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CONJUGACY SEPARABILITY OF GENERALIZED FREE PRODUCTS OF FINITELY GENERATED NILPOTENT GROUPS

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Abstract. In this paper, we prove a criterion of conjugacy separability of generalized free products of polycyclic-by-finite groups with a non-cyclic amalgamated subgroup. Applying this criterion, we prove that certain generalized free products of polycyclic-by-finite groups are conjugacy separable.

1. Introduction

Let S be a subset of a group G. Then G is said to be S-separable if, for each $x \in G \backslash S$, there exists a normal subgroup N_x of finite index in G such that $x \notin N_x S$. If $S = \{1\}$, then G is *residually finite*. If for each $x \in G$, G is $\{x\}^G$-separable, where $\{x\}^G$ is the conjugacy class of x in G, then G is said to be *conjugacy separable*. Residual and separability properties are of interest to both group theorists and topologists. They are related to the solvability of the word problem, conjugacy problem (Mal'cev [10], Mostowski [12]). Topologically they are related to problems on the embeddability of equivariant subspaces in their regular covering spaces (Scott [15], Niblo [13]).

Known classes of conjugacy separable groups are not too many. In [4], Blackburn proved that finitely generated nilpotent groups are conjugacy separable. Formanek [7] showed that polycyclic groups are conjugacy separable. Fine and Rosenberger [6] showed that Fuchsian groups are conjugacy separable.

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In this paper we use generalized free products to get more classes of conjugacy separable groups. Most papers constructed by this way use cyclic amalgamated subgroups (see [14], [5]). Baumslag [2] constructed a generalized free product of two finitely generated nilpotent groups, amalgamating a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$, which is not residually finite. However generalized free products of polycyclic-by-finite groups, amalgamating central subgroups, are conjugacy separable [8]. Recently, Allenby, Kim, and Tang [1] considered the case when the amalgamated subgroup is a direct product of two cyclic groups and showed that most of Seifert groups are conjugacy separable.

In this paper, we consider the conjugacy separability of generalized free products of polycyclic-by-finite groups amalgamating a finite extension of central subgroup. In particular we prove the following criterion:

Corollary 2.8. *Let $G = A *_H B$, where A, B are polycyclic-by-finite groups. Let $C \subset Z(G)$ with $|H : C| < \infty$. Then G is conjugacy separable if and only if,*

(C1) *For $u \in H\backslash C$ and $c \in C$, if $uc \not\sim_G u$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and, in $\hat{G} = \hat{A} *_{\hat{H}} \hat{B}$, $\hat{u}\hat{c} \not\sim_{\hat{G}} \hat{u}$, where $\hat{A} = A/M$, $\hat{B} = B/N$, $\hat{H} = HM/M = HN/N$.*

Using this criterion, we shall prove that certain generalized free products are conjugacy separable in Section 3.

2. A criterion

Throughout this paper we use standard terms and notations.

The letter G always denotes a group.

If $x \in G$, $\{x\}^G$ denotes the set of all conjugates of x in G.

$x \sim_G y$ means x, y are conjugate in G.

$x \not\sim_G y$ means x, y are not conjugate in G.

$Z(G)$ means the center of G and $Z_2(G)$ satisfies $Z_2(G)/Z(G) = Z(G/Z(G))$.

$N \lhd_f G$ means N is a normal subgroup of G with finite index.

If $G = A *_H B$ then $||x||$ denotes the free product length of x in G.

The following results are important for the study of the conjugacy separability of generalized free product, which will be used extensively in this paper:

Theorem 2.1 ([9, Theorem 4.6])**.** *Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclic reduced, and that $x \sim_G y$.*

(1) *If $||x|| = 0$, then $||y|| \leq 1$ and, if $y \in A$, then there exists a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x$.*

(2) *If $||x|| = 1$, then $||y|| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*

(3) *If $||x|| \geq 2$, then $||x|| = ||y||$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

Theorem 2.2 ([5, Theorem 4])**.** *If A and B are conjugacy separable and H is finite, then $G = A *_H B$ is conjugacy separable.*

By this theorem, we can easily see that the generalized free product of finite groups is conjugacy separable, which will be used later. In order to study the conjugacy separability of generalized free product with a non-cyclic amalgamated subgroup, we need the followings.

Definition 2.3. A group G is *H-conjugacy separable* if, for each $x \in G$ such that $\{x\}^G \cap H = \emptyset$, there exists a normal subgroup N of finite index in G such that $\{\overline{x}\}^{\overline{G}} \cap \overline{H} = \emptyset$ where $\overline{G} = G/N$.

Lemma 2.4. *Let A be a polycyclic-by-finite group and $C \lhd A$. If $C \leq H \leq A$ and $|H/C|$ is finite, then A is H-conjugacy separable.*

Proof. Let $a \in A$ and $\{a\}^A \cap H = \emptyset$. Let $\overline{A} = A/C$. Then $\{\overline{a}\}^{\overline{A}} \cap \overline{H} = \emptyset$. Since $\overline{A}$ is also a polycyclic-by-finite group, $\overline{A}$ is conjugacy separable. As $\overline{H}$ is finite, there exists $\overline{N} \lhd_f \overline{A}$ such that $\overline{N}\overline{a} \not\sim_{\overline{A}/\overline{N}} \overline{N}\overline{h}$ for all $\overline{h} \in \overline{H}$. Let N be the preimage of $\overline{N}$ in A. Then we have $\{Na\}^{A/N} \cap NH/N = \emptyset$. □

A group G is *central subgroup separable* if G is H-separable for any finitely generated subgroup H in $Z(G)$. Here we note that $Z(A *_H B) = Z(A) \cap Z(B) \subset A \cap B = H$.

Lemma 2.5. *Let $G = A *_H B$ where A, B are central subgroup separable groups. Let $C \subset Z(G)$ such that $|H : C| < \infty$. Then, for each $M_1 \lhd_f A$ and $N_1 \lhd_f B$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ with $M \subseteq M_1$ and $N \subseteq N_1$.*

Proof. For every $S <_f C$, we consider $\overline{G} = G/S = A/S *_{H/S} B/S$. Since A, B are central subgroup separable, A/S and B/S are residually finite. Since H/S is finite, $\overline{G}$ is residually finite [3]. Let $C_1 = M_1 \cap N_1 \cap C$. Then $C_1 <_f C$. Hence H/C_1 is finite and $\overline{G} = G/C_1$ is residually finite. Thus there exists $\overline{L} \lhd_f \overline{G}$ such that $\overline{L} \cap \overline{H} = 1$. Let L be the preimage of $\overline{L}$ in G. Let $M = M_1 \cap L$ and $N = N_1 \cap L$. Since $L \cap H = C_1$, we have $M \cap H = (M_1 \cap L) \cap H = M_1 \cap C_1 = C_1$ and, similarly, $N \cap H = C_1$. □

Lemma 2.6. *Let $G = A *_H B$, where A, B are central subgroup separable groups. Let $C \subset Z(G)$ with $|H : C| < \infty$. Then G is residually finite and H-separable.*

Proof. Let $1 \neq x \in G$. If $x \notin C$, then, in $\overline{G} = A/C *_{H/C} B/C$, $\overline{x} \neq 1$. Since $\overline{G}$ is residually finite, there exists $\overline{L} \lhd_f \overline{G}$ such that $\overline{x} \notin \overline{L}$. Let L be the preimage of $\overline{L}$ in G. Then $L \lhd_f G$ such that $x \notin L$. If $x \in C$, there exists $S \lhd_f C$ such that $x \notin S$. Let $\overline{G} = A/S *_{H/S} B/S$. Then $\overline{x} \neq 1$. Since $\overline{G}$ is again residually finite, as before we can find $L \lhd_f G$ such that $x \notin L$. Hence G is residually finite.

To show that G is H-separable, let $x \in G\backslash H$. Clearly $\overline{x} \notin \overline{H}$ in $\overline{G} = A/C *_{H/C} B/C$, where $\overline{H} = H/C$. Since $\overline{G}$ is residually finite and $\overline{H}$ is finite, there exists $\overline{L} \lhd_f \overline{G}$ such that $\overline{L} \cap \overline{x}\overline{H} = \emptyset$. Let L be the preimage of $\overline{L}$ in G. Then $x \notin LH$. Hence G is H-separable. □

Theorem 2.7. *Let $G = A *_H B$, where A, B are central subgroup separable groups. Let $C \subset Z(G)$ with $|H : C| < \infty$. Suppose that A, B are H-conjugacy separable and $A/C, B/C$ are conjugacy separable. Then G is conjugacy separable if and only if,*

(C1) *For $u \in H\backslash C$ and $c \in C$, if $uc \not\sim_G u$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and, in $\hat{G} = \hat{A} *_{\hat{H}} \hat{B}$, $\hat{u}\hat{c} \not\sim_{\hat{G}} \hat{u}$, where $\hat{A} = A/M$, $\hat{B} = B/N$, $\hat{H} = HM/M = HN/N$.*

Proof. Suppose that G is conjugacy separable. Let $u \in H\backslash C$ and $c \in C$ such that $uc \not\sim_G u$. Then there exists $T \lhd_f G$ such that in $\overline{G} = G/T$, $\overline{uc} \not\sim_{\overline{G}} \overline{u}$. Let $M = T \cap A$ and $N = T \cap B$. Then $M \lhd_f A$ and $N \lhd_f B$. By Theorem 2.2, $\hat{G} = A/M *_{\hat{H}} B/N$ is conjugacy separable, where $\hat{H} = HM/M = HN/N$. Since there is a natural homomorphism $\phi : \hat{G} \to \overline{G}$, we have $\hat{u}\hat{c} \not\sim_{\hat{G}} \hat{u}$.

Conversely, suppose that condition (C1) is satisfied in G.

Let $x, y \in G$ such that $x \not\sim_G y$. Without loss of generality we can assume that x and y are of minimal length in their conjugacy classes in G. Since G is residually finite (Lemma 2.6), we may assume that $x \neq 1 \neq y$. To prove the theorem, we shall find $M \lhd A$ and $N \lhd B$ such that $M \cap H = N \cap H$ such that $\overline{x} \not\sim_{\overline{G}} \overline{y}$, where $\overline{G} = A/M *_{\overline{H}} B/N$ is conjugacy separable. If this is done, there exists $T \lhd_f G$ such that $xT \not\sim_{G/T} yT$, which means that G is conjugacy separable.

Case 1. $||x|| = ||y|| = 0$.

Subcase 1. $x \in C$ and $y \in H$ (or $y \in C$ and $x \in H$). Suppose that $x \in C$. Clearly, $x \neq y$. Since G is residually finite by Lemma 2.6, there exists $L \lhd_f G$ such that $x^{-1}y \notin L$. Let $\overline{A} = A/(A \cap L)$ and $\overline{B} = B/(B \cap L)$. Consider $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{H} \cong HL/L$. Obviously $\overline{x} \not\sim_{\overline{G}} \overline{y}$ because $x \in C \subset Z(G)$ and $\overline{x} \neq \overline{y}$ by the choice of L. By Theorem 2.2, $\overline{G}$ is conjugacy separable, as required.

Subcase 2. $x, y \in H\backslash C$. Now consider $\tilde{G} = A/C *_{H/C} B/C$. By assumption $A/C, B/C$ are conjugacy separable. Hence $\tilde{G}$ is conjugacy separable. If $\tilde{x} \not\sim_{\tilde{G}} \tilde{y}$, then there is nothing to prove. So we suppose that $\tilde{x} \sim_{\tilde{G}} \tilde{y}$. Then $\tilde{y} = \tilde{g}^{-1}\tilde{x}\tilde{g}$ for some $g \in G$. Hence $y = g^{-1}xgc = g^{-1}xcg$ for some $c \in C$. Thus, since $x \not\sim_G y$, $x \not\sim_G xc$. By condition (C1), there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and, in $\overline{G} = A/M *_{\overline{H}} B/N$, $\overline{xc} \not\sim_{\overline{G}} \overline{x}$, where $\overline{H} = HM/M = HN/N$. Then $\overline{G}$ is conjugacy separable, and $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 2. $||x|| = 1$ and $||y|| = 0$ (or $||y|| = 1$ and $||x|| = 0$).

Without loss of generality, let $x \in A\backslash H$ and $y \in H$. Since x is of the minimal length 1 in its conjugacy class, $\{x\}^G \cap H = \emptyset$. Hence $\{x\}^A \cap H = \emptyset$.

By assumption, A is H-conjugacy separable. There exists $M_1 \lhd_f A$ such that, in A/M_1, $\{xM_1\}^{A/M_1} \cap HM_1/M_1 = \emptyset$. By Lemma 2.5, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \subseteq M_1$, $N \subseteq B$, and $M \cap H = N \cap H$. Let $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = HM/M = HN/N$. Since $A/M, B/N$ are finite, $\overline{G}$ is conjugacy separable. By the choice of $M \subset M_1$, we have $\{\overline{x}\}^{\overline{A}} \cap \overline{H} = \emptyset$. Hence $\overline{x}$ is of the minimal length 1 in its conjugacy class in $\overline{G} = A/M *_{\overline{H}} B/N$. Since $\overline{y} \in \overline{H}$, by Theorem 2.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 3. $||x|| = ||y|| = 1$.

Subcase 1. $x, y \in A\backslash H$ (or $x, y \in B\backslash H$). Since x is of minimal length in its conjugacy class, $\{x\}^G \cap H = \emptyset$. Hence $\{x\}^A \cap H = \emptyset$. By assumption, A is H-conjugacy separable. There exists $M_1 \lhd_f A$ such that, in A/M_1, $\{xM_1\}^{A/M_1} \cap HM_1/M_1 = \emptyset$. Since A is conjugacy separable, there exists $M_2 \lhd_f A$ such that $xM_2 \not\sim_{A/M_2} yM_2$. By Lemma 2.5, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \subseteq M_1 \cap M_2$, $N \subseteq B$, and $M \cap H = N \cap H$. Let $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = HM/M = HN/N$. Since $A/M, B/N$ are finite, $\overline{G}$ is conjugacy separable. By the choice of M, N, we have $\{\overline{x}\}^{\overline{A}} \cap \overline{H} = \emptyset$ and $\overline{x} \not\sim_{\overline{A}} \overline{y}$. Hence, by Theorem 2.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Subcase 2. Suppose that $x \in A\backslash H$ and $y \in B\backslash H$. As in Subcase 1, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$, $\{xM\}^{A/M} \cap HM/M = \emptyset$ and $\{yN\}^{B/N} \cap HN/N = \emptyset$. Let $\overline{G} = A/M *_{\overline{H}} B/N$, where $\overline{H} = HM/M = HN/N$. Then $\overline{x}$ and $\overline{y}$ are of the minimal length 1 in their conjugacy classes, respectively. Hence, by Theorem 2.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 4. $||x|| \neq ||y||$ and $||x|| \geq 2$ (or $||y|| \geq 2$). Since x is of minimal length of its conjugacy class, we can suppose that $x = a_1 b_1 \cdots a_m b_m$, where $a_i \in A\backslash H$ and $b_i \in B\backslash H$. Without loss of generality, suppose that $y = c_1 d_1 \cdots c_n d_n$ where $c_j \in A\backslash H$ and $d_j \in B\backslash H$. Since G is H-separable by Lemma 2.6, there exists $L \lhd_f G$ such that all $a_i, c_j, b_i, d_j \notin HL$. Let $M = A \cap L$ and $N = B \cap L$. Then $M \lhd_f A$ and $N \lhd_f B$, and $\overline{G} = A/M *_{\overline{H}} B/N$ is conjugacy separable, where $\overline{H} = HM/M = HN/N$. Moreover we have $||\overline{x}|| = ||x||, ||\overline{y}|| = ||y||$. Thus $\overline{x}, \overline{y}$ are cyclically reduced and $||\overline{x}|| \geq 2$. It follows that $\overline{x}$ is of minimal length in its conjugacy class. Since $||\overline{x}|| \neq ||\overline{y}||$, by Theorem 2.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required.

Case 5. $||x|| = ||y|| \geq 2$. As before, suppose that $x = a_1 b_1 \cdots a_m b_m$ and $y = c_1 d_1 \cdots c_m d_m$, where $a_i, c_i \in A\backslash H$ and $b_i, d_i \in B\backslash H$. Let $X = \{h^{-1} x^* h |\ h \in H$ and x^* is a cyclic permutation of $x\}$. Since $C \subset Z(G) \subset H$, $|H : Z(G)|$ is finite. It follows that X is finite and $y \notin X$. Since G is residually finite, there exists $L_1 \lhd_f G$ such that $yL_1 \cap \{zL_1 |\ z \in X\} = \emptyset$. Now G is H-separable by Lemma 2.6. As before, there exists $L_2 \lhd_f G$ such that all $a_i, c_i, b_i, d_i \notin HL_2$. By Lemma 2.5, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and $M, N \subseteq L_1 \cap L_2$. Then, in $\overline{G} = A/M *_{\overline{H}} B/N$, we have $||\overline{x}|| = ||x|| = ||y|| = ||\overline{y}||$. Since A/M and B/N are finite, $\overline{G}$ is conjugacy separable. By the choice of L_1, we have $\overline{y} \not\sim_{\overline{H}} \overline{x^*}$. Hence, by Theorem 2.1, $\overline{x} \not\sim_{\overline{G}} \overline{y}$, as required. $\square$

Note that quotient groups of polycyclic-by-finite groups are polycyclic-by-finite and polycyclic-by-finite groups are subgroup separable [11] and conjugacy separable [7]. Hence, by Lemma 2.4, we have the following from Theorem 2.7:

Corollary 2.8. *Let $G = A *_H B$, where A, B are polycyclic-by-finite groups. Let $C \subset Z(G)$ with $|H : C| < \infty$. Then G is conjugacy separable if and only if,*

(C1) *For $u \in H\backslash C$ and $c \in C$, if $uc \not\sim_G u$, there exist $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and, in $\hat{G} = \hat{A} *_{\hat{H}} \hat{B}$, $\hat{u}\hat{c} \not\sim_{\hat{G}} \hat{u}$, where $\hat{A} = A/M$, $\hat{B} = B/N$, $\hat{H} = HM/M = HN/N$.*

3. Conjugacy separability of certain generalized free products

In this section, we prove certain generalized free products of polycyclic-by-finite groups are conjugacy separable.

Theorem 3.1. *Let A, B be isomorphic polycyclic-by-finite groups under the isomorphism ϕ and let $G = A *_H B$, where $H = \phi(H)$. Suppose that $C \subset Z(G)$ with $|H : C| < \infty$. Then G is conjugacy separable.*

Proof. To use Corollary 2.8, we prove that (C1) holds. Let $u \not\sim_G uc$, where $u \in H\backslash C$ and $c \in C$. Then $u \not\sim_A uc$. Since A is conjugacy separable, there exists $M \lhd_f A$ such that $\overline{u} \not\sim_{\overline{A}} \overline{uc}$, where $\overline{A} = A/M$. Then $N = \phi(M) \lhd_f B$ and in $\overline{G} = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{B} = B/\phi(M)$ and $\overline{H} = H/N = \phi(H)/\phi(M)$, we show that $\overline{u} \not\sim_{\overline{G}} \overline{uc}$.

If $\overline{u} \sim_{\overline{G}} \overline{uc}$, then there exist $\overline{h}_i \in \overline{H} = \overline{\phi(H)}$ such that

$$\overline{u} \sim_{\overline{A}} \overline{h}_1 \sim_{\overline{B}} \overline{h}_2 \sim_{\overline{A}} \cdots \sim_{\overline{A}} \overline{h}_r \sim_{\overline{B}} \overline{uc}. \tag{1}$$

Since $h_i = \phi(h_i)$, each of $\overline{h}_s \sim_{\overline{B}} \overline{h}_{s+1}$ implies $\overline{h}_s = \phi^{-1}(\overline{h}_s) \sim_{\phi^{-1}(\overline{B})} \phi^{-1}(\overline{h}_{s+1}) = \overline{h}_{s+1}$. Hence (1) implies

$$\overline{u} \sim_{\overline{A}} \overline{h_1} \sim_{\overline{A}} \overline{h}_2 \sim_{\overline{A}} \overline{h}_3 \sim_{\overline{A}} \cdots \sim_{\overline{A}} \overline{h_r} \sim_{\overline{A}} \overline{uc}. \tag{2}$$

Thus $\overline{u} \sim_{\overline{A}} \overline{uc}$, which contradicts our choice of N. Therefore $\overline{u} \not\sim_{\overline{G}} \overline{uc}$. It follows from Corollary 2.8 that G is conjugacy separable. $\square$

Theorem 3.2. *Let $G = A *_H B$, where A, B are polycyclic-by-finite groups and $H = K \times C$ such that $C \subseteq Z(G)$ and K is finite. Then G is conjugacy separable.*

Proof. Since C is a finitely generated abelian group, $C = K_1 \times C_1$ where K_1 is finite and C_1 is torsion-free. Hence we may assume that C is torsion free. Under this assumption, we show that (C1) holds.

Let $x \in H\backslash C$ and $c \in C$ such that $x \not\sim_G xc$. Since $H = K \times C$, we may assume that $x \in K$. Let $S <_f C$ such that $c \notin S$. Let $\overline{G} = G/S = A/S *_{H/S} B/S$. We shall show that $\overline{x} \not\sim_{\overline{G}} \overline{xc}$. Suppose that $\overline{x} \sim_{\overline{G}} \overline{xc}$. Then, by Theorem 2.1, there is a sequence $\overline{h_1}, \ldots, \overline{h_r}$ of elements in $\overline{H}$ such that $\overline{x} \sim_{\overline{A}} \overline{h_1} \sim_{\overline{B}} \overline{h_2} \sim_{\overline{A}} \cdots \sim_{\overline{A}} \overline{h_r} \sim_{\overline{B}} \overline{xc}$. Since $\overline{x} \sim_{\overline{A}} \overline{h_1}$ for some $a_1 \in A$, we have $\overline{x} = \overline{a_1}^{-1}\overline{h}_1\overline{a}_1$. Let $\overline{h}_1 = \overline{k}_1\overline{c}_1$ where $k_1 \in K$ and $c_1 \in C$. Then $\overline{x}^{-1}\overline{a_1}^{-1}\overline{k}_1\overline{a}_1 = \overline{c}_1^{-1}$. Hence $x^{-1}a_1^{-1}k_1a_1 \in C$. Let $x^{-1}a_1^{-1}k_1a_1 = z \in C$. Then $a_1^{-1}k_1a_1 = xz$. Since $k_1, x \in K$ are of finite orders, let $m = \mathrm{lcm}\{|x|, |k_1|\}$. Then $1 = a_1^{-1}k_1^m a_1 = x^m z^m = z^m$. However C is torsion-free. Hence $z = 1$. Thus $x^{-1}a_1^{-1}k_1a_1 = 1$. It follows that $\overline{c}_1 = 1$ and $\overline{h}_1 = \overline{k}_1$. Similarly, since $\overline{h_1} \sim_{\overline{B}} \overline{h_2}$, if $\overline{h}_2 = \overline{k}_2\overline{c}_2$, $c_2 \in C$ and $k_2 \in K$, then $\overline{c}_2 = 1$ and $\overline{h}_2 = \overline{k}_2$. Inductively, we can see that if $\overline{h}_r = \overline{k}_r\overline{c}_r$, then $\overline{c}_r = 1$ and $\overline{h}_r = \overline{k}_r$. Now, since $\overline{k}_r = \overline{h_r} \sim_{\overline{B}} \overline{y} = \overline{xc}$, as before, we have $\overline{c} = 1$, contradicting the choice of S.

Consequently, $\overline{x} \not\sim_{\overline{G}} \overline{xc}$. Since $\overline{H}$ is finite, $\overline{G}$ is conjugacy separable (Theorem 2.2). Hence there exists $\overline{L} \lhd_f \overline{G}$ such that $\overline{L}\overline{x} \not\sim_{\overline{G}/\overline{L}} \overline{L}\overline{xc}$. Let L be the preimage of $\overline{L}$ in G. Let $M = L \cap A$ and $N = L \cap B$. Then $M \lhd_f A$, $N \lhd_f B$ and $M \cap H = N \cap H$ and, in $\hat{G} = \hat{A} *_{\hat{H}} \hat{B}$, $\hat{x} \not\sim_{\hat{G}} \hat{x}\hat{c}$, where $\hat{A} = A/M, \hat{B} = B/N$, $\hat{H} = HM/M = HN/N$.

This proves that (C1) holds. Hence, by Corollary 2.8, G is conjugacy separable. □

Theorem 3.3. *Let $G = A *_H B$, where A, B are polycyclic-by-finite groups. Let $C \subset Z(G)$ with $|H : C| < \infty$. If $H \subset Z(A)$, then G is conjugacy separable.*

Proof. To show that (C1) holds, let $x \in H\backslash C$ and $c \in C$ such that $x \not\sim_G xc$. Since B is conjugacy separable and $x \not\sim_B xc$, there exists $N_1 \lhd_f B$ such that $N_1x \not\sim_{B/N_1} N_1xc$. Let $N_1 \cap C = S$. Then $S <_f C$. Let $\overline{G} = G/S = A/S *_{H/S} B/S$. We shall show that $\overline{x} \not\sim_{\overline{G}} \overline{xc}$. Suppose that $\overline{x} \sim_{\overline{G}} \overline{xc}$. Then, by Theorem 2.1, there is a sequence $\overline{h_1}, \ldots, \overline{h_r}$ of elements in $\overline{H}$ such that $\overline{x} \sim_{\overline{A}} \overline{h_1} \sim_{\overline{B}} \overline{h_2} \sim_{\overline{A}} \cdots \sim_{\overline{A}} \overline{h_r} \sim_{\overline{B}} \overline{xc}$. Since $H \subset Z(A)$, if $\overline{h} \sim_{\overline{A}} \overline{k}$ for $h, k \in H$ then $\overline{h} = \overline{k}$. Hence we have $\overline{x} = \overline{h}_1 \sim_{\overline{B}} \overline{h}_2 = \overline{h}_3 \sim_{\overline{B}} \cdots \sim_{\overline{B}} \overline{h}_{r-1} = \overline{h}_r \sim_{\overline{B}} \overline{xc}$. Thus $\overline{x} \sim_{\overline{B}} \overline{xc}$ which contradicts our choice of N_1. Therefore, $\overline{x} \not\sim_{\overline{G}} \overline{xc}$. Then, as in the proof of previous theorem, we can find $M \lhd_f A$ and $N \lhd_f B$ such that $M \cap H = N \cap H$ and, in $\hat{G} = \hat{A} *_{\hat{H}} \hat{B}$, $\hat{x} \not\sim_{\hat{G}} \hat{x}\hat{c}$, where $\hat{A} = A/M, \hat{B} = B/N$, $\hat{H} = HM/M = HN/N$.

This proves that (C1) holds. Hence, by Corollary 2.8, G is conjugacy separable. □

Theorem 3.4. *Suppose that $G = A *_H B$, where A, B are finitely generated nilpotent groups. Suppose that $H \subseteq Z_2(A)$, $H \subseteq Z_2(B)$ and $C = Z(A) \cap H = Z(B) \cap H = \langle c \rangle$ with $|H : C| < \infty$. Then G is conjugacy separable.*

Proof. Suppose that $x, y \in H$. If $x \sim_A y$, then there exists $a \in A$ such that $a^{-1}xa = y$. Since $H \subset Z_2(A)$, there exists $z \in Z(A)$ such that $a^{-1}xa = xz$. Thus we have $y^{-1}x = z^{-1} \in H \cap Z(A) = \langle c \rangle$. This means that if the elements x, y of H are conjugate in A then x, y have to be in the same coset of C in H. In the same way, in $\overline{A} = A/\langle c^k \rangle$, if $\overline{x} \sim_{\overline{A}} \overline{y}$, then x, y are in the same coset of C in H. Similarly, in $\overline{B} = B/\langle c^k \rangle$, if $\overline{x} \sim_{\overline{B}} \overline{y}$, then x, y are in the same coset of C in H.

To prove (C1), suppose that $h \in H\backslash C$ and $h \not\sim_G hc^l$. Then $h \not\sim_A hc^l$ and $h \not\sim_B hc^l$.

(1) Suppose that $h \not\sim_A hc^i$ (similarly $h \not\sim_B hc^i$) for all $i \neq 0$. In this case, we have $hc^i \sim_A hc^j$ if and only if $i = j$. Moreover, for any $k > 0$, in $\overline{A} = A/\langle c^k \rangle$, $\overline{h}\overline{c}^i \sim_{\overline{A}} \overline{h}\overline{c}^j$ if and only if $\overline{c}^i = \overline{c}^j$. Since $h \not\sim_B hc^l$, there exists $N \lhd_f B$ such that $Nh \not\sim_{B/N} Nhc^l$. Let $N \cap \langle c \rangle = \langle c^k \rangle$ for some $k > 0$. We shall show that, in $\overline{G} = G/\langle c^k \rangle$, $\overline{h} \not\sim_{\overline{G}} \overline{h}\overline{c}^l$.

If $\overline{h} \sim_{\overline{G}} \overline{h}\overline{c}^l$, then, by Theorem 2.1, there exist $\overline{h}_1, \ldots, \overline{h}_r \in H$ such that $\overline{h} \sim_{\overline{A}} \overline{h}_1 \sim_{\overline{B}} \overline{h}_2 \sim_{\overline{A}} \cdots \sim_{\overline{A}} \overline{h}_r \sim_{\overline{B}} \overline{h}\overline{c}^l$.

We have proved that if $\overline{h}_i \sim_{\overline{A}(\overline{B})} \overline{h}_{i+1}$, then h_i and h_{i+1} are in same coset of C in H. Hence $h, h_1, h_2, \ldots, h_r, hc^l$ are in the coset hC. Therefore, we have $h_i = hc^{l_i}$ for some l_i. Note $\overline{h}\overline{c}^i \sim_{\overline{A}} \overline{h}\overline{c}^j$ if and only if $\overline{c}^i = \overline{c}^j$. Hence, if $\overline{h}_i \sim_{\overline{A}} \overline{h}_{i+1}$, then $\overline{h}\overline{c}^{l_i} \sim_{\overline{A}} \overline{h}\overline{c}^{l_{i+1}}$ which implies $\overline{c}^{l_i} = \overline{c}^{l_{i+1}}$. Thus $\overline{h}_i = \overline{h}_{i+1}$. Therefore we have $\overline{h} = \overline{h}_1 \sim_{\overline{B}} \overline{h}_2 = \cdots = \overline{h}_r \sim_{\overline{B}} \overline{h}\overline{c}^l$. Thus $\overline{h} \sim_{\overline{B}} \overline{h}\overline{c}^l$ which contradicts our choice of N. Therefore $\overline{h} \not\sim_{\overline{G}} \overline{h}\overline{c}^l$.

(2) Suppose that there exist positive integers k_1, k_2 such that $h \sim_A hc^{k_1}$ and $h \sim_B hc^{k_2}$. Since $h \not\sim_A hc^l$, we can choose m to be the minimal positive integer such that $h \not\sim_A hc^m$ but $h \sim_A hc^{m+1}$. Thus we have

$$h \not\sim_A hc^1,\ h \not\sim_A hc^2,\ \ldots,\ h \not\sim_A hc^m, \text{ but } h \sim_A hc^{m+1}.$$

This implies that $h \sim_A hc^i$ if and only if $c^i \in \langle c^{m+1} \rangle$. Similarly, there exists the smallest positive integer n such that $h \not\sim_B hc^n$ and $h \sim_B hc^{n+1}$. This also implies that $h \sim_B hc^i$ if and only if $c^i \in \langle c^{n+1} \rangle$.

Since $h \sim_A hc^{m+1}$ and $h \sim_B hc^{n+1}$, we have

$$h \sim_A hc^{\lambda(m+1)} \sim_B hc^{\lambda(m+1)+\mu(n+1)}$$

for all integers λ, μ. Let $d = \gcd\{m+1, n+1\}$. Hence $h \sim_G hc^{dk}$ for all integer k. Thus, since $h \not\sim_G hc^l$, we have $c^l \notin \langle c^d \rangle$.

Now consider $\overline{G} = G/\langle c^d \rangle = \overline{A} *_{\overline{H}} \overline{B}$, where $\overline{A} = A/\langle c^d \rangle$, $\overline{B} = B/\langle c^d \rangle$, and $\overline{H} = H/\langle c^d \rangle$. We note that

$$\overline{h} \sim_{\overline{A}(\overline{B})} \overline{h}\overline{c}^i \text{ if and only if } \overline{c}^i = 1. \tag{3}$$

We shall show that, in $\overline{G}$, $\overline{h} \not\sim_{\overline{G}} \overline{h}\overline{c}^l$. Suppose that $\overline{h} \sim_{\overline{G}} \overline{h}\overline{c}^l$. By Theorem 2.1, there exist $\overline{h}_1, \ldots, \overline{h}_r \in \overline{H}$ such that $\overline{h} \sim_{\overline{A}} \overline{h}_1 \sim_{\overline{B}} \cdots \sim_{\overline{A}} \overline{h}_r \sim_{\overline{B}} \overline{h}\overline{c}^l$. As before, $h, h_1, h_2, \ldots, h_r, hc^l$ are in the coset hC. Hence $h_i = hc^{l_i}$ for some l_i. Since $\overline{h} \sim_{\overline{A}} \overline{h}_1 = \overline{h}\overline{c}^{l_1}$, we have $\overline{c}^{l_1} = 1$ by (3) and $\overline{h}_1 = \overline{h}$. Since $\overline{h} = \overline{h}_1 \sim_{\overline{B}} \overline{h}_2 = \overline{h}\overline{c}^{l_2}$, we have $\overline{c}^{l_2} = 1$ by (3) and $\overline{h}_2 = \overline{h}$. Similarly, we have $\overline{c}^{l_3} = 1 = \cdots = \overline{c}^{l_r}$ and $\overline{h}_3 = \cdots = \overline{h}_r = \overline{h}$. Finally, since $\overline{h} = \overline{h}_r \sim_{\overline{B}} \overline{h}\overline{c}^l$, we have $\overline{c}^l = 1$ by (3), which contradicts the fact that $c^l \notin \langle c^d \rangle$. Hence we have $\overline{h} \not\sim_{\overline{G}} \overline{h}\overline{c}^l$.

Since $\overline{H}$ is finite, $\overline{G}$ is conjugacy separable (Theorem 2.2). Hence there exists $\overline{T} \lhd_f \overline{G}$ such that $\overline{hT} \not\sim_{\overline{G}/\overline{T}} \overline{h}\overline{c}^l\overline{T}$. Let T be the preimage of $\overline{T}$ in G and let $M = T \cap A$ and $N = T \cap B$. Then $M \cap H = N \cap H$ and, in $\hat{G} = A/M *_{\hat{H}} B/M$, we have $\hat{h} \not\sim_{\hat{G}} \hat{h}\hat{c}^l$. This proves that (C1) holds. Thus, by Corollary 2.8, G is conjugacy separable. □

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