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AN EXTENSION OF AN ANALYTIC FORMULA OF THE DETERMINISTIC EPIDEMICS MODEL PROBLEM THROUGH LIE GROUP OF OPERATORS

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Abstract. In the present paper, we evaluate an analytic formula as a solution of Susceptible Infective (SI) model problem for communicable disease in which the daily contact rate ($C(N)$) is supposed to be varied linearly with population size $N(t)$ that is large so that it is considered as a continuous variable of time t. Again, we introduce some Lie group of operators to make an extension of above analytic formula of the deterministic epidemics model problem. Finally, we discuss some of its particular cases.

1. Introduction

In population dynamics, various deterministic epidemics model problems involving differential equations were studied and obtained due to Kermack and McKendrick [8, 9] and Bailey [1] etc. Kapur [7] has been introduced and solved several mathematical models of epidemics through systems of ordinary differential equations of first order. Recently, Joshi [6] has presented a solution of the deterministic epidemics model in terms of the hypergeometric functions ${}_0F_1(\cdot)$ (See Rainville [10]). On epidemics modeling a study has also done by Daley and Gani [2]. Hethcote [4] has presented mathematical interpretations on infectious diseases and lateron he [5] has described basic epidemiological models.

An epidemic usually describes as occurrence of a diseases in excess to normal expectations. Since contiguousness is one of the main causes of spread of such epidemics, the term epidemiology has been applied to general study and deriving measures of controlling all communicable diseases. The general model for communicable disease in which an infected person does not recover is known as SI model. In this model at a time t, total population $N(t)$ is divided into two disjoint classes namely infective class $I(t)$ consisting of totally

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of infected individuals who can transmit disease and the susceptible class $S(t)$ of the individuals who can incur disease by contact the infected individuals.

The confluent hypergeometric differential equation, satisfying by the confluent hypergeometric function $u ={}_1 F_1 \left(\begin{smallmatrix} a; \\ c; \end{smallmatrix} z\right)$, is given by (See Rainville [10])

$$z\frac{d^2u}{dz^2} + (c - z)\frac{du}{dz} - au = 0, \tag{1.1}$$

where, c is not an integer.

Again, the asymptotic estimates of ${}_1F_1 \left(\begin{smallmatrix} a; \\ c; \end{smallmatrix} z\right)$ are given by (See Srivastava and Manocha [11])

$${}_1F_1 \begin{pmatrix} a; \\ c; \end{pmatrix} z = \left\{ \begin{array}{l} \frac{\Gamma(c)}{\Gamma(a)} e^z z^{a-c}\{1 + O(|z|^{-1})\}, \mathrm{Re}(z) \to \infty \\ \frac{\Gamma(c)}{\Gamma(c-a)}(-z)^{-a}\{1 + O(|z|^{-1})\}, \mathrm{Re}(z) \to -\infty \end{array} \right\}, \tag{1.2}$$

where, a and c are bounded complex numbers.

Here, in our investigations, we evaluate an analytic formula as a solution of Susceptible Infective (SI) model problem for communicable disease in which the daily contact rate $(C(N))$ is supposed to be varied linearly with population size $N(t)$ that is large so that it is considered as a continuous variable of time t. Again, we introduce some Lie group of operators to make an extension of above analytic formula of the deterministic epidemics model problem. Finally, we discuss some of its particular cases.

2. Deterministic mathematical model of epidemics

In this section, we construct an epidemics model problem and then convert it in terms of the differential equation satisfied by the Kummer's confluent hypergeometric function ${}_1F_1 \left(\begin{smallmatrix} a; \\ c; \end{smallmatrix} z\right)$ (See Rainville [10]).

In this model, the population size $N(t)$ is so large that it can be considered as a continuous variable of time t. The population is changing on account of immigration, births, emigration and deaths (due to disease in question or other causes). Let β be the rate at which the population is receiving new individuals due to immigration and birth and μ be the rate at which individuals are being removed on account of emigration and death. Hence all the new entrants are assumed to be susceptible. The population is assumed to be uniform or homogeneous. The daily contact rate $C(N)$, at which number of susceptibles are become infected, is taken by $\alpha IS - nN(\beta - \mu), n \in \{0, 1, 2, \ldots\}$ and $\alpha, \beta, \mu \in R$ (the set of real numbers).

The initial value problem for the SI model can be put in the form:

$$\frac{dS}{dt} = \beta N - \mu S - \alpha IS + nN(\beta - \mu), \tag{2.1}$$

$$\frac{dI}{dt} = \alpha IS - \mu I - nN(\beta - \mu), \tag{2.2}$$

$$N = I + S; \tag{2.3}$$

where, n is a parameter such that $n \in \{0, 1, 2, \ldots\}$ and $\alpha, \beta, \mu \in R$ (set of real numbers) such that $(\beta - \mu) > 0$; $N(0) = N_0 > 0, S(0) = S_0 > 0$ and $I(0) = I_0 > 0$.

Now adding the equations (2.1) and (2.2) and then making an appeal to the equation (2.3), we find

$$\frac{dN}{dt} = N(\beta - \mu). \tag{2.4}$$

So that the equation (2.4) gives us

$$N = N_0 e^{(\beta-\mu)t} \tag{2.5}$$

provided that all conditions of the equation (2.3) are followed and $t > 0$.

Further, setting $\frac{du}{dt} = -\alpha S u$ in the equation (2.1) and then making an application of equation (2.3), we find that

$$\frac{d^2u}{dt} + (\mu + \alpha N)\frac{du}{dt} + ((n+1)\alpha\beta N - n\alpha\mu N)u = 0. \tag{2.6}$$

Now, making an appeal to the equations (2.4), (2.5) and (2.6), we get a transformed equation in the form

$$N^2\frac{d^2u}{dN^2} + \left(\frac{\beta}{\beta-\mu} + \frac{\alpha}{\beta-\mu}N\right)N\frac{du}{dN} + \left(\frac{\beta}{\beta-\mu} + n\right)\frac{\alpha}{\beta-\mu}Nu = 0. \tag{2.7}$$

Again, set $\frac{\alpha}{\beta-\mu}N = -M$ in the equation (2.7), we find another transformed equation in the form of confluent hypergeometric differential equation as

$$\left[M\frac{d^2u}{dM^2} + \left(\frac{\beta}{\beta-\mu} - M\right)\frac{du}{dM} - \left(\frac{\beta}{\beta-\mu} + n\right)u\right] = 0. \tag{2.8}$$

Hence, with the aid of the equation (1.1), the general solution of the equation (2.8) may be written in the form

$$u(g^{\beta,\mu}; M) = \sum_{n=0}^{\infty} C_{n\,1}F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu} + n; \\ \frac{\beta}{\beta-\mu};\end{matrix} M\right), \tag{2.9}$$

where, $\frac{\beta}{\beta-\mu}$ is bounded and $(\beta - \mu) > 0$ and all C_n, where $n \in \{0, 1, 2, \ldots\}$ are arbitrary constants, $g^{\beta,\mu}$ is the function of β and μ only.

With the help of the equations (1.2) and (2.9), we may evaluate the asymptotic estimates of the function $u(g^{\beta,\mu}; M)$ and then, analyze asymptotically of our epidemics model problem.

For further our investigations, replacing n by $y\frac{\partial}{\partial y}$ in the equation (2.8), we construct a partial differential equation

$$\left[M\frac{\partial^2}{\partial M^2} + \left(\frac{\beta}{\beta-\mu} - M\right)\frac{\partial}{\partial M} - \left(\frac{\beta}{\beta-\mu} + y\frac{\partial}{\partial y}\right)\right]v = 0,$$

whose general solution is

$$v(g^{\beta,\mu};M,y)=\sum_{n=0}^{\infty}C_n\,{}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n;\\ \frac{\beta}{\beta-\mu};\end{matrix}M\right)y^n, \tag{2.10}$$

where, $|y|<1$ and all conditions of the equation (2.9) are included.

3. An extension formula

On introducing more parameters in the equation (2.10), and setting $C_n=1$, $\forall n=1,2,\dots$ we may write

$$\begin{aligned}\theta(g^{\beta,\mu};M,y,z)&=y^{\frac{\beta}{\beta-\mu}}z^{\frac{\beta}{\beta-\mu}}v(g^{\beta,\mu};M,y,z)\\ &=\sum_{n=0}^{\infty}{}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n;\\ \frac{\beta}{\beta-\mu};\end{matrix}M\right)y^{\frac{\beta}{\beta-\mu}+n}z^{\frac{\beta}{\beta-\mu}},\end{aligned} \tag{3.1}$$

where, y and z are complex numbers and here all conditions of the equation (2.9) are included.

In the equation (3.1), the function $\theta(g^{\beta,\mu};M,y,z)$ satisfies the partial differential equation

$$\left[M\frac{\partial^2}{\partial M^2}+\left(z\frac{\partial}{\partial z}-M\right)\frac{\partial}{\partial M}-\left(y\frac{\partial}{\partial y}\right)\right]\theta=0,$$

with simultaneous equations

$$z\frac{\partial\theta}{\partial z}=\frac{\beta}{\beta-\mu}\theta \text{ and } y\frac{\partial\theta}{\partial y}=\left(\frac{\beta}{\beta-\mu}+n\right)\theta. \tag{3.1a}$$

Now, to obtain an extension formula of the function $\theta(g^{\beta,\mu};M,y,z)$, given in the equation (3.1), we introduce following Lie group of operators

$$A\equiv y\left[M\frac{\partial}{\partial M}+y\frac{\partial}{\partial y}\right] \text{ and } B\equiv z^{-1}\left[M\frac{\partial}{\partial M}+z\frac{\partial}{\partial z}-1\right]. \tag{3.2}$$

Then, we evaluate following group of actions of the above Lie group of operators on the basis function taken by

$$\theta(g^{\beta,\mu}(n),g^{\beta,\mu};M,y,z)={}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n;\\ \frac{\beta}{\beta-\mu};\end{matrix}M\right)y^{\frac{\beta}{\beta-\mu}+n}z^{\frac{\beta}{\beta-\mu}}. \tag{3.3}$$

Now, let $g^{\beta,\mu}(n)=\frac{\beta}{\beta-\mu}+n$, and $g^{\beta,\mu}(0)=g^{\beta,\mu}=\frac{\beta}{\beta-\mu}$, then

(3.4)

$$A\theta(g^{\beta,\mu}(n),g^{\beta,\mu};M,y,z)=(g^{\beta,\mu}(n))\theta(g^{\beta,\mu}(n+1),g^{\beta,\mu};M,y,z),$$
$$B\theta(g^{\beta,\mu}(n),g^{\beta,\mu};M,y,z)=(g^{\beta,\mu}-1)\theta(g^{\beta,\mu}(n),g^{\beta,\mu}-1;M,y,z),$$
$$\exp[\lambda A]\theta(g^{\beta,\mu}(n),g^{\beta,\mu};M,y,z)=\theta(g^{\beta,\mu}(n),g^{\beta,\mu};\frac{M}{1-\lambda y},\frac{y}{1-\lambda y},z)$$

$$\exp[\xi B]\theta(g^{\beta,\mu}(n), g^{\beta,\mu}; M, y, z) = \left(\frac{z}{z+\xi}\right)\theta(g^{\beta,\mu}(n), g^{\beta,\mu}; \frac{M}{z}(z+\xi), y, z+\xi),$$

$$\exp[\lambda A + \xi B]\theta(g^{\beta,\mu}(n), g^{\beta,\mu}; M, y, z) = \left(\frac{z}{z+\xi}\right)\theta\left(g^{\beta,\mu}(n), g^{\beta,\mu}; \frac{M}{z}\left(\frac{z+\xi}{1-\lambda y}\right), \frac{y}{1-\lambda y}, z+\xi\right).$$

Therefore, from the equations (3.3) and (3.4), we evaluate the relation in the form

(3.5)

$$\begin{aligned}&\left(\frac{z}{z+\xi}\right)\theta\left(g^{\beta,\mu}(n), g^{\beta,\mu}; \frac{M}{z}\left(\frac{z+\xi}{1-\lambda y}\right), \frac{y}{1-\lambda y}, z+\xi\right)\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(g^{\beta,\mu}(n))_r(1-g^{\beta,\mu})_s\frac{(\lambda)^r}{r!}\frac{(-\xi)^s}{s!}\theta(g^{\beta,\mu}(n+r), g^{\beta,\mu}-s; M, y, z).\end{aligned}$$

Finally, with the help of the equations (3.1), (3.3) and (3.5), we obtain the extension formula

(3.6)

$$\begin{aligned}&\left(\frac{z}{z+\xi}\right)\theta\left(g^{\beta,\mu}; \frac{M}{z}\left(\frac{z+\xi}{1-\lambda y}\right), \frac{y}{1-\lambda y}, z+\xi\right)\\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(g^{\beta,\mu}(n))_r(1-g^{\beta,\mu})_s\frac{(\lambda)^r}{r!}\frac{(-\xi)^s}{s!}\theta(g^{\beta,\mu}(n+r), g^{\beta,\mu}-s; M, y, z)\\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{\beta}{\beta-\mu}+n\right)_r\left(\frac{-\mu}{\beta-\mu}\right)_s\frac{(\lambda)^r}{r!}\frac{(-\xi)^s}{s!}\\ &\quad {}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n+r;\\ \frac{\beta}{\beta-\mu}-s;\end{matrix}M\right)y^{\frac{\beta}{\beta-\mu}+n+r}z^{\frac{\mu}{\beta-\mu}-s}.\end{aligned}$$

4. Special cases

Set $\lambda = 0$ in the equation (3.6), particularly, we get

$$\begin{aligned}&\left(\frac{z}{z+\xi}\right)\theta\left(g^{\beta,\mu}; \frac{M}{z}(z+\xi), y, z+\xi\right)\\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}(1-g^{\beta,\mu})_s\frac{(-\xi)^s}{s!}\theta(g^{\beta,\mu}(n), g^{\beta,m\mu}-s; M, y, z)\\ &= \sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\left(\frac{-\mu}{\beta-\mu}\right)_s\frac{(-\xi)^s}{s!}{}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n;\\ \frac{\beta}{\beta-\mu}-s;\end{matrix}M\right)y^{\frac{\beta}{\beta-\mu}+n}z^{\frac{\mu}{\beta-\mu}-s}.\end{aligned} \tag{4.1}$$

With the help of the equations (3.1) and (4.1), we write

$$\left(\frac{z}{z+\xi}\right)\sum_{n=0}^{\infty}{}_1F_1\left(\begin{matrix}\frac{\beta}{\beta-\mu}+n;\\ \frac{\beta}{\beta-\mu};\end{matrix}\frac{M}{z}(z+\xi)\right)y^{\frac{\beta}{\beta-\mu}+n}(z+\xi)^{\frac{\mu}{\beta-\mu}} \tag{4.2}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{-\mu}{\beta-\mu} \right)_s \frac{(-\xi)^s}{s!} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n; \\ \frac{\beta}{\beta-\mu}-s; \end{matrix} M \right) y^{\frac{\beta}{\beta-\mu}+n} z^{\frac{\mu}{\beta-\mu}-s}.$$

Now, equate the n-th term in both sides of the equation (4.2), we find the well known result equivalent to Erde'lyi et al. [3, Section (6.14), eqn. (2)]

$$\tag{4.3} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n; \\ \frac{\beta}{\beta-\mu}; \end{matrix} \frac{M}{z}(z+\xi) \right)$$
$$= (z+\xi)^{\frac{-\mu}{\beta-\mu}} \sum_{s=0}^{\infty} \left(\frac{-\mu}{\beta-\mu} \right)_s \frac{(-\xi)^s}{s!} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n; \\ \frac{\beta}{\beta-\mu}-s; \end{matrix} M \right) z^{\frac{\mu}{\beta-\mu}-s-1}.$$

Again, set $\xi = 0$ in the equation (3.6), we find that

$$\theta \left(g^{\beta,\mu}; \frac{M}{1-\lambda y}, \frac{y}{1-\lambda y}, z \right)$$
$$\tag{4.4} = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (g^{\beta,\mu}(n))_r \frac{(\lambda)^r}{r!} \theta(g^{\beta,\mu}(n+r), g^{\beta,\mu}; M, y, z)$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \left(\frac{\beta}{\beta-\mu}+n \right)_r \frac{(\lambda)^r}{r!} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n+r; \\ \frac{\beta}{\beta-\mu}; \end{matrix} M \right) y^{\frac{\beta}{\beta-\mu}+n+r} z^{\frac{\mu}{\beta-\mu}}.$$

Then, with the help of the equations (3.1) and (4.4), we write

$$\tag{4.5} \sum_{n=0}^{\infty} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n; \\ \frac{\beta}{\beta-\mu}; \end{matrix} \frac{M}{1-\lambda y} \right) \left(\frac{y}{1-\lambda y} \right)^{\frac{\beta}{\beta-\mu}+n} z^{\frac{\mu}{\beta-\mu}}$$
$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \left(\frac{\beta}{\beta-\mu}+n \right)_r \frac{(\lambda)^r}{r!} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n+r; \\ \frac{\beta}{\beta-\mu}; \end{matrix} M \right) y^{\frac{\beta}{\beta-\mu}+n+r} z^{\frac{\mu}{\beta-\mu}}.$$

Now, equate the n-th term in both sides of the equation (4.5), we find the well known result equivalent to Erde'lyi et al. [3, Section (6.14), eqn. (3)]

$$\tag{4.6} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n; \\ \frac{\beta}{\beta-\mu}; \end{matrix} \frac{M}{1-\lambda y} \right)$$
$$= (1-\lambda y)^{\frac{\beta}{\beta-\mu}+n} \sum_{r=0}^{\infty} \left(\frac{\beta}{\beta-\mu}+n \right)_r \frac{(\lambda y)^r}{r!} {}_1F_1 \left(\begin{matrix} \frac{\beta}{\beta-\mu}+n+r; \\ \frac{\beta}{\beta-\mu}; \end{matrix} M \right).$$

Remarks. It is noted that when $n = 0$ our model problem, described in equations (2.1)-(2.3), terminates into the epidemics model problem due to Joshi [6]. Again, the solution of our epidemics model problem as an extension formula has great importance and is the powerful tool so that by which we may plot various structures of population dynamics and then analyze it to use in Health Sciences.

5. Discussions

Recently, at our Kanpur city in India, an epidemic (Diarrhea) is spread at its different places, 1, 2, 3,.... We collect some samples and observe that the contact rate between susceptible and infected persons obeys the rule near to the formula $\alpha IS - nN(\beta - \mu), n \in \{0, 1, 2, \ldots\}$, here α is constant and $\beta, \mu \in \mathbb{R}$ (the set of real numbers).

Again, on introducing Lie groups with new parameters λ and ξ such that λ and ξ lie in a sufficiently small neighborhood of $0 \in \mathbb{C}$ (the set of complex numbers), we evaluate some transformations in the characteristic function. Then, due to Lie group $\exp[\lambda A]$ the transformation in the characteristic function $\theta(g^{\beta,\mu}(n), g^{\beta,\mu}; M, y, z)$ occurs and we have that

$$\begin{bmatrix} M \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \frac{M}{1-\lambda y} \\ \frac{y}{1-\lambda y} \\ z \end{bmatrix}. \tag{5.1}$$

Further by the Lie group $\exp[\xi B]$ we find the transformation in the characteristic function $\theta(g^{\beta,\mu}(n), g^{\beta,\mu}; M, y, z)$ in the form

$$\begin{bmatrix} M \\ y \\ z \end{bmatrix} \to \begin{bmatrix} \frac{M}{z}(z+\xi) \\ y \\ z+\xi \end{bmatrix} \tag{5.2}$$

with the multiplier $\left(\frac{z}{z+\xi}\right)$.

Therefore by introducing Lie groups the region of the domain is extended and thus obtained analytic solution allows us to examine how an epidemic behavior changes with variation in model parameters near the origin and to characterize the threshold level of the disease where an epidemic become irrelevant. This model has collective approach to describe the distribution of disease (How the disease spreads).

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