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EXPONENTS OF CARTESIAN PRODUCTS OF TWO DIGRAPHS OF SPECIAL ORDERS

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Abstract. In this paper, we find the maximum exponent of $D \times E$, the cartesian product of two digraphs D and E on n, $n+2$ vertices, respectively for an even integer $n \geq 4$. We also characterize the extremal cases.

1. Introduction

Let $D = (V, A)$ be a digraph on n vertices and $u, v \in V$. A $u \to v$ *walk* is a walk from u to v. We use the notation $u \xrightarrow{k} v$ when there is a $u \to v$ walk of length k. A digraph $D = (V, A)$ is said *primitive* if for some k, $u \xrightarrow{k} v$ for all pair of vertices u, v of D. In this case, the smallest such k is called the *exponent* of D and denoted by $\exp(D)$. For a matrix A, the minimal k such that all the entries of A^k are positive is called the *exponent* of A. The exponent of a primitive digraph D is equal to the exponent of its adjacency matrix. Wielandt [7] found that the maximum exponent of primitive digraphs on n vertices is $W_n = n^2 - 2n + 2$ and characterized all the digraphs attaining this bound, which are called Wielandt graphs. Shao [6] improved this bound to $2n - 2$ and Liu, McKay, Wormald and Zhang [4] characterized all the digraphs attaining this improved bound.

Let $D = (V_D, A_D)$ and $E = (V_E, A_E)$ be digraphs such that $| V_D |= n$, $| V_E |= m$. The cartesian product of D and E is defined as $D \times E = (V, A)$ where $V = V_D \times V_E$ and

$$A = \{((u_1, v_1), (u_2, v_2)) \mid ((u_1, u_2) \in A_D \quad \text{and} \quad v_1 = v_2) \quad \text{or} \\ (u_1 = u_2 \quad \text{and} \quad (v_1, v_2) \in A_E)\}.$$

R. Lamprey and B. Barnes [3] showed that if $D \times E$ is primitive, then

$$\exp(D \times E) \leq (n+m)^2 - 4(n+m) + 5.$$

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The first author, Song and Hwang [2] improved this upper bound as

$$\exp(D \times E) \leq nm - 1. \tag{1}$$

They also showed that the upper bound in (1) is attained when $(m, n) = 1$ and characterized all the digraphs D and E which attain it. Moreover, when $m = n$, they proved that

$$\exp(D \times E) \leq n^2 - n + 1,$$

where the equality holds if and only if D and E are isomorphic to a directed cycle and a Wielandt graph. In this paper, we prove that if D and E are digraphs on n and $n+2$ vertices respectively for an even integer $n \geq 4$, and $D \times E$ is primitive, then

$$\exp(D \times E) \leq n^2 + n$$

and we characterize all the extremal digraphs.

2. Main result

From now on we assume that $D = (V_D, A_D)$ and $E = (V_E, A_E)$ are digraphs on n and m vertices, respectively and $D \times E$ is primitive. Let l_1 be the smallest length of a directed cycle of D and l_i be the smallest length of a directed cycle of D which is not a multiple of $(l_1, \ldots, l_{i-1})$ for $i \geq 2$. Let h be the last index of such i. Now let l_j be the smallest length of a directed cycle of E which is not a multiple of $(l_1, \ldots, l_{j-1})$ for $j \geq h+1$. Let k be the last index of such j. For $1 \leq i \leq h$, let $d_i = (l_1, \ldots, l_i)$ and C_i be a directed cycle of D whose length is l_i. For $1 \leq j \leq k$, let $d_j = (l_1, \ldots, l_j)$ and C_j be a directed cycle of E whose length is l_j. Since $D \times E$ is primitive, $k \geq 2$, $d_k = 1$ and $\frac{d_1}{d_2}, \frac{d_2}{d_3}, \ldots, \frac{d_{k-1}}{d_k} \geq 2$.

For relatively prime positive numbers $l_1, \ldots, l_k$, the Frobenius number $g(l_1, l_2, \ldots, l_k)$ is the largest number G such that the equation $l_1 x_1 + \cdots + l_k x_k = G$ is not solvable for non-negative integers $x_1, \ldots, x_k$. Classical results on Frobenius numbers are as follows.

Lemma 1 ([1])**.** *For relatively prime positive numbers $l_1, \ldots, l_k$,*

$$g(l_1, l_2, \ldots, l_k) \leq l_2 \frac{d_1}{d_2} + l_3 \frac{d_2}{d_3} + \cdots + l_k \frac{d_{k-1}}{d_k} - l_1 - l_2 - \cdots - l_k,$$

where $d_i = (l_1, \ldots, l_i)$ for $1 \leq i \leq k$.

Lemma 2 ([1, 5])**.** *If $(a, d) = 1$,*

$$g(a, a+d, a+2d, \ldots, a+kd) = (\lfloor \frac{a-2}{k} \rfloor + d)a - d.$$

The followings are from the first author, Song and Hwang [2].

Lemma 3 ([2])**.** *Let D and E be digraphs on n and m vertices, respectively with h, k, $l_1, \ldots, l_k$ as above. Then*

$$\exp(D \times E) \leq g(l_1, l_2, \ldots, l_k) - l_1 - \cdots - l_k + (h+1)m + (k-h+1)n - 1.$$

Lemma 4 ([2]). *Let D and E be digraphs on n and m vertices, respectively with k as above. If $k \geq 3$, then*

$$\exp(D \times E) \leq \frac{nm}{2} + m - 1.$$

Assume $k \geq 4$. Let $V_k = \{0, 1, \ldots, k-1\}$, $(A_0)_k = \{(i, i+1) | 0 \leq i \leq k-2\} \cup \{(k-2, 0)\}$ and $\tilde{B_k} = \{(k-1, 1), (k-1, 2), (k-2, 1), (k-3, 0)\}$ and $\tilde{E_k} = (V_k, \tilde{A_k})$ be a digraph where $\tilde{A_k} = (A_0)_k \cup \tilde{B_k}$ as shown in Figure 1. Also let $\mathcal{E}_k$ be the set of digraphs $E_k = (V_k, A_k)$ such that $A_k = (A_0)_k \cup B_k$ where B_k is a subset of $\tilde{B_k}$ which contains at least one of $(k-1, 1)$ or $(k-1, 2)$. Then $\tilde{E_k} \in \mathcal{E}_k$ and every element of $\mathcal{E}_k$ is a subgraph of $\tilde{E_k}$. Note that $Z_n = (V_n, \{(i, i+1) | 0 \leq i \leq n-2\} \cup \{(n-1, 0)\})$.

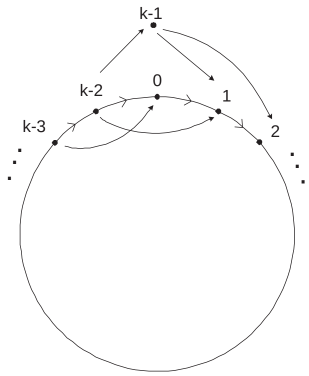

Figure 1

Lemma 5. *Let $E = (V, A)$ be a digraph with m vertices and $v, w \in V$. If the distance from v to w is $m-1$ and $l_1, l_2, \ldots, l_k$ are all the lengths of directed cycles in E, then each length of a path from v to w is represented by $m - 1 + l_1x_1 + l_2x_2 + \cdots + l_kx_k$ where $x_1, x_2, \ldots, x_k$ are nonnegative integers.*

Proof. If there is a path from v to w whose length is not of the form $m - 1 + l_1x_1 + l_2x_2 + \cdots + l_kx_k$, then we can choose a path whose length is minimal among the paths with this property. Let $v = v_0 \to v_1 \to \cdots \to v_t = w$ be such a path. If there are no repeated vertices among $v_0, v_1, \ldots, v_t$, then $t + 1 \leq m$. While considering that the distance from v to w is $m-1$, $t \geq m-1$ and hence $t = m-1$. This is a contradiction. If there are repeated vertices among $v_0, v_1, \ldots, v_t$, then we can take a pair i, j such that $0 \leq i < j \leq t$, $v_i = v_j$ and $j - i$ is minimal among the pairs with this property. Then $v_i \to v_{i+1} \to \cdots \to v_j = v_i$ is a directed cycle, $j - i = l_h$ for some $1 \leq h \leq k$. Therefore

$$v = v_0 \to v_1 \to \cdots \to v_i = v_j \to v_{j+1} \to \cdots \to v_t = w$$

is a path from v to w with length $t - j + i < t$. By the minimality of t, $t - j + i = t - l_h = m - 1 + l_1x_1 + l_2x_2 + \cdots + l_kx_k$ for some nonnegative integers

$x_1, x_2, \ldots, x_k$ and hence $t = m-1+l_1x_1+l_2x_2+\cdots+l_{h-1}x_{h-1}+l_h(x_h+1)+l_{h+1}x_{h+1}+\cdots+l_kx_k$. This is a contradiction. Thus the lemma is proved. □

Lemma 6. *Let $E = (V, A)$ be a digraph on $n+2$ vertices. If $Z_n \times E$ is primitive,* $\mathrm{diam}(E) = n+1$ *and all the lengths of directed cycles in E is n and $n+1$, then*

$$\exp(Z_n \times E) \geq n^2 + n.$$

Proof. Since $\mathrm{diam}(E) = n+1$, there are $v, w \in V$ such that the distance from v to w is $n+1$. It is enough to show that $(0, v) \overset{n^2+n-1}{\not\longrightarrow} (n-1, w)$. Suppose $(0, v) \overset{n^2+n-1}{\longrightarrow} (n-1, w)$. Then $(0, v) = (u_0, v_0) \longrightarrow (u_1, v_1) \longrightarrow \cdots \longrightarrow (u_{n^2+n-1}, v_{n^2+n-1}) = (n-1, w)$ for some vertices (u_i, v_i) of $Z_n \times E$. Let

$S = \{i | 1 \leq i \leq n^2+n-1, u_{i-1} \neq u_i\}$ and $T = \{i | 1 \leq i \leq n^2+n-1, v_{i-1} \neq v_i\}$

with $|S| = s$ and $|T| = t$. Then $S \cup T = \{i | 1 \leq i \leq n^2+n-1\}$ and $S \cap T = \phi$. Therefore $s + t = n^2 + n - 1$. By Lemma 5, $s = n - 1 + nx$ and $t = n + 1 + ny + (n+1)z$ for some nonnegative integers x, y, z. Thus $n^2 + n - 1 = s + t = n - 1 + nx + n + 1 + ny + (n+1)z$ and hence $n^2 - n - 1 = n(x+y) + (n+1)z$. Considering $g(n, n+1) = n^2 - n - 1$, this is impossible. □

Theorem 1. *If $E = (V, A) \in \mathcal{E}_{n+2}$, then*

$$\exp(Z_n \times E) \geq n^2 + n.$$

Proof. Considering that $0 \overset{n}{\longrightarrow} n \to n+1$, $\mathrm{diam}(E) = n+1$. $Z_n \times E_{n+2}$ is primitive as Z_n and E_{n+2} contain cycle of length n and $n+1$, respectively. Also E_{n+2} contains directed cycles of length n and $n+1$ only. Thus by Lemma 6, the theorem is proved. □

Definition 1. Let $D = (V, A)$ be a digraph. Denote $A^T = \{(v, w) | (w, v) \in A\}$. we call a digraph (V, A^T) the *transpose* of D and denote it by D^T.

Remark 1. $\exp(D) = \exp(D^T)$.

Theorem 2. *Let $n \geq 4$ be even. Assume D and E are digraphs on n and $n+2$ vertices respectively and $D \times E$ is primitive. Then*

(i)

$$\exp(D \times E) \leq n^2 + n$$

and

(ii) *the equality holds if and only if D is isomorphic to Z_n, and E or E^T belongs to $\mathcal{E}_{n+2}$.*

Proof. (i) Take h, k, $l_1, \ldots, l_k$ as above. If $k \geq 3$, then from Lemma 4,

$$\exp(D \times E) \leq \frac{n(n+2)}{2} + n + 2 - 1 < n^2 + n.$$

Assume that $k = 2$. From Lemma 1 and Lemma 3,

$$\begin{aligned}\exp(D \times E) &\leq g(l_1, l_2) - l_1 - l_2 + 2n + 2(n+2) - 1\\ &\leq l_1 l_2 - 2l_1 - 2l_2 + 4n + 3\\ &= (l_1 - 2)(l_2 - 2) + 4n - 1.\end{aligned}$$

If $l_1 \leq n-1$, then $\exp(D \times E) \leq n^2 + n - 1$ as $l_2 \leq n+2$. Assume that $l_1 = n$. Then D is isomorphic to the directed cycle of length n, which is Z_n. Since $(l_1, l_2) = 1$, $l_2 = n+1$ and hence

$$\exp(D \times E) \leq (n-2)(n-1) + 4n - 1 = n^2 + n + 1.$$

Assume that C_1 is a directed cycle $u_0 \to u_1 \to \cdots \to u_{n-1} \to u_0$ in D and C_2 is a directed cycle $v_0 \to v_1 \to \cdots \to v_n \to v_0$ in E. Let (x_1, y_1) and (x_2, y_2) be vertices on $D \times E$ such that $(x_1, y_1) \overset{\alpha-1}{\not\longrightarrow} (x_2, y_2)$ where $\alpha = \exp(D \times E)$. Then there are integers s and t such that $x_1 \overset{s}{\longrightarrow} x_2$, $y_1 \overset{t}{\longrightarrow} y_2$, $0 \leq s \leq n-1$ and $0 \leq t \leq n+1$. We consider the following two cases.

Case (i)a: There is a directed cycle of length $n+2$ in E.
Let $\beta \geq n^2 + n - 1$. If $y_1 \neq y_2$, then at least one of y_1, y_2 belongs to the directed cycle of length $n+1$. We may assume that y_1 does and hence $y_1 \overset{n+1}{\longrightarrow} y_1$. By Lemma 2,

$$\begin{aligned}\beta - s - t &\geq n^2 + n - 1 - (n-1) - (n+1) = n^2 - n - 1\\ &> \frac{n^2}{2} - 1 = g(n, n+1, n+2)\end{aligned}$$

and hence there are $p, q, r \geq 0$ such that $np + (n+1)q + (n+2)r = \beta - s - t$. Then, since

$$(x_1, y_1) \overset{np}{\longrightarrow} (x_1, y_1) \overset{s}{\longrightarrow} (x_2, y_1) \overset{(n+1)q}{\longrightarrow} (x_2, y_1) \overset{t}{\longrightarrow} (x_2, y_2) \overset{(n+2)r}{\longrightarrow} (x_2, y_2)$$

and $np + s + (n+1)q + t + (n+2)r = (\beta - s - t) + s + t = \beta$, $(x_1, y_1) \overset{\beta}{\longrightarrow} (x_2, y_2)$. So $\exp(D \times E) \leq n^2 + n - 1$. If $y_1 = y_2$, then $y_1 \overset{t_1}{\longrightarrow} v_0 \overset{t_2}{\longrightarrow} y_1$ for some $0 \leq t_1, t_2 \leq n+1$ such that $t_1 + t_2 = n+2$ as E is strongly connected.

$$\begin{aligned}\beta - s - (n+2) &\geq n^2 + n - 1 - (n-1) - (n+2) = n^2 - n - 2\\ &> \frac{n^2}{2} - 1 = g(n, n+1, n+2)\end{aligned}$$

by Lemma 2. So there are $p, q, r \geq 0$ such that $np + (n+1)q + (n+2)r = \beta - s - (n+2)$. Then, since

$$\begin{aligned}&(x_1, y_1) \overset{np}{\longrightarrow} (x_1, y_1) \overset{s}{\longrightarrow} (x_2, y_1) \overset{t_1}{\longrightarrow} (x_2, v_0) \overset{(n+1)q}{\longrightarrow} (x_2, v_0)\\ &\overset{t_2}{\longrightarrow} (x_2, y_1) \overset{(n+2)r}{\longrightarrow} (x_2, y_1)\end{aligned}$$

and $np + s + t_1 + (n+1)q + t_2 + (n+2)r = (\beta - s - (n+2)) + s + n + 2 = \beta$, $(x_1, y_1) \overset{\beta}{\longrightarrow} (x_2, y_1)$. So $\exp(D \times E) \leq n^2 + n - 1$.

Case (i)b: There is no directed cycle of length $n+2$ in E.
If $t \geq 1$, then since there is only one vertex $x \in V_E$ such that $x \neq v_i$ for $i = 0, 1 \ldots, n$, there is an intermediate vertex v_j such that $y_1 \xrightarrow{t_1} v_j \xrightarrow{t-t_1} y_2$. Since

$$n^2+n-s-t \geq n^2+n-(n-1)-(n+1) = n^2-n > (n-1)n-1 = g(n, n+1),$$

there are $p, q \geq 0$ such that $np+(n+1)q = n^2+n-s-t$. Then, since

$$(x_1, y_1) \xrightarrow{np} (x_1, y_1) \xrightarrow{s} (x_2, y_1) \xrightarrow{t_1} (x_2, v_j) \xrightarrow{(n+1)q} (x_2, v_j) \xrightarrow{t-t_1} (x_2, y_2)$$

and $np + s + t_1 + (n+1)q + t - t_1 = (n^2 + n - s - t) + s + t = n^2 + n$, $(x_1, y_1) \xrightarrow{n^2+n} (x_2, y_2)$. If $t = 0$, then $y_1 = y_2$. As E is strongly connected, $y_1 \xrightarrow{t_1} v_0 \xrightarrow{t_2} y_1$ for some $0 \leq t_1, t_2 \leq n+1$ such that $t_1 + t_2 \leq n+1$. Since

$$n^2+n-s-(t_1+t_2) \geq n^2+n-(n-1)-(n+1) = n^2-n > n^2-n-1 = g(n, n+1),$$

there are $p, q \geq 0$ such that $np+(n+1)q = n^2+n-s-(t_1+t_2)$. Then, since

$$(x_1, y_1) \xrightarrow{np} (x_1, y_1) \xrightarrow{s} (x_2, y_1) \xrightarrow{t_1} (x_2, v_0) \xrightarrow{(n+1)q} (x_2, v_0) \xrightarrow{t_2} (x_2, y_1)$$

and $np+s+t_1+(n+1)q+t_2 = (n^2+n-s-(t_1+t_2))+s+t_1+t_2 = n^2+n$, $(x_1, y_1) \xrightarrow{n^2+n} (x_2, y_2)$. So $\exp(D \times E) \leq n^2+n$.

(ii) Now we characterize all the extremal graphs. Let $\exp(D \times E) = n^2+n$. It is possible only in Case (i)b where E has no directed cycle of length $n+2$. If $s+t \leq 2n-1$, then similarly as above, $\exp(D \times E) \leq n^2+n-1$. So $s = n-1$ and $t = n+1$. Let (x_1, y_1) and (x_2, y_2) be vertices of $D \times E$ such that $x_1 \xrightarrow{s} x_2$, $y_1 \xrightarrow{t} y_2$. Also let w_0 be the only vertex of E such that $w_0 \neq v_i$ for all $i = 0, 1, \ldots, n$. Since E is strongly connected, there are i, j such that $(v_i, w_0), (w_0, v_j) \in A_E$. So we may also assume $(v_n, w_0) \in A_E$. As $w_0 \to v_j \to v_{j+1} \to \cdots \to v_n \to w_0$ is a directed cycle of length $n \leq n+2-j < n+2$, j is 1 or 2.

Case (ii)a: $j = 1$.
As there can be only directed cycles of order n or $n+1$ in E, among the arcs which are adjacent to w_0, at most one of (w_0, v_2), (v_{n-1}, w_0) may exist. Since $y_1 \xrightarrow{n+1} y_2$, (y_1, y_2) is (v_0, w_0) in the first case, and (w_0, v_0) in the second case. Let $0 \leq k < l \leq n$. If $(v_l, v_k) \in A_E$, then $v_k \xrightarrow{l-k} v_l \to v_k$ is a directed cycle of length $l-k+1$, and hence $l = k+n-1$ or $l = k+n$. So (k, l) is $(0, n-1)$, $(0, n)$ or $(1, n)$. When (k, l) is $(0, n-1)$, (y_1, y_2) is (v_0, w_0) and when (k, l) is $(1, n)$, (y_1, y_2) is (v_0, w_0) or (w_0, v_0). If $(v_k, v_l) \in A_E$, then $v_k \to v_l \xrightarrow{n+1-l} v_0 \xrightarrow{k} v_k$ is a directed cycle of length $n-l+k+2$, and hence $l = k+1$ or $l = k+2$. Let $l = k+2$. If $k \geq 1$, then $(y_1.y_2)$ does not exist. So (k, l) is $(0, 2)$ and (y_1, y_2) is (w_0, v_0). In conclusion, there are two possible forms of digraphs. Firstly, when (y_1, y_2) $= (v_0, w_0)$, the only arcs which might be added to $(A_0)_{n+2} \cup \{(w_0, v_1)\}$ are (w_0, v_2), (v_{n-1}, v_0), and (v_n, v_1). As $\tilde{B}_{n+2} = \{(w_0, v_1), (w_0, v_2), (v_{n-1}, v_0), (v_n,$

$v_1)\}$, E is a subgraph of $\tilde{E_{n+2}}$. Secondly, when $(y_1, y_2) = (w_0, v_0)$, the only arcs which might be added to $(A_0)_{n+2} \cup \{(w_0, v_1)\}$ are (v_{n-1}, w_0), (v_n, v_1), and (v_0, v_2). If we switch v_0 and w_0, we get the same arcs as above and hence E is a subgraph of $\tilde{E_{n+2}}$.

Case (ii)b: $j = 2$.
If $(w_0, v_1) \in A_E$, then it reduces to Case (ii)a. Assume $(w_0, v_1) \notin A_E$. As there can be only directed cycles of order n or $n+1$ in E, among the arcs which are to w_0, only (v_0, w_0) may exist. Assume $(v_0, w_0) \in A_E$. After exchanging v_i with v_{n+2-i} for $2 \leq i \leq n$, and v_0 with v_1, we get E^T where E is described in Case (ii)a which has arc (w_0, v_2) and arcs of $(A_0)_{n+2} \bigcup \{(w_0, v_1)\}$. Thus by similar argument, E^T is a subgraph of $\tilde{E_{n+2}}$. Assume $(v_0, w_0) \notin A_E$. As there can be only directed cycles of order n or $n+1$ in E, among the arcs which are from w_0, only (v_n, w_0) may exist where (y_1, y_2) is (v_0, w_0) or (w_0, v_1). Let $0 \leq k < l \leq n$. If $(v_l, v_k) \in A_E$ and $(k, l) \neq (0, n)$, then by the same argument as in Case (ii)a, (k, l) is $(0, n-1)$, or $(1, n)$ and (y_1, y_2) is (v_0, w_0). If $(v_k, v_l) \in A_E$ where $l \neq k+1$, then by the same argument as in Case (ii)a, (k, l) is $(0, 2)$ or $(1, 3)$ and (y_1, y_2) is (w_0, v_1). In conclusion, there are two possible forms of digraphs. Firstly when $(y_1, y_2) = (v_0, w_0)$, the only arcs which might be added to $(A_0)_{n+2} \cup \{(w_0, v_2)\}$ are (v_{n-1}, v_0) and (v_n, v_1) and hence E is a subgraph of $\tilde{E_{n+2}}$. Secondly when $(y_1, y_2) = (w_0, v_1)$, the only arcs which might be added to $(A_0)_{n+2} \cup \{(w_0, v_2)\}$ are (v_0, v_2) and (v_1, v_3). After exchanging v_i with v_{n+2-i} for $2 \leq i \leq n$ and v_0 with v_1, E^T is a subgraph of $\tilde{E_{n+2}}$. □

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