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ON THE ZEROS OF SELF-RECIPROCAL POLYNOMIALS SATISFYING CERTAIN COEFFICIENT CONDITIONS

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Abstract. Kim and Park investigated the distribution of zeros around the unit circle of real self-reciprocal polynomials of even degrees with five terms, where the absolute value of middle coefficient equals the sum of all other coefficients. In this paper, we extend some of their results to the same kinds of polynomials with arbitrary many nonzero terms.

1. Introduction and statement of results

It what follows, U denotes the unit circle and n is a positive integer. All polynomials in this paper will be assumed to have real coefficients. A polynomial $P(z)$ of degree n is said to be self-inversive if it satisfies $P(z) = \pm z^n P(1/z)$. In particular, if $P(z) = z^n P(1/z)$, $P(z)$ is called self-reciprocal. Thus the zeros of a self-reciprocal polynomial either lie on U or occur in pairs conjugate to U. Since the class of self-inversive polynomials of degree n includes polynomials of degree n which have all their zeros on U, it is interesting to mention the condition for a self-reciprocal polynomial having all its zeros on U. There have been a number of literatures (see [1]-[2] and [4]-[12]) about the distribution of zeros of self-reciprocal polynomials.

The simple self-reciprocal polynomials

$$Az^{2m} \pm 2Az^m + A$$

have all their zeros on U. This indicates that zeros of self-reciprocal polynomials (where the absolute value of the middle coefficient equals the sum of all other coefficients) may behave similarly. With this motivation, Kim and Park [5] investigated the distribution of zeros around U of such self-reciprocal polynomials of even degrees with five terms. Our purpose in this paper is to extend some of their results to polynomials with arbitrary many nonzero terms. For

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m a positive integer, we consider self-reciprocal polynomials

$$P(z) = \sum_{k=0}^{2m} a_k z^k, \quad a_{2m} > 0,$$

where the absolute value of middle coefficient equals the sum of all other coefficients, that is,

$$|a_m| = \sum_{k=0}^{2m} a_k - a_m. \tag{1}$$

In addition, we assume that the coefficients strictly between the leading and the middle term have the same signs. More specifically, we investigate the distribution of zeros of $P(z)$ around U when

(A) $a_m \geq 0$, where $a_k \geq 0$ for $1 \leq k \leq m-1$,
(B) $a_m < 0$, where $a_k \geq 0$ for $1 \leq k \leq m-1$,
(C) $a_m \geq 0$, where $a_k \leq 0$ for $1 \leq k \leq m-1$,
(D) $a_m < 0$, where $a_k \leq 0$ for $1 \leq k \leq m-1$.

These four types of polynomials generalize the polynomials studied in [5] in the case when the coefficients strictly between the leading and the middle term have the same signs. Firstly, we may check that all zeros of $P(z)$ in cases (C) and (D) lie on U. This is an easy consequence of Theorem 1 of [8] which is the following.

Theorem 1. *If* $P(z) = \sum_{k=0}^{n} a_k z^k$ *is self-inversive and*

$$|a_n| \geq \frac{1}{2} \sum_{k=0}^{n-1} a_k, \tag{2}$$

holds, then all zeros of $P(z)$ *are on* U.

In fact, one can reformulate the coefficient condition (1) as

$$\frac{|a_m|}{2} = \sum_{k=0}^{m-1} a_k = a_0 + \sum_{k=1}^{m-1} a_k,$$

and in cases (C) and (D)

$$|a_0| = a_0 = -\sum_{k=1}^{m-1} a_k + \frac{|a_m|}{2} = \sum_{k=1}^{m-1} |a_k| + \frac{|a_m|}{2} = \frac{1}{2} \sum_{k=1}^{2m-1} |a_k|.$$

Thus

$$|a_0| = \frac{1}{2} \sum_{k=0}^{2m-1} |a_k|$$

is exactly the condition (2) with equality. Hence we only need to prove cases (A) and (B). In fact, we will prove Theorem 2 below in Section 2.

Theorem 2. *Let d be the greatest common divisor of positive integers k among*

$$1, 2, \ldots, m-1, m+1, \ldots, 2m,$$

where $a_k \neq 0$. Then we have the following.

(a) *If $d \mid m$, then $P(z)$ in case* (A) *has no zeros on U.*

(b) *If $d \mid m$, then, in case* (B), *$P(z)$ has exactly d zeros on U without counting multiplicities. Such zeros are the d-th roots of 1.*

(c) *If $d \nmid m$, then, in cases* (A) *and* (B), *d is even and $P(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1 for* (A) *and the $d/2$-th roots of 1 for* (B), *respectively.*

2. Proof

Proof of Theorem 2. Firstly, we prove (a) and the case (A) of (c). We may assume that $a_m > 0$ because $P(z)$ is the zero polynomial when $a_m = 0$. For $\epsilon > 0$, we define the polynomial

$$P_\epsilon(z) = \sum_{k=0}^{m-1} a_k z^k + (a_m + \epsilon) z^m + \sum_{k=m+1}^{2m} a_k z^k.$$

It follows from the triangle inequality and from the positivity of a_m that, for $|z| = 1$,

$$|P_\epsilon(z)| \geq (a_m + \epsilon) - 2 \sum_{k=0}^{m-1} a_k = \epsilon > 0.$$

Thus $P_\epsilon(z)$ does not have a zero on U, and so $P_\epsilon(z)$ has exactly m zeros strictly inside U, say $\alpha_1, \ldots, \alpha_m$. Assume on the contrary to (a) that P has at least one zero on U. Then for at least one j we have

$$\alpha_j \to z_j := e^{i\theta_j} \in U \ \ (\theta_j \in \mathbb{R}) \ \text{ as } \epsilon \to 0$$

and $P(z_j) = 0$. Since $P(z_j) = 0$, we have

$$\sum_{k=0}^{2m} a_k e^{ik\theta_j} = 0$$

and thus

$$\begin{aligned} 0 = P(z_j) &= e^{im\theta_j} \left(\sum_{k=0}^{m-1} a_k \left(e^{i(m-k)\theta_j} + e^{-i(m-k)\theta_j} \right) + a_m \right) \\ &= e^{im\theta_j} \left(\sum_{k=0}^{m-1} 2a_k \cos(m-k)\theta_j + \sum_{k=0}^{m-1} 2a_k \right). \end{aligned}$$

Therefore

$$\sum_{k=0}^{m-1} a_k \left(\cos(m-k)\theta_j + 1 \right) = 0.$$

Since all terms of the above sum are nonnegative, they must be zero. Hence $a_k \neq 0$ $(k = 0, 1, \ldots, m-1)$ implies $\cos(m-k)\theta_j + 1 = 0$. Using this for $k = 0$ and another $k = 1, 2, \ldots, m-1$, one can conclude that

$$k\theta_j \equiv 0 \,(\mathrm{mod}\ 2\pi) \tag{3}$$

for all k, $0 \leq k \leq 2m$, $k \neq m$ with $a_k \neq 0$. Let d be the greatest common divisor of those positive integers k among

$$1, 2, \ldots, m-1, m+1, \ldots, 2m$$

for which $a_k \neq 0$. Then $z_j = e^{i\theta_j}$ is a d-th root of unity. In case of $d|m$,

$$P(z_j) = 2\sum_{k=0}^{m-1} a_k + a_m z_j^m = a_m(1 + z_j^m) = 2a_m \neq 0,$$

which is a contradiction, proving (a). For the proof of the case (A) of (c), we suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \leq r \leq d-1$. Then d must be even since, for d odd, $d \mid 2m$ and so $d \mid m$. Also $d \mid 2m$ implies that $d \mid 2r$. Put $2r = du$ for some positive integer u. Then $du/2 < d$ and so $u = 1$, that is, $d = 2r$. So

$$P(z_j) = 2\sum_{k=0}^{m-1} a_k + a_m z_j^m = a_m(1 + z_j^m) = a_m(1 + z_j^r) = a_m(1 + z_j^{d/2}).$$

Since the $d/2$-th roots of -1 are contained in the d-th roots of unity, $P(z)$ has zeros on U, and it has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of -1. Next we prove (b) and the case (B) of (c) even though the proofs for these are very similar to the above. For $\epsilon > 0$, we define the polynomial

$$P_{\epsilon,1}(z) = \sum_{k=0}^{m-1} a_k z^k + (a_m - \epsilon)z^m + \sum_{k=m+1}^{2m} a_k z^k.$$

It follows from the triangle inequality and from the negativeness of a_m that, for $|z| = 1$,

$$|P_{\epsilon,1}(z)| \geq (-a_m + \epsilon) - 2\sum_{k=0}^{m-1} a_k = \epsilon > 0.$$

Thus $P_{\epsilon,1}(z)$ does not have a zero on U, and so $P_{\epsilon,1}(z)$ has exactly m zeros strictly inside U, say $\beta_1, \ldots, \beta_m$. Suppose that P has at least one zero on U. Then for at least one j we have

$$\beta_j \to w_j := e^{i\rho_j} \in U \ \ (\rho_j \in \mathbb{R}) \ \text{ as } \epsilon \to 0$$

and $P(w_j) = 0$. Since $P(w_j) = 0$, we have

$$\sum_{k=0}^{2m} a_k e^{ik\rho_j} = 0.$$

We follow the process to get (3) in the proof of (a) so that we obtain

$$k\rho_j \equiv 0 \pmod{2\pi}$$

for all k, $0 \le k \le 2m$, $k \ne m$ with $a_k \ne 0$. So $w_j = e^{i\rho_j}$ is a d-th root of unity, where d was the greatest common divisor of positive integers among all k's, where $0 \le k \le 2m$, $k \ne m$ with $a_k \ne 0$. In case of $d|m$,

$$P(w_j) = 2\sum_{k=0}^{m-1} a_k + a_m w_j^m = a_m(w_j^m - 1).$$

Thus $P(z)$ has zeros on U, and it has exactly d zeros on U that are the d-th roots of unity. We now suppose that $d \nmid m$ and $m = dk + r$ for some integers k, r with $1 \le r \le d-1$. Then we note that $d = 2r$ as before. So

$$P(w) = 2\sum_{k=0}^{m-1} a_k + a_m w^m = a_m(w^m - 1) = a_m(w^r - 1).$$

Since the $d/2$-th roots of unity are contained in the d-th roots of unity, $P(z)$ has exactly $d/2$ zeros on U without counting multiplicities. Such zeros are the $d/2$-th roots of unity. This completes the proof of Theorem 2. □

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