex value $b \in \overline{\mathbf{C}}$ except at most $[\frac{(2v-2)(2a_n^3)^\rho a_n^3}{C_{a_n} \rho \ln 2}] + 2$ possible exceptions enclosed by spherical circles of radius $\delta = V^{-1/40}(p_n \ln a_n)$ on the Riemann sphere. Hence D_n are the filling discs in the strong Borel direction $L(\theta_0)$ of $w(z)$. $\square$

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