Bull. Korean Math. Soc. **47** (2010), No. 6, pp. 1163–1170
DOI 10.4134/BKMS.2010.47.6.1163

FINITE TYPE CURVE IN 3-DIMENSIONAL SASAKIAN MANIFOLD

Cetin Camci and H. Hilmi Hacisalihoglu

Abstract. We study finite type curve in $R^3(-3)$ which lies in a cylinder $N^2(c)$. Baikousis and Blair proved that a Legendre curve in $R^3(-3)$ of constant curvature lies in cylinder $N^2(c)$ and is a 1-type curve, conversely, a 1-type Legendre curve is of constant curvature. In this paper, we will prove that a 1-type curve lying in a cylinder $N^2(c)$ has a constant curvature. Furthermore we will prove that a curve in $R^3(-3)$ which lies in a cylinder $N^2(c)$ is finite type if and only if the curve is 1-type.

1. Introduction

If a differentiable $(2n+1)$-dimensional manifold M carries a global differential 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M, it is called contact manifold and η is called contact structure [3]. It is well known that there exists a unique vector ξ such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$. ξ is called characteristic vector field of the contact structure on η. D is said to be contact distribution defined by

$$D = \{X \in \chi(M) : \eta(X) = 0\}.$$

1-dimensional integral submanifold of M is called a Legendre curve [2]. (ϕ, ξ, η) is called an almost contact structure on M where ϕ, η and ξ are type $(1,1)$, $(0,1)$ and $(1,0)$, respectively, satisfying the equations $\phi^2 X = -X + \eta(X)\xi$ and $\phi(\xi) = 0$ [3]. (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is almost contact structure and g is a Riemannian metric satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

(ϕ, ξ, η, g) is called a contact metric structure on M if it is an almost contact metric on M and such that $d\eta(X, Y) = g(X, \phi Y)$ [3]. It is well known that a contact metric structure is a contact structure. A $(2n+1)$-dimensional manifold is said to be a Sasakian manifold if it is endowed with a normal contact

Received March 11, 2009; Revised November 10, 2009.
2000 *Mathematics Subject Classification.* Primary 53C15; Secondary 53C25.
Key words and phrases. Sasakian Manifold, Legendre curve, finite type curve.

metric structure $(\phi, \xi, \eta, g, \xi)$ [3]. It is well known that an almost contact metric structure $(\phi, \xi, \eta, g, \xi)$ on M is a Sasakian structure if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{1.1}$$

where $X, Y \in \chi(M)$ [3]. Let (x, y, z) be standard coordinates on R^3. Consider the 1-form $\eta = \frac{1}{2}(dz - ydx)$ on R^3. In this case if we put $\xi = 2\frac{\partial}{\partial z}$, then we obtain $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Thus ξ is a characteristic vector field. We consider the endomorphism of ϕ defined by matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{bmatrix}.$$

Thus we have $\phi^2 X = -X + \eta(X)\xi$. Hence (ϕ, ξ, η) is an almost contact structure on R^3 [3]. If the Riemannian metric is defined by

$$ds^2 = \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta,$$

then it is well know that (ϕ, ξ, η, g) is a Sasakian structure. In this space, the sectional curvature of all plane sections orthogonal to ξ is equal to -3. $(R^3, \phi, \xi, \eta, g,)$ is denoted shorty by $R^3(-3)$ [2]. It is well known that

$$\left\{ e = e_1 = 2\frac{\partial}{\partial y}, \quad \phi e = e_2 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad \xi = e_3 = 2\frac{\partial}{\partial z} \right\}$$

is orthonormal basis with respect to g.

In a 3-dimensional contact metric manifold, if the angle between tangent vector of the curve and the Reeb vector field is a constant, then it is said that the curve is a slant curve [7]. Cho et al. have investigated slant curves in a Sasakian 3-manifold and stated the following theorem. A non-geodesic curve in a Sasakian 3-manifold M is a slant curve if and only if $\eta(n) = g(n, \xi) = 0$, where n is a normal vector of the curve [7].

Let $R^3(-3)$ be an isometric immersion, where M is a compact submanifold of $R^3(-3)$. Then we have spectral decomposition

$$x^A - (x^A)_0 = \sum_{t=p_A}^{q_A} (x^A)^t, \tag{1.2}$$

where $p_A = \{\inf t : (x_A)_t \neq 0\}$ and $q_A = \{\sup t : (x_A)_t \neq 0\}$. If $p = \inf\{p_A\}$ and $q = \sup\{q_A\}$, then we have the following spectral decomposition

$$\phi^2 x = \phi^2 x_0 + \sum_{t=p}^{q} \phi^2 x_t, \tag{1.3}$$

where $\Delta g(x_t, e_A) = \lambda_t g(x_t, e_A)$ ([1], [2]). In spectral decomposition, if there are exactly k nonzero of $\phi^2 x_t$, then we say that the submanifold M of $R^3(-3)$ is k-type ([1], [2]).

Furthermore, in $R^3(-3)$ Sasakian space, the cylinder is defined by

$$N^2(r) = \{X \in R^3(-3) : g(x - x_0, x - x_0) - \eta(x - x_0) = r^2\}$$

or

$$N^2(r) = \{X = (x, y, z) \in R^3(-3) : (x - x_0)^2 + (y - y_0) = (2r)^2\}, \tag{1.4}$$

where r is a constant [2].

Takahashi [8] showed that a submanifold of Euclidean space is 1-type if and only if it is a minimal submanifold of a hypersphere. As a result of Takahashi's theorem, a curve in 2-sphere in Euclidean 3-space is 1-type if and only if it is circle. Furthermore Chen et al. [6] showed that a finite type curve 2-sphere in Euclidean 3-space is a circle.

Extension of Takahashi's theorem is given by Baikousis and Blair [1]. They showed that a compact m-dimensional integral submanifold of $R^{2n+1}(-3)$ is of 1-type if and only if it is minimal in cylinder $N^{2n}(c)$ [1]. Furthermore, they showed that a Legendre curve in $R^3(-3)$ of constant curvature lying in a cylinder $N^2(r)$ is a 1-type curve. Conversely, a 1-type Legendre curve is of constant curvature. They conjectured that a finite type curve lying in a cylinder $N^2(c)$ is of constant curvature in the metric of $R^3(-3)$ [2]. In this paper, we prove this conjecture.

2. Finite type curve in cylinder $N^2(r)$ of $R^3(-3)$

Let γ be a curve with unit coordinate neighborhood (I, γ) and s be arc-parameter for the curve. Define the curve by

$$\gamma(s) = (x(s), y(s), z(s))$$

in $N^2(r)$, then we have

$$(x - x_0)^2 + (y - y_0)^2 = (2r)^2.$$

Thus we write $x(s) = 2r \cos\theta(s) + x_0$, $y(s) = 2r \sin\theta(s) + y_0$,

$$\gamma(s) - \gamma_0 = (2r\cos\theta(s), 2r\sin\theta(s), z(s) - z_0)$$

and

$$\gamma - \gamma_0 = r\sin\theta e + r\cos\theta\phi e + \left(\frac{z(s) - z_0}{2}\right)\xi, \tag{2.1}$$

where $\gamma_0 = (x_0, y_0, z_0)$.

Lemma 2.1 ([4]). *Let γ be a curve which lies in a cylinder $N^2(r)$. The curve is 1-type if and only if θ' is a constant.*

Proof. If the curve is a 1-type, then we obtain

$$\triangle g(\gamma - \gamma_0, e_i) = \lambda g(\gamma - \gamma_0, e_i),\ i = 1, 2.$$

It is well known that the Laplacian of the curve is given by

$$\triangle f = -X_1 X_1 f = -f'', \tag{2.2}$$

where $X_1 = \gamma'$ is the unique tangent vector of the curve [1]. Thus from the equations (2.1) and (2.2), we see that

$$-r\theta'' cos\theta + r(\theta')^2 \sin\theta = \lambda r \sin\theta$$

and

$$r\theta'' \sin\theta + r(\theta')^2 \cos\theta = \lambda r \cos\theta.$$

Thus we easily see that θ' is a constant.

Conversely, suppose that $\theta' = c$ is a constant. Thus we write

$$\theta(s) = cs + c_1.$$

From the equation (2.1), we have

$$g(\gamma - \gamma_0, e) = r\sin\theta, \quad g(\gamma - \gamma_0, \phi e) = r\cos\theta.$$

Thus from the equations (2.1) and (2.2), we obtain

$$\triangle g(\gamma - \gamma_0, e) = c^2 g(\gamma - \gamma_0, e)$$

and

$$\triangle g(\gamma - \gamma_0, \phi e) = c^2 g(\gamma - \gamma_0, \phi e).$$

Then we see that the curve is 1-type. □

Theorem 2.1 ([4])**.** *Let M be a curve which lies in cylinder $N^2(r)$ with (I, γ) unit coordinate neighborhood. If the curve is 1-type, then it has a constant curvature.*

Proof. From the equation (2.1), if we take the derivative along γ, then we obtain

$$t = \gamma'(s) = (-2r\theta'(s)\sin\theta(s),\ 2r\theta'(s)\cos\theta(s),\ z'(s)),$$

where $x' = -2r\theta'\sin\theta$ and $y' = 2r\theta'\cos\theta$. Furthermore because of

$$\frac{\partial}{\partial y} = \frac{1}{2}e,\ \frac{\partial}{\partial x} = \frac{1}{2}(\phi e - y\xi),\ \frac{\partial}{\partial z} = \frac{1}{2}\xi,$$

then we have

$$t = \frac{1}{2}(y'e + x'\phi e) + \sigma\xi \tag{2.3}$$

and

$$1 = r^2(\theta')^2 + \sigma^2. \tag{2.4}$$

Thus we have

$$\nabla_t t = (\frac{1}{2}y'' + \sigma x')e + (\frac{1}{2}x'' - \sigma y')\phi e + \sigma'\xi. \tag{2.5}$$

From the equation (2.5) we obtain

$$\kappa^2 = \frac{1}{4}((x'')^2 + (y'')^2) + \sigma(y''x^{'} - x''y') + \sigma^2((x')^2 + (y')^2) + (\sigma')^2, \tag{2.6}$$

where

$$x' = -2r\theta^{'}\sin\theta,\ x^{''} = -2r\theta^{''}\sin\theta - 2r(\theta^{'})^2\cos\theta \tag{2.7}$$

and

$$y' = 2r\theta^{'}\cos\theta,\ y^{''} = 2r\theta^{''}\cos\theta - 2r(\theta^{'})^2\sin\theta. \tag{2.8}$$

From the equations (2.6), (2.7) and (2.8), we have

$$\kappa^2 = r^2(\theta')^4 + 4r^2(\theta^{'})^3\sqrt{(1 - r^2(\theta')^2} + 4r^2(\theta')^2(1 - r^2(\theta')^2) + \frac{r^2(\theta'')^2}{1 - r^2(\theta')^2}.$$

Assume that the curve is a 1-type then from Lemma 1 and the equation above, θ' is a constant. Therefore the curvature of the curve is a constant. □

Remark 2.1 ([4]). The converse of the above theorem is not true. If we define the function F from open interval $(-\frac{1}{r}, \frac{1}{r})$ to $(-\frac{1}{r}, \frac{1}{r})$ by

$$F(y) = \int_{y_0}^{y} \frac{du}{\sqrt{1 - r^2u^2}\sqrt{\frac{k^2}{r^2} + (4r^2 - 1)u^4 - 5u^3 - 4u^3\sqrt{1 - r^2u^2}}},$$

then we have

$$F'(y) = \frac{1}{\sqrt{1 - r^2y^2}\sqrt{\frac{k^2}{r^2} + (4r^2 - 1)y^4 - 5y^3 - 4y^3\sqrt{1 - r^2y^2}}},$$

where k is a constant and $y, y_0 \in (-\frac{1}{r}, \frac{1}{r})$. Since $F'(y) > 0$, there exists an inverse function of F. Define $\theta(s)$ by

$$\theta(s) = \int_{s_0}^{s} F^{-1}(h)dh,$$

then we see that κ is equal to k. Since θ' is not constant, the curve is not of 1-type.

Corollary 2.1. *Let γ be a curve which lies in a cylinder $N^2(r)$ in $R^3(-3)$. From the equation* (2.4)*, we easily see that the curve is a slant curve if and only if θ' is a constant.*

Theorem 2.2 ([4])**.** *Let γ be a curve which lies in a cylinder $N^2(r)$ in $R^3(-3)$. Then the curve is of finite type if and only if it is of 1-type.*

Proof. Let γ be a curve with unit coordinate neighborhood (I, γ) and s be arc-parameter for the curve. If the curve finite k-type, then we have

$$\phi^2(\gamma - \gamma_0) = \phi^2\gamma_1 + \phi^2\gamma_2 + \cdots + \phi^2\gamma_k$$

and

$$\triangle g(\gamma_i, e_j) = \lambda_i g(\gamma_i, e_j),$$

where $i = 1, 2, \ldots, k$ and $j = 1, 2$. From the above equation and (2.2), we have

$$f'' + \lambda_i f = 0,$$

where $f = g(\gamma_i, e_j)$. Integrating the above equation, we have

$$\begin{aligned}\gamma_i(s) = {} & (A_{i1}\cos(\sqrt{\lambda_i}s) + A_{i2}\sin(\sqrt{\lambda_i}s))e \\ & + (B_{i1}\cos(\sqrt{\lambda_i}s) + B_{i2}\sin(\sqrt{\lambda_i}s))\phi e + \mu\xi.\end{aligned}$$

Thus we obtain

$$g(\gamma-\gamma_0,e)=\sum_{i=1}^{k}(A_{i1}\cos(\sqrt{\lambda_i}s)+A_{i2}\sin(\sqrt{\lambda_i}s)) \tag{2.9}$$

and

$$g(\gamma-\gamma_0,\phi e)=\sum_{i=1}^{k}(B_{i1}\cos(\sqrt{\lambda_i}s)+B_{i2}\sin(\sqrt{\lambda_i}s)). \tag{2.10}$$

From the equations (2.1), (2.9) and (2.10), we have

$$\begin{aligned}&(\sum_{i=1}^{k}(A_{i1}\cos(\sqrt{\lambda_i}s)+A_{i2}\sin(\sqrt{\lambda_i}s)))^2\\&+(\sum_{i=1}^{k}(B_{i1}\cos(\sqrt{\lambda_i}s)+B_{i2}\sin(\sqrt{\lambda_i}s)))^2=r^2.\end{aligned}$$

If we take the derivative of the equation above, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\sqrt{\lambda_i}((A_{i1}A_{i2}+B_{i1}B_{i2})\cos 2\sqrt{\lambda_i}s+\frac{1}{2}(A_{i2}^2+B_{i2}^2-A_{i1}^2-B_{i1}^2)\sin 2\sqrt{\lambda_i}s)\\&+\sum_{i\neq j}(-\sqrt{\lambda_j}(A_{i1}A_{j1}+B_{i1}B_{j1})+\sqrt{\lambda_i}(A_{i2}A_{j2}+B_{i2}B_{j2}))\sin\sqrt{\lambda_i}s\cos\sqrt{\lambda_j}s\\&+\sum_{i\neq j}\sqrt{\lambda_j}(A_{i1}A_{j2}+B_{i1}B_{j2})\cos\sqrt{\lambda_i}s\cos\sqrt{\lambda_j}s\\&-\sum_{i\neq j}\sqrt{\lambda_j}(A_{i2}A_{j1}+B_{i2}B_{j1})\sin\sqrt{\lambda_i}s\sin\sqrt{\lambda_j}s)=0.\end{aligned}$$

Since $\cos(2\sqrt{\lambda_i}s)$, $\cos(2\sqrt{\lambda_j}s)$, $\sin(\sqrt{\lambda_i}s)$, $\cos(\sqrt{\lambda_j}s)$, $\sin(\sqrt{\lambda_i}s)\sin(\sqrt{\lambda_j}s)$, $\cos(\sqrt{\lambda_i}s)\cos(\sqrt{\lambda_j}s)$ are linear independent, we obtain

$$A_{i2}^2+B_{i2}^2=A_{i1}^2+B_{i1}^2, \tag{2.11}$$

$$A_{i1}A_{i2}+B_{i1}B_{i2}=0, \tag{2.12}$$

$$\sqrt{\lambda_j}(A_{i1}A_{j2}+B_{i1}B_{j1})=\sqrt{\lambda_i}(A_{i2}A_{j2}+B_{i2}B_{j2}), \tag{2.13}$$

$$A_{i1}A_{j2}+B_{i1}B_{j2}=0, \tag{2.14}$$

where $i,j=1,2,\dots,k$ and $i\neq j$. Assume that $A_{i1}\neq 0$ and using the equation (2.12), we have

$$A_{i2}=-\frac{B_{i1}B_{i2}}{A_{i1}}. \tag{2.15}$$

From the equations (2.11) and (2.15), we obtain

$$\frac{B_{i2}^2}{A_{i1}^2}(A_{i1}^2+B_{i1}^2)=A_{i1}^2+B_{i1}^2$$

and

$$\mid B_{i2} \mid=\mid A_{i1} \mid, \quad \mid B_{i1} \mid=\mid A_{i2} \mid .$$

Thus without loss of generality, we assume that

$$B_{i2} = A_{i1}, \quad B_{i1} = -A_{i2}. \tag{2.16}$$

From the equations (2.13) and (2.16), we have

$$(\sqrt{\lambda_j} - \sqrt{\lambda_i})(A_{i1}A_{j1} + A_{i2}A_{j2}) = 0.$$

Since $\sqrt{\lambda_j} \neq \sqrt{\lambda_i}$ while $i \neq j$, we obtain

$$A_{i1}A_{j1} + A_{i2}A_{j2} = 0. \tag{2.17}$$

From the equations (2.14) and (2.16), we have

$$A_{i1}A_{j2} - A_{i2}A_{j1} = 0. \tag{2.18}$$

Since $A_{i1} \neq 0$, from the equations (2.17) and (2.18) we obtain

$$A_{j1}^2 + A_{j2}^2 = 0. \tag{2.19}$$

Thus we have $A_{j1} = A_{j2} = 0$ for $j = 1, 2, \ldots, k$, a contradiction. So, this curve is 1-type. □

Theorem 2.3. *Let γ be a curve which lies in a cylinder $N^2(r)$ in $R^3(-3)$. Then the curve is geodesic* (*minimal*) *in cylinder if and only if θ' is a constant.*

Proof. Let N be a normal vector of the cylinder. It is known that $N = -2\phi^2(x - x_0)$ [1]. If the curve is a geodesic curve in cylinder, then we have $n = \lambda N$. Thus we obtain $g(n, \xi) = 0$ and be a constant. From Corollary 2.1, we easily see that θ' is a constant.

Conversely, let θ' be a constant. From Corollary 2.1 we easily see that the curve is a slant curve and

$$\sigma(s) = g(t, \xi) = \text{constant}.$$

Camci and Ozgur ([5]) showed that the binormal vector of the curve is equal to

$$b = -\frac{\beta}{\kappa}\frac{\phi t}{\sqrt{1 - \sigma^2}} + \frac{\alpha}{\kappa}\frac{\xi - \sigma t}{\sqrt{1 - \sigma^2}},$$

where $\beta = \frac{\sigma'}{\sqrt{1-\sigma^2}}$ and $\alpha^2 + \beta^2 = \kappa^2$. Thus we have

$$b = \frac{\alpha}{\kappa}\frac{\xi - \sigma t}{\sqrt{1 - \sigma^2}}.$$

From the equation above, we see that b lies on the tangent space of the cylinder. Thus the normal vector of the curve and the normal of the cylinder are collinear and the curve is geodesic (minimal). □

Corollary 2.2. *Let γ be a curve which lies in a cylinder $N^2(r)$ in $R^3(-3)$. As a result of Theorem 2.1, Corollary 2.1 and Theorem 2.3, the following are equivalent*:

i) *the curve is finite type*;
ii) *the curve is 1-type*;
iii) *the curve is a slant curve*;
iv) θ' *is a constant*;
v) *the curve is geodesic* (*minimal*) *in cylinder.*

References

[1] C. Baikoussis and D. E. Blair, *Finite type integral submanifolds of the contact manifold* $R^{2n+1}(-3)$, Bull. Inst. Math. Acad. Sinica **19** (1991), no. 4, 327–350.

[2] ———, *On Legendre curves in contact* 3*-manifolds*, Geom. Dedicata **49** (1994), no. 2, 135–142.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, Berlin-New York, 1976.

[4] C. Camci, *A curves theory in contact geometry*, Ph. D. Thesis, Ankara University, 2007.

[5] C. Camci and C. Ozgur, *On a curve with torsion is equal to 1 in 3-dimensional Sasakian manifold*, Preprint.

[6] B. Y. Chen, J. Dillen, F. Verstraelen, and L. Vrancken, *Curves of finite type*, Geometry and topology of submanifolds, II (Avignon, 1988), 76–110, World Sci. Publ., Teaneck, NJ, 1990.

[7] J. T. Cho, J.-I. Inoguchi, and J.-E. Lee, *On slant curves in Sasakian 3-manifolds*, Bull. Austral. Math. Soc. **74** (2006), no. 3, 359–367.

[8] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

Cetin Camci
Department of Mathematics
Onsekiz Mart University
17020 Çanakkale, Turkey
E-mail address: ccamci@comu.edu.tr

H. Hilmi Hacisalihoglu
Department of Mathematics
Faculty of Science
Bilecik University
11210 Bilecik, Turkey
E-mail address: hacisali@science.ankara.edu.tr