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SOME NEW CHARACTERIZATIONS OF WEIGHTED BERGMAN SPACES

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Abstract. In this paper we obtain some new characterizations for weighted Bergman spaces in the unit ball of $\mathbb{C}^n$.

1. Introduction

Let D be the unit disk in the complex plane and B be the unit ball in the n-dimensional complex Euclidean space $\mathbb{C}^n$. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,\ |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Thus $B = \{z \in \mathbb{C}^n : |z| < 1\}$. Let dv be volume measure on B, normalized so that $v(B) = 1$. We denote by $H(B)$ the class of all holomorphic functions on B.

Suppose $0 < p < \infty$ and $\alpha > -1$, recall that the weighted Bergman space A^p_α consists of those functions $f \in H(B)$ for which

$$\|f\|^p_{A^p_\alpha} = \int_B |f(z)|^p dv_\alpha(z) < \infty,$$

where $dv_\alpha(z) = (1 - |z|^2)^\alpha dv(z)$. When $\alpha = 0$, we get the classical Bergman spaces, which will be denoted by A^p. In the setting of the unit disk, we denote $dv_\alpha(z)$ by $dA_\alpha(z)$.

There are many papers considered the characterization of weighted Bergman spaces, see [2, 3, 6, 7, 9, 11]. In [2], among other results we obtained the following characterization for weighted Bergman spaces.

Theorem A. *Assume that* $f \in H(B)$ *and* $\alpha > -1$. *If* β *and* γ *are real parameters satisfying*

$$(1)\qquad \beta + \gamma = \alpha + p - (n + 1)$$

and

$$(2)\qquad -1 < \beta < p - (n + 1),\ -1 < \gamma < p - (n + 1),$$

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then $f \in A^p_\alpha$ *if and only if*

$$\int_B \int_B \left(\frac{|f(z)-f(w)|}{|1-\langle z,w\rangle|} \right)^p dv_\beta(z) dv_\gamma(w) < \infty. \tag{3}$$

It is very important to give more characterizations for a function space, which is very useful to study operator theory on function spaces. For example, an $f \in H(B)$ is said to belong to the α-Bloch space, denoted by $\mathcal{B}^\alpha(B)$, if

$$\sup_{z\in B} (1-|z|^2)^\alpha |\nabla f(z)| < \infty.$$

However it is difficult to study composition operators on α-Bloch space by the last formula.

In [10], Zhang and Xu introduced a new metric by modifying the Bergman metric, i.e., they gave the following metric (see [10] for more details)

$$F^\alpha_z(w) = \sqrt[2]{\frac{n+1}{2}} \frac{\sqrt[2]{\lambda_\alpha(|z|)|w|^2 + (1-\lambda_\alpha(|z|))|\langle w,z\rangle|^2/|z|^2}}{(1-|z|^2)^\alpha}.$$

Using this new metric, they proved that $f \in \mathcal{B}^\alpha(B)$ if and only if

$$\sup_{z,w\in\mathbb{C}^n\setminus\{0\}} \frac{|\nabla f(z) w|}{F^\alpha_z(w)} < \infty. \tag{4}$$

Using (4), the boundedness and compactness of composition operators on α-Bloch spaces have been completely characterized.

In this paper, we add some other characterizations, including derivative-free characterizations and mixture of derivative characterizations for weighted Bergman spaces in the unit ball of $\mathbb{C}^n$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $b/C \le a \le Cb$.

2. Preliminaries and auxiliary results

Let $\mathrm{Aut}(B)$ be the group of all biholomorphic maps B into B. It is well known that $\mathrm{Aut}(B)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1-\langle z,a\rangle},$$

where $s_a = (1-|a|^2)^{1/2}$, P_a is the orthogonal projection into the space spanned by a, i.e.,

$$P_a z = \frac{\langle z,a\rangle a}{|a|^2},\ |a|^2 = \langle a,a\rangle,\ P_0 z = 0$$

and $Q_a = I - P_a$ (see, e.g. [11]). Moreover, φ_a has the following well-known properties:

$$\varphi_a(0) = a,\ \varphi_a(a) = 0,\ 1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}, \tag{5}$$

$$1 - \langle \varphi_w(z), w\rangle = \frac{1-|w|^2}{1-\langle z, w\rangle}. \tag{6}$$

For $f \in H(B)$, $z \in B$, let

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\Big)$$

denote the complex gradient of f and let $\widetilde{\nabla} f$ denote the invariant gradient of B, i.e.,

$$(\widetilde{\nabla} f)(z) = \nabla(f \circ \varphi_z)(0).$$

Let $d\lambda(z) = (1-|z|^2)^{-n-1} dv(z)$. Then $d\lambda(z)$ is a Möbius invariant measure, that is, for any $\psi \in \mathrm{Aut}(B)$ and $f \in L^1(B)$,

$$\int_B f(z) d\lambda(z) = \int_B f \circ \psi(z) d\lambda(z). \tag{7}$$

To prove the main results of this paper, we need some lemmas. We begin with the following result (see [11]).

Lemma 1. *Assume that $f \in H(B)$, $p > 0$ and $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if $|\widetilde{\nabla} f(z)|$ belongs to $L^p(B, dv_\alpha)$ and if and only if*

$$\int_B (1-|z|^2)^p |\nabla f(z)|^p dv_\alpha < \infty.$$

Lemma 2 ([7])**.** *Suppose $p > 0$, $\alpha > -1$, $0 \le q < p+2$ and $f \in H(B)$. Then $f \in A^p_\alpha$ if and only if*

$$\int_B |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q dv_\alpha(z) < \infty. \tag{8}$$

Furthermore, the quantities

$$\int_B (1-|z|^2)^p |\nabla f(z)|^p dv_\alpha \ \text{ and } \ |f(0)| + \int_B |f(z)|^{p-q} |\widetilde{\nabla} f(z)|^q dv_\alpha(z)$$

are comparable for $f \in H(B)$.

The following result is well known (see Proposition 1.4.10 [8] or Theorem 1.12 of [11]).

Lemma 3. *Suppose $\alpha > -1$ and $t > 0$. Then there exists a constant $C > 0$ such that*

$$\int \frac{(1-|w|^2)^\alpha \, dv(w)}{|1-\langle z, w\rangle|^{n+1+\alpha+t}} \le \frac{C}{(1-|z|^2)^t}$$

for all $z \in B$.

3. Main results and proofs

Now we are in a position to state and prove the main results in this paper.

Theorem 1. *Assume that* $f \in H(B)$, $p > 0$ *and* $\alpha > -1$. *If* β *and* γ *are real parameters satisfying* (1) *and* (2), *then* $f \in A^p_\alpha$ *if and only if*

$$\int_B \int_B \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} \right)^p dv_\beta(z) dv_\gamma(w) < \infty. \tag{9}$$

Proof. Suppose that (9) holds. Since

$$\begin{aligned} & \int_B \int_B \left(\frac{|f(z) - f(w)|}{|1 - \langle z, w \rangle|} \right)^p dv_\beta(z) dv_\gamma(w) \\ \leq & \int_B \int_B \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} \right)^p dv_\beta(z) dv_\gamma(w), \end{aligned} \tag{10}$$

from Theorem A we see that $f \in A^p_\alpha$.

Conversely, suppose that $f \in A^p_\alpha$. Let

$$\begin{aligned} I &= \int_B dv_\beta(z) \int_B \left(\frac{|f(w) - f(z)|}{|w - P_w z - s_w Q_w z|} \right)^p dv_\gamma(w) \\ &= \int_B dv_\beta(z) \int_B \frac{|f(w) - f(z)|^p}{|\varphi_z(w)|^p} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^p}. \end{aligned}$$

Making the change of variables $w \mapsto \varphi_z(w)$ we have

$$I = \int_B dv_\alpha(z) \int_B \frac{|f \circ \varphi_z(w) - f \circ \varphi_z(0)|^p}{|w|^p} \frac{dv_\gamma(w)}{|1 - \langle z, w \rangle|^{2(n+1+\gamma)-p}}.$$

From the proof of Proposition 10 of [2], we obtain

$$I \leq C \int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^p} dv_\beta(z) dv_\gamma(w) < \infty.$$

The proof of the theorem is complete. □

Remark 1. Results of this type can be found in [5], where it was shown that an $f \in H(B)$ belongs to the Besov space $B_p (p > 2n)$ if and only if

$$\int_B \int_B \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} (1 - |z|^2)^{\frac{1}{2}} (1 - |w|^2)^{\frac{1}{2}} \right)^p d\lambda(z) d\lambda(w) < \infty.$$

In the setting of the unit disk, $|w - P_w z - s_w Q_w z|$ is just $|z - w|$. From Theorem 1, we get the following corollary.

Corollary 1. *Assume that* f *is an analytic function in* D, $p > 0$ *and* $\alpha > -1$. *If* β *and* γ *are real parameters satisfying* $\beta + \gamma = \alpha + p - 2$ *and*

$$-1 < \beta < p - 2, \; -1 < \gamma < p - 2,$$

then $f \in A^p_\alpha(D)$ *if and only if*

$$\int_D \int_D \left(\frac{|f(z) - f(w)|}{|z - w|} \right)^p dA_\beta(z) dA_\gamma(w) < \infty.$$

Remark 2. In [2], we proved one of the limit case of the above result, i.e., we proved that $f \in A^p_{-1}(D)(p > 1)$ if and only if

$$\int_D \int_D \left(\frac{|f(z) - f(w)|}{|z - w|} \right)^p dA_\beta(z) dA_\gamma(w) < \infty,$$

where $\beta = \gamma = (p - 3)/2$.

Taking $\beta = \gamma = \frac{p+\alpha-(n+1)}{2}$ in Theorems A and 1 we easily get the following corollary.

Corollary 2. *Assume that* $f \in H(B)$, $\alpha > -1$ *and* $\alpha + n + 1 < p < \infty$. *Then* $f \in A^p_\alpha$ *if and only if*

$$\int_B \int_B \left(\frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} (1 - |z|^2)^{\frac{1}{2}} (1 - |w|^2)^{\frac{1}{2}} \right)^p dv_k(z) dv_k(w) < \infty \tag{11}$$

and if and only if

$$\int_B \int_B \left(\frac{|f(z) - f(w)|}{|1 - \langle z, w \rangle|} (1 - |z|^2)^{\frac{1}{2}} (1 - |w|^2)^{\frac{1}{2}} \right)^p dv_k(z) dv_k(w) < \infty, \tag{12}$$

where $k = (\alpha - (n + 1))/2$.

Remark 3. An example in [9] shows that the formula (12) is no longer true in general when $p = n + 1 + \alpha$. When $0 < p < n + 1 + \alpha$, in [3] we proved that $f \in A^p_\alpha$ if and only if

$$\int_B \int_B \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^p} dv_\alpha(z) dv_\alpha(w) < \infty.$$

Note that the characterization for weighted Bergman space in Theorem A and Lemma 1 are different. Now, motivated by [1, 4], we find a new condition which interpolates these two conditions, i.e., we obtain the following theorem.

Theorem 2. *Assume that* $f \in H(B)$, $0 \le q \le p$, $\alpha > -1$ *and* $p > 2\alpha + 2(n+1)$. *Let* $\beta = p/2 - (n + 1)$ *and* $\gamma = \alpha + p/2$. *Then* $f \in A^p_\alpha$ *if and only if*

$$L := \int_B \int_B \frac{|f(z) - f(w)|^{p-q}}{|1 - \langle z, w \rangle|^p} |\widetilde{\nabla} f(z)|^q dv_\beta(z) dv_\gamma(w) < \infty. \tag{13}$$

Proof. First of all $p > 2\alpha + 2(n+1)$ implies $\beta > \alpha$. Now suppose that (13) holds. Making the change of variables $z \mapsto \varphi_w(z)$ and note that $\alpha = \gamma + \beta - p + n + 1$ and $\beta = p/2 - (n + 1)$, using the formula (6), we have

(14)
$$L = \int_B \int_B |f(z) - f(w)|^{p-q} |\widetilde{\nabla} f(z)|^q \frac{(1 - |z|^2)^{\beta+n+1} (1 - |w|^2)^\gamma}{|1 - \langle z, w \rangle|^p} d\lambda(z) dv(w)$$

$$= \int_B \int_B |f\circ\varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f\circ\varphi_w(z)|^q (1-|z|^2)^{\beta+n+1}$$
$$\times (1-|w|^2)^{\beta+\gamma+n+1-p} d\lambda(z) dv(w)$$
$$= \int_B \int_B |f\circ\varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f\circ\varphi_w(z)|^q dv_\beta(z) dv_\alpha(w).$$

Let $r > 0$ and let $D(z, r)$ denote the Bergman metric ball at z with radius r. By Lemma 2.4 of [11], there exists a positive constant C such that

$$|\nabla g(0)|^p \le C \int_{D(0,r)} |g(z) - g(0)|^p dv_\beta(z)$$

for all $g \in H(B)$. Replace g by $f \circ \varphi_w$, we obtain

$$\begin{aligned} |\widetilde{\nabla} f(w)|^p &\le C \int_{D(0,r)} |f\circ\varphi_w(z) - f(w)|^p dv_\beta(z) \\ &\le C \int_B |f\circ\varphi_w(z) - f(w)|^p dv_\beta(z). \end{aligned} \tag{15}$$

Fix $w \in B$ for a moment and let $F_w(z) = f \circ \varphi_w(z) - f(w)$. Noting that $F_w(0) = 0$, by Lemma 2, we have

$$\int_B |F_w(z)|^p dv_\beta(z) \le C \int_B |F_w(z)|^{p-q} |\widetilde{\nabla} F_w(z)|^q dv_\beta(z). \tag{16}$$

In view of (14-16), we obtain

$$\begin{aligned} &\int_B |\widetilde{\nabla} f(w)|^p dv_\alpha(w) \\ \le\ & C \int_B \int_B |F_w(z)|^p dv_\beta(z) dv_\alpha(w) \\ \le\ & C \int_B \int_B |f\circ\varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f\circ\varphi_w(z)|^q dv_\beta(z) dv_\alpha(w) \\ =\ & C \int_B \int_B \frac{|f(z) - f(w)|^{p-q}}{|1 - \langle z, w\rangle|^p} |\widetilde{\nabla} f(z)|^q dv_\beta(z) dv_\gamma(w) \\ =\ & CL < \infty. \end{aligned}$$

It follows from Lemma 1 that $f \in A^p_\alpha$.

Conversely, suppose that $f \in A^p_\alpha$. Since $\beta > \alpha$, we see that $f \in A^p_\beta$. By Lemma 2 and Theorem 2.16 of [11], it follows that

$$\begin{aligned} &\int_B |f\circ\varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f\circ\varphi_w(z)|^q dv_\beta(z) \\ \le\ & C \int_B |f\circ\varphi_w(z) - f(w)|^p dv_\beta(z) \\ \le\ & C \int_B |\widetilde{\nabla} f\circ\varphi_w(z)|^p dv_\beta(z) \end{aligned}$$

$$= C\int_B |\widetilde{\nabla} f(z)|^p (1-|\varphi_w(z)|^2)^{\beta+n+1} d\lambda(z). \tag{17}$$

From (14) and (17) we obtain

$$\begin{aligned} L &= \int_B\int_B |f\circ\varphi_w(z)-f(w)|^{p-q}|\widetilde{\nabla} f\circ\varphi_w(z)|^q dv_\beta(z)dv_\alpha(w) \\ &\le C\int_B\int_B |\widetilde{\nabla} f(z)|^p(1-|\varphi_w(z)|^2)^{\beta+n+1}d\lambda(z)dv_\alpha(w). \end{aligned}$$

By using Fubini's Theorem, Lemma 3 and the condition $p > 2\alpha+2(n+1)$ we obtain

$$\begin{aligned} L &\le C\int_B |\widetilde{\nabla} f(z)|^p d\lambda(z)\int_B (1-|\varphi_w(z)|^2)^{\beta+n+1}(1-|w|^2)^\alpha dv(w) \\ &= C\int_B |\widetilde{\nabla} f(z)|^p dv_\beta(z)\int_B \frac{(1-|w|^2)^{\alpha+\beta+n+1}}{|1-\langle z,w\rangle|^{2(\beta+n+1)}}dv(w) \\ &\le C\int_B |\widetilde{\nabla} f(z)|^p dv_\alpha(z) < \infty, \end{aligned}$$

as desired. □

Theorem 3. *Suppose that $f\in H(B)$, $0<p<\infty$ and $\alpha>-1$. Then $f\in A^p_\alpha$ if and only if*

$$K := \int_B\int_B \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2n+2}}dv_{\frac{n+1+\alpha}{2}}(z)dv_{\frac{n+1+\alpha}{2}}(w) < \infty. \tag{18}$$

Proof. First we assume that (18) holds. For a fixed $r\in(0,1)$, let

$$E(a,r) = \{z\in B : |\varphi_a(z)| < r\}.$$

Set $|E(a,r)| = v(E(a,r))$. From [11] we see that

$$(1-|z|^2)^{n+1} \asymp (1-|a|^2)^{n+1} \asymp |1-\langle a,z\rangle|^{n+1} \asymp |E(a,r)| \tag{19}$$

when $z\in E(a,r)$. It is easy to see (using Cauchy's estimate for example) that there exists a positive constant C such that

$$|\nabla f(0)|^p \le C\int_{E(0,r)} |f(z)-f(0)|^p\, dv(z)$$

for all $f\in H(B)$. Replace f by $f\circ\varphi_w$ and make a change of variables, we get

$$|\widetilde{\nabla} f(w)|^p \le C\int_{E(w,r)} |f(w)-f(z)|^p \frac{(1-|w|^2)^{n+1}}{|1-\langle z,w\rangle|^{2(n+1)}}\, dv(z)$$

for all $f\in H(B)$ and $w\in B$. From the last inequality, (19) and Lemma 2.14 of [11], we have

$$\begin{aligned} &(1-|w|^2)^{p+\alpha}|\nabla f(w)|^p \\ \le\ &(1-|w|^2)^\alpha|\widetilde{\nabla} f(w)|^p \end{aligned}$$

$$\leq \frac{C}{(1-|w|^2)^{n+1-\alpha}} \int_{E(w,r)} |f(z)-f(w)|^p dv(z)$$
$$\leq C \int_{E(w,r)} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2n+2}} (1-|z|^2)^{\frac{n+1+\alpha}{2}} (1-|w|^2)^{\frac{n+1+\alpha}{2}} dv(z).$$

Integrating both sides with respect to dv on B, by (19) we get

$$\begin{aligned}
&\int_B (1-|w|^2)^p |\nabla f(w)|^p dv_\alpha \\
= &\int_B (1-|w|^2)^{p+\alpha} |\nabla f(w)|^p dv \\
\leq\ & C \int_B \int_{E(w,r)} \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2n+2}} (1-|z|^2)^{\frac{n+1+\alpha}{2}} (1-|w|^2)^{\frac{n+1+\alpha}{2}} dv(z)dv(w) \\
\leq\ & C \int_B \int_B \frac{|f(z)-f(w)|^p}{|1-\langle z,w\rangle|^{2n+2}} dv_{\frac{n+1+\alpha}{2}}(z) dv_{\frac{n+1+\alpha}{2}}(w) \\
= &\ CK < \infty.
\end{aligned}$$

Hence $f \in A^p_\alpha$.

Conversely, suppose that $f \in A^p$. By Lemma 3 we obtain,

$$\begin{aligned}
K \leq\ & C \int_B \int_B \frac{|f(z)|^p}{|1-\langle z,w\rangle|^{2(n+1)}} (1-|z|^2)^{\frac{n+1+\alpha}{2}} (1-|w|^2)^{\frac{n+1+\alpha}{2}} dv(z)dv(w) \\
& + C \int_B \int_B \frac{|f(w)|^p}{|1-\langle z,w\rangle|^{2(n+1)}} (1-|z|^2)^{\frac{n+1+\alpha}{2}} (1-|w|^2)^{\frac{n+1+\alpha}{2}} dv(z)dv(w) \\
\leq\ & C \int_B |f(z)|^p (1-|z|^2)^{\frac{n+1+\alpha}{2}} dv(z) \int_B \frac{(1-|w|^2)^{\frac{n+1+\alpha}{2}}}{|1-\langle z,w\rangle|^{2(n+1)}} dv(w) \\
& + C \int_B |f(w)|^p (1-|w|^2)^{\frac{n+1+\alpha}{2}} dv(w) \int_B \frac{(1-|z|^2)^{\frac{n+1+\alpha}{2}}}{|1-\langle z,w\rangle|^{2(n+1)}} dv(z) \\
\leq\ & C \int_B |f(z)|^p dv_\alpha(z) + C \int_B |f(w)|^p dv_\alpha(w) \\
\leq\ & C \int_B |f(z)|^p dv_\alpha(z) < \infty.
\end{aligned}$$

The proof is complete. □

Motivated by Theorem 3, similarly to Theorem 2 we obtain the following theorem.

Theorem 4. *Assume that $f \in H(B)$, $0 < p < \infty$, $0 \leq q < p+2$ and $\alpha > -1$. Then $f \in A^p_\alpha$ if and only if*

$$M := \int_B \int_B \frac{|f(z)-f(w)|^{p-q}}{|1-\langle z,w\rangle|^{2n+2}} |\widetilde{\nabla} f(z)|^q dv(z) dv_{n+1+\alpha}(w) < \infty. \tag{20}$$

Proof. Suppose that (20) holds. By the proof of Theorem 2 we see that

$$|\widetilde{\nabla} f(w)|^p \leq C \int_B |f \circ \varphi_w(z) - f(w)|^p dv(z). \tag{21}$$

After some computation, we get

(22)

$$\begin{aligned} M &= \int_B \int_B |f(z) - f(w)|^{p-q} |\widetilde{\nabla} f(z)|^q \frac{(1-|w|^2)^{n+1}(1-|z|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} d\lambda(z) dv_\alpha(w) \\ &= \int_B \int_B |f \circ \varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f \circ \varphi_w(z)|^q dv(z) dv_\alpha(w). \end{aligned}$$

Fix $w \in B$ at this moment, let $F_w(z) = f \circ \varphi_w(z) - f(w)$. Note that $F_w(0) = 0$, by Lemma 2 we have

$$\int_B |F_w(z)|^p dv(z) \leq C \int_B |F_w(z)|^{p-q} |\widetilde{\nabla} F_w(z)|^q dv(z). \tag{23}$$

From (22) and (23), similarly to the proof of Theorem 2, we get

$$\begin{aligned} & \int_B |\widetilde{\nabla} f(w)|^p dv_\alpha(w) \\ \leq\ & C \int_B \int_B |f \circ \varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f \circ \varphi_w(z)|^q dv(z) dv_\alpha(w) \\ =\ & C \int_B \int_B \frac{|f(z) - f(w)|^{p-q}}{|1-\langle z, w\rangle|^{2n+2}} |\widetilde{\nabla} f(z)|^q dv(z) dv_{n+1+\alpha}(w) = CM < \infty. \end{aligned}$$

Hence $f \in A^p_\alpha$.

Conversely, suppose that $f \in A^p_\alpha$. From (17) and (23) we obtain

$$\begin{aligned} M &= \int_B \int_B |f \circ \varphi_w(z) - f(w)|^{p-q} |\widetilde{\nabla} f \circ \varphi_w(z)|^q dv(z) dv_\alpha(w) \\ &\leq C \int_B \int_B |\widetilde{\nabla} f(z)|^p \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} dv(z) dv_\alpha(w). \end{aligned}$$

Applying Fubini's Theorem, Lemmas 1 and 3 we have

$$\begin{aligned} M &\leq C \int_B |\widetilde{\nabla} f(z)|^p dv(z) \int_B \frac{(1-|w|^2)^{n+1}}{|1-\langle z, w\rangle|^{2(n+1)}} dv_\alpha(w) \\ &\leq C \int_B |\widetilde{\nabla} f(z)|^p dv_\alpha(z) < \infty. \end{aligned}$$

The proof is complete. □

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