Bull. Korean Math. Soc. **47** (2010), No. 6, pp. 1105–1119
DOI 10.4134/BKMS.2010.47.6.1105

ENDOGENOUS DOWNWARD JUMP DIFFUSION AND BLOW UP PHENOMENA BEFORE CRASH

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Abstract. We consider jump processes which has only downward jumps with size a fixed fraction of the current process. The jumps of the processes are interpreted as crashes and we assume that the jump intensity is a nondecreasing function of the current process say $\lambda(X)$ ($X = X(t)$: process).

For the case of $\lambda(X) = X^\alpha, \alpha > 0$, we show that the process X should explode in finite time, say t_e, conditional on no crash.

For the case of $\lambda(X) = (\ln X)^\alpha$, we show that $\alpha = 1$ is the borderline of two different classes of processes. We generalize the model by adding a Brownian noise and examine the blow up properties of the sample paths.

1. Introduction

In this paper, we introduce diffusion processes with only downward jumps with size a fixed fraction of the current process. Moreover the jump intensity is increasing as the process goes up. The motive of research of such processes is modeling of the bubble and crash phenomenon which is characterized by the rapid increase of an asset price followed by a sudden collapse (e.g., Smith et al. [19], Camerer [5], and Porter and Smith [17]). Blanchard [3] and Blanchard and Watson [4] have developed a rational expectation theory of deterministic and stochastic bubbles in discrete time.

The unrealistic features of the model, such as the exponential blow-up phenomenon in the deterministic model and the never growing phenomenon in the stochastic model, have been modified by many authors. Evans [7] proposed periodic collapsing bubbles, and Fukuta [8] developed a model with incompletely collapsing bubbles, which was generalized by Lux and Sornette [15] and Malevergne and Sornette [16]. Sornette and Anderson [20] and Anderson and Sornette [2] have studied the nonlinear rational expectation model, a modification of the Geometric Brownian Motion, under the continuous time

Received June 25, 2008; Revised July 9, 2010.
2000 *Mathematics Subject Classification.* 60J60, 60J45.
Key words and phrases. crash, explosion, jump diffusion.
This research was financially supported in part by Hansung University during sabbatical year (2009).

formulation. In their papers, the crash probability increase according to the size of bubble. Our process is also a modeling of the idea that the larger the bubble size is, the greater the probability of crash seems to be. With this idea, we call downward jumps as crash.

We assume that the process is the sum of a pure jump process and its continuous compensation so that the process should be a martingale. Later, we add a Brownian noise to the process to make a diffusion process with jumps.

One of our main contributions is that, with such assumptions, the blow up curve before crash has a different nature. Indeed, for some reasonable functions $\lambda(X)$, we can figure out the expected blow up curve explicitly, and it turns out that it explodes in finite time, say t_e, depending only on the initial price, the jump intensity $\lambda(X)$, and the jump size. More precisely, for the case of $\lambda(X) = X^\alpha$, conditional on no crash, the process $X = X(t)$ follows

$$X(t) = \frac{C_1}{(C_2 - t)^{\frac{1}{\alpha}}} \tag{1.1}$$

for some explicit constants C_1 and C_2. This shows that the crash should occur before $t = C_2$, since, if not, the process should be infinity at $t = C_2$. It is worth noting that this explosion comes from the nonlinearity of the process. By allowing the rate to be dependent on the process, we have extended the process to be a solution of a nonlinear stochastic differential equation.

There is another interesting result for the case of $\lambda(X) = (\ln X)^\alpha$. It turns out that conditional on no crash the process $X(t)$ explodes in finite time if $\alpha > 1$ and grows super exponentially but never explodes in finite time if $0 < \alpha < 1$. Here we say that $X(t)$ grows super exponentially if $X(t)/\exp(at) \to \infty$ for any $a < \infty$ as $t \to \infty$. We extend the above model by adding a Brownian noise and show the surprising result that, for the case of $\lambda(X) = X^\alpha$, there exist α and δ such that with positive probability the process does not explode in finite time. Moreover, the process tends to zero with positive probability. Recall that in such a case, if there is no noise, then the process explodes in finite time with probability one as in (1.1). This shows that the noise is an important factor of the crash phenomena.

This paper is organized as follows. After this introduction, in Section 2, we describe our process as a solutions of some stochastic differential equations containing jumps. Basically, we consider two types of martingale processes. Some of them will be generalized by adding a Brownian noise to the jump diffusion process. We show the existence and uniqueness results of the stochastic differential equations discussed. In Section 3, we show the exact form of the process before crash and the distribution of crash.

2. Existence and uniqueness

We are seeking a stochastic process which is a martingale and makes only downward jumps. One simple example is $\lambda t - N_t$ where N_t is a Poisson process with jump intensity λ. The term $-N_t$ represents a pure downward jump process, where jump size is 1 and λt is the nondecreasing compensation making the whole process a martingale. We are interested in the process where the jump intensity depends on the state and the jump size is a fixed fraction of the state. The general version of such processes can be expressed as the solution of the following stochastic differential equation (SDE, in short), with $X_0 > 0$:

$$dX_t = \int c(X_{t-}, u)[\nu(dudt) - \pi(du)dt], \tag{2.1}$$

where $c(x, u)$ is a Borel measurable function defined on $\mathbb{R}^+ \times \mathbb{R}$, ν is a Poisson random measure on $\mathbb{R} \times \mathbb{R}^+$ and π is a σ-finite measure on $\mathbb{R}$ such that

$$E\left(\nu(A \times [0, t])\right) = \pi(A)t$$

for any measurable $A \subset \mathbb{R}$. Denoting $\tilde{\nu}(duds) = \nu(duds) - \pi(du)ds$. This type of SDE is derived from the work of Skorohod [18] and Gihmann and Skorohod [9]. The solution of (2.1),

$$X_t = X_0 + \int_0^t \int c(X_{s-}, u)\tilde{\nu}(duds),$$

is a martingale with jumps. For any measurable function f defined on $\mathbb{R}$ such that $f(x) > 0$ for $x > 0$, we choose $\pi(du) = du$, the Lebesgue measure on $\mathbb{R}$ and

$$c(x, u) = -\delta x 1_{(0,f(x))}(u), \quad 0 < \delta < 1 \tag{2.2}$$

in (2.1) to obtain a martingale process with jump rate at x of $f(x)$ and jump size $-\delta x$. Here, for $x > 0$, $1_{(0,x)}(u) = 1$, if $u \in (0, x)$, 0 if $u \notin (0, x)$. In this paper, we consider functions of the form

$$f(x) = x^\alpha, \quad \text{or} \quad f(x) = (\ln x)^\alpha, \quad \alpha > 0.$$

By this formulation, we can add endogenous variables into consideration. We now add a Brownian noise to get the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + \int c(X_{t-}, u)\tilde{\nu}(dudt). \tag{2.3}$$

Now we set up mathematical background and show the existence and uniqueness of processes that we are interested in.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, a non-decreasing family of σ-algebra $\mathcal{F}_t$ belonging to $\mathcal{F}$ be given and let $D_{\mathbb{R}}[0, \infty)$ be the set of processes on $\mathbb{R}$ which are right continuous with left limit. We consider a Poisson random measure ν on $\mathbb{R} \times \mathbb{R}^+$ satisfying the following two properties:

(i) if $C_1, C_2, \ldots, C_n$ are pairwise disjoint Borel sets of $\mathbb{R}\times\mathbb{R}^+$, then the random variables $\nu(C_1), \nu(C_2), \ldots, \nu(C_n)$ are mutually independent Poisson random variables;

(ii) for a Borel set $A \subset \mathbb{R}$, $\nu(A \times [0,t])$ is $\mathcal{F}_t$-measurable and in $D_{\mathbb{R}}[0,\infty)$, having a Poisson distribution with

$$E(\nu(A \times [0,t])) = \pi(A)t,$$

where π is a σ-finite measure on $\mathbb{R}$.

For a Borel set $\Delta \subset \mathbb{R}^+$, set

$$\tilde{\nu}(A \times \Delta) = \nu(A \times \Delta) - \pi(A)|\Delta|.$$

Here $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$. Note that, for a fixed set A with $\pi(A) < \infty$, $\nu(A \times [0,t])$ is a Poisson process with parameter $\pi(A)$, i.e., $\tilde{\nu}(A \times [0,t])$ is a martingale.

Now for a Brownian motion starting from 0, $\{B_t\}$ which is independent of ν, consider the stochastic differential equation (2.3) where $\sigma(x), b(x)$ and $c(x,u)$ are nonrandom functions. Then, under the following **Condition C**, (2.3) has a unique solution, which is in $D_{\mathbb{R}}[0,\infty)$, and we call the solution 'a diffusion with jump'.

Condition C

(C1) There is a constant K such that for any $x \in \mathbb{R}$,

$$|\sigma(x)|^2 + |b(x)|^2 + \int |c(x,u)|^2 \pi(du) \leq K(1+|x|^2).$$

(C2) (Local Lipschitz condition) For any $M > 0$, there is a constant C_M such that for $|x| < M$, $|y| < M$,

$$|\sigma(x)-\sigma(y)|^2 + |b(x)-b(y)|^2 + \int |c(x,u)-c(y,u)|^2 \pi(du) \leq C_M |x-y|^2.$$

Now, this section is devoted to prove the following Theorem 2.1 which gives the existence and uniqueness of the solution of SDE (2.4), not covered by general theory of jump diffusion above. In above setting, we let $\pi(du) = du$ and consider the following SDE:

$$X_t = X_0 + \int_0^t \sigma X_s dB_s + \int_0^t \int -\delta X_{s-} 1_{(0,X_{s-}^{\alpha})}(u) \tilde{\nu}(duds). \tag{2.4}$$

Theorem 2.1. *Let $\sigma \geq 0$, $0 < \delta < 1$, $\alpha > 0$ be given and $X_0 > 0$ with $E(X_0) < \infty$. Then there exists a unique solution of* (2.4).

Proof. Let X_t, Y_t be solutions of (2.4). Then

$$X_t - Y_t = \int_0^t \sigma(X_s - Y_s) dB_s + \int_0^t \int -\delta[X_{s-} 1_{(0,X_{s-}^{\alpha})}(u) - Y_{s-} 1_{(0,Y_{s-}^{\alpha})}(u)] \tilde{\nu}(duds).$$

Take $N > 0$ large, then put

$$\tau_N = \inf\{t \geq 0 : |X_t| \geq N\},$$
$$\tau_N' = \inf\{t \geq 0 : |Y_t| \geq N\}.$$

Note that the coefficients do not satisfy Lipschitz conditions. Following the ideas in Theorem 3.2 of Ikeda and Watanabe [10], define a decreasing sequence of numbers $\{a_n\}_{n=1}^{\infty}$ such that $1 = a_0 > a_1 > a_2 > \cdots > a_n > \cdots > 0$ and

$$\int_{a_n}^{a_{n-1}} u^{-2} du = n \quad \text{for} \quad n \geq 1.$$

Clearly, $a_n \to 0$ as $n \to \infty$. For each $n = 1, 2, \ldots$, let $\psi_n(u)$ be a continuous function such that its support is contained in (a_n, a_{n-1}) and

$$0 \leq \psi_n(u) \leq 2\frac{u^{-2}}{n}, \quad \text{and} \quad \int_{a_n}^{a_{n-1}} \psi_n(u) du = 1.$$

Set

$$\phi_n(x) = \int_0^{|x|} \int_0^y \psi_n(u) du dy, \quad x \in \mathbb{R}.$$

Then $\phi_n \in C^2(\mathbb{R}^1)$, $|\phi_n'(x)| \leq 1$ and $\phi_n(x) \uparrow |x|$ as $n \to \infty$. By Ito formula (2.5)

$$\begin{aligned}
&\phi_n(X_{t\wedge\tau_N\wedge\tau_N'} - Y_{t\wedge\tau_N\wedge\tau_N'}) \\
&= \int_0^{t\wedge\tau_N\wedge\tau_N'} \sigma\phi_n'(X_s - Y_s)(X_s - Y_s) dB_s \\
&\quad + \frac{\sigma^2}{2}\int_0^{t\wedge\tau_N\wedge\tau_N'} \phi_n''(X_s - Y_s)(X_s - Y_s)^2 ds \\
&\quad + \int_0^{t\wedge\tau_N\wedge\tau_N'} \int [\phi_n\left(X_{s-} - Y_{s-} - \delta(X_{s-}1_{(0,X_{s-}^\alpha)}(u) - Y_{s-}1_{(0,Y_{s-}^\alpha)}(u))\right) \\
&\quad - \phi_n(X_{s-} - Y_{s-})]\tilde{\nu}(duds) \\
&\quad + \int_0^{t\wedge\tau_N\wedge\tau_N'} \int \{\phi_n(X_{s-} - Y_{s-} - \delta(X_{s-}1_{(0,X_{s-}^\alpha)}(u) - Y_{s-}1_{(0,Y_{s-}^\alpha)}(u)) \\
&\quad - \phi_n(X_{s-} - Y_{s-}) \\
&\quad + \delta\phi_n'(X_{s-} - Y_{s-})[X_{s-}1_{(0,X_{s-}^\alpha)}(u) - Y_{s-}1_{(0,Y_{s-}^\alpha)}(u)]\} duds.
\end{aligned}$$

Let us denote the four integrals in (2.5) by I, II, III and IV in the order and let us estimate their expectations. For any $s \leq t \wedge \tau_N \wedge \tau_N'$, X_s and Y_s are positive and jumps downward only and $|X_s| \leq N$ and $|Y_s| \leq N$. Therefore we can use truncation method as continuous case; for all t,

$$E(\text{I}) = E\int_0^{t\wedge\tau_N\wedge\tau_N'} \sigma\phi_n'(X_s - Y_s)(X_s - Y_s) dB_s = 0$$

and similarly $E(\text{III}) = 0$. By the construction of ϕ_n

$$\begin{aligned}
0 \leq E(\text{II}) &\leq \frac{\sigma^2}{2} E\int_0^{t\wedge\tau_N\wedge\tau_N'} \frac{2}{n}|X_s - Y_s|^{-2}|X_s - Y_s|^2 ds \\
&\leq \frac{\sigma^2 t}{n}.
\end{aligned}$$

By the Mean Value Theorem, for some X^* between $X_{s-} - Y_{s-}$ and

$$X_{s-} - Y_{s-} - \delta(X_{s-}1_{(0,X_{s-}^\alpha)}(u) - Y_{s-}1_{(0,Y_{s-}^\alpha)}(u)),$$

we have

$$\begin{aligned} &\text{IV} \\ &= \int_0^{t\wedge\tau_N\wedge\tau_N'} \int [-\phi_n'(X_s^*) + \phi_n'(X_s - Y_s)]\delta(X_s1_{(0,X_s^\alpha)}(u) - Y_s1_{(0,Y_s^\alpha)}(u))duds \\ &= \delta\int_0^{t\wedge\tau_N\wedge\tau_N'} [\phi_n'(X_s - Y_s) - \phi_n'(X_s^*)](X_s^{1+\alpha} - Y_s^{1+\alpha})ds. \end{aligned}$$

Since $|\phi_n'(x)| \leq 1$ for all x, by the Mean Value Theorem on the function $x^{\alpha+1}$, there is X^{**} such that

$$\begin{aligned} |\text{IV}| &\leq 2\delta\int_0^{t\wedge\tau_N\wedge\tau_N'} |X_s^{\alpha+1} - Y_s^{\alpha+1}|ds \\ &= 2\delta\int_0^{t\wedge\tau_N\wedge\tau_N'} |(\alpha+1)(X_s^{**})^\alpha(X_s - Y_s)|ds, \end{aligned}$$

where X_s^{**} lies between X_s and Y_s and $|X_s^{**}| \leq C_N$ for some constant C_N depending on N. Hence

$$E(|\text{IV}|) \leq \delta C_{\alpha,N}\int_0^t E(|X_{s\wedge\tau_N\wedge\tau_N'} - Y_{s\wedge\tau_N\wedge\tau_N'}|)ds$$

for some constant $C_{\alpha,N}$ depending on α and N and

$$E[\phi_n(X_{t\wedge\tau_N\wedge\tau_N'} - Y_{t\wedge\tau_N\wedge\tau_N'})] \leq \delta C_{\alpha,N}\int_0^t E(|X_{s\wedge\tau_N\wedge\tau_N'} - Y_{s\wedge\tau_N\wedge\tau_N'}|)ds + \frac{\sigma^2 t}{n}.$$

Therefore as $n \to \infty$,

$$E[|X_{t\wedge\tau_N\wedge\tau_N'} - Y_{t\wedge\tau_N\wedge\tau_N'}|] \leq \delta C_{\alpha,N}\int_0^t E(|X_{s\wedge\tau_N\wedge\tau_N'} - Y_{s\wedge\tau_N\wedge\tau_N'}|)ds$$

and by the Gronwall's Lemma and the right continuity of the processes, $X_t = Y_t$ a.s. for all $t \leq \tau_N \wedge \tau_N'$.

For the construction of the solution, for given N, $X_0 > 0$, let

$$X_t^0 = X_0,$$

$$X_t^n = \begin{cases} X_0 + \int_0^t \sigma X_s^{n-1}dB_s - \int_0^t\int \delta X_{s-}^{n-1}1_{(0,(X_{s-}^{n-1})^\alpha)}(u)\tilde{\nu}(duds), \\ \qquad \text{if } t \leq \tau_N^n \\ N, \qquad \text{if } t \geq \tau_N^n, \end{cases}$$

where $\tau_N^n = \inf\{t \geq 0 : |X_t^n| \geq N\}$. For $t \geq 0, n \geq 1$, by the construction, $X_t^n \leq N$ and X_t^n is right continuous.

$$X_t^{n+1} - X_t^n = \int_0^t \sigma(X_s^n - X_s^{n-1})dB_s$$

$$- \int_0^t \int \delta(X_{s-}^n 1_{(0,(X_{s-}^n)^\alpha)}(u) - X_{s-}^{n-1} 1_{(0,(X_{s-}^{n-1})^\alpha)}(u))\tilde{\nu}(duds).$$

Apply Ito formula to $\phi_k(x)$, let $k \to \infty$, and use the same argument as uniqueness to get

$$E|X_t^{n+1} - X_t^n| \le \delta C_{\alpha,N} \int_0^t E|X_s^n - X_s^{n-1}|ds.$$

By iteration, we have

$$E|X_t^{n+1} - X_t^n| \le \frac{(\delta C_{\alpha,N})^n t^n}{n!}.$$

From the construction, for each n, X_t^n is a martingale and right continuous with left limit. By the Doob's inequality

$$\frac{1}{2^n} P(\sup_{0\le s\le t} |X_s^{n+1} - X_s^n| \ge \frac{1}{2^n}) \le E|X_t^{n+1} - X_t^n|,$$

so

$$P(\sup_{0\le s\le t} |X_s^{n+1} - X_s^n| \ge \frac{1}{2^n}) \le \frac{(2\delta C_{\alpha,N})^n t^n}{n!}.$$

Therefore by Borel-Cantelli lemma

$$P(\sup_{0\le s\le t} |X_s^{n+1} - X_s^n| \ge \frac{1}{2^n} \quad \text{i.o.}) = 0.$$

This implies that, almost surely, X_s^n converges to a limit Y_s uniformly on $[0, t]$. Since $t > 0$ is arbitrary, it follows that, almost surely, X^n converges uniformly on any bounded time interval to a limit process Y. Furthermore the process Y is right continuous with left limit.

Now we need to show that Y_t is a solution of (2.4) up to $\sigma_N = \inf\{t \ge 0 : |Y_t| \ge N\}$. Let

$$Z_t = X_0 + \int_0^t \sigma Y_s dB_s - \int_0^t \int \delta Y_{s-} 1_{(0,Y_{s-}^\alpha)}(u)\tilde{\nu}(duds).$$

Then

$$\begin{aligned} Z_t - X_t^{n+1} &= \int_0^t \sigma(Y_s - X_s^n)dB_s \\ &\quad - \int_0^t \int \delta(Y_{s-} 1_{(0,Y_{s-}^\alpha)}(u) - X_{s-}^n 1_{(0,(X_{s-}^n)^\alpha)}(u))\tilde{\nu}(duds). \end{aligned}$$

By the same argument as uniqueness, we have

$$E|Z_t - X_t^{n+1}| \le \delta C_{\alpha,N} \int_0^t E|Y_s - X_s^n|ds$$

and there exists $m > 0$ such that $E|Z_t - X_t^{n+1}| < \epsilon$ if $n > m$ for given $\epsilon > 0$. Therefore Z_t is the pointwise limit of X_t^n and $Z_t = Y_t$ a.s. Hence, let X the unique solution of (2.4) up to τ_N, $\tau_0 = \inf\{t > 0 : X_t \le 0\}$ and $X_0 > 0$.

Then $\tau_0 > 0$ since X_t is right continuous. Therefore by applying Ito formula to $f(x) = \ln x$ for $x > 0$ before τ_0, we have

$$
\begin{aligned}
X_t = X_0 \exp[\sigma B_t - \frac{1}{2}\sigma^2 t + \int_0^t \int \ln(1 - \delta 1_{(0,X_{s-}^\alpha)}(u))\tilde{\nu}(duds) \\
+ \int_0^t \int (\ln(1 - \delta 1_{(0,X_{s-}^\alpha)}(u)) + \delta 1_{(0,X_{s-}^\alpha)}(u))duds].
\end{aligned} \tag{2.6}
$$

Hence, $X_t > 0$ if $X_0 > 0$ and X_t is a process only with stochastic integrals. Therefore X_t is a positive local martingale and for arbitrary N, $X_{t\wedge\tau_N}$ is a martingale and we have $E(X_{t\wedge\tau_N}) = E(X_0)$. Then by Fatou's Lemma, for any t, $E(X_t) \le \lim_{N->\infty} E(X_{t\wedge\tau_N})$. Therefore $\sup_t E[X_t] < M$ for some $M > 0$. Hence X_t does not explode in finite time a.s. and we are done. □

Remark 2.1. Consider the following SDE

$$
X_t = X_0 + \int_0^t \sigma X_s dB_s + \int_0^t \int -\delta X_{s-} 1_{(X_{s-}>1)} 1_{(0,(\ln X_{s-})^\alpha)}(u)\tilde{\nu}(duds). \tag{2.7}
$$

The existence and uniqueness of the solution of (2.7) can be proved by a similar argument. But in this case we take

$$
\begin{aligned}
\tau_{N,\epsilon} &= \inf\{t > 0 : |X_t| > N \text{ or } |X_t| < 1 + \epsilon\}, \\
\tau'_{N,\epsilon} &= \inf\{t > 0 : |Y_t| > N \text{ or } |Y_t| < 1 + \epsilon\}
\end{aligned}
$$

for $N > 0$ and $\epsilon > 0$. Then $X_{s-}1_{(0,(X_{s-})^\alpha)}(u)$ is changed to

$$
X_{s-}1_{(X_{s-}>1)}1_{(0,(\ln(X_{s-}))^\alpha)}(u)
$$

and (2.5) to

$$
\begin{aligned}
\text{IV} &\le 2\delta \int_0^{t\wedge\tau_{N,\epsilon}\wedge\tau'_{N,\epsilon}} \int |X_{s-}1_{(X_{s-}>1)}1_{(0,(\ln(X_{s-}))^\alpha)}(u) \\
&\quad - Y_{s-}1_{(Y_{s-}>1)}1_{(0,(\ln(Y_{s-}))^\alpha)}(u)|duds \\
&= 2\delta \int_0^{t\wedge\tau_{N,\epsilon}\wedge\tau'_{N,\epsilon}} |X_s(\ln X_s)^\alpha - Y_s(\ln Y_s)^\alpha| ds \\
&\le 2\delta C \int_0^{t\wedge\tau_{N,\epsilon}\wedge\tau'_{N,\epsilon}} |X_s - Y_s| ds
\end{aligned}
$$

by the Mean Value Theorem for $f(x) = x(\ln x)^\alpha$, $f'(x) = (\ln x)^\alpha + (\ln x)^{\alpha-1}$ and so $|f'(x)| \le C$ if $1 + \epsilon < x < N$.

For the construction of the solution, let

$$
X_t^0 = x,
$$

$$
X_t^n = \begin{cases} x + \int_0^t \sigma X_s^{n-1} dB_s \quad - \int_0^t \int \delta X_{s-}^{n-1} 1_{(X_{s-}^{n-1}>1)} 1_{(0,(\ln X_{s-}^{n-1})^\alpha)}(u)\tilde{\nu}(duds), \\ \qquad\qquad \text{if} \quad t < \tau_{N,\epsilon}^n \\ X_{(\tau_{N,\epsilon}^n)-}, \qquad\qquad \text{if} \quad t \ge \tau_{N,\epsilon}^n, \end{cases}
$$

where $\tau^n_{N,\epsilon} = \inf\{s \geq 0 : |X^n_s| \geq N \text{ or } |X^n_s| < 1+\epsilon\}$. Then for all $k \leq n$ and for all t, $1+\epsilon \leq X^k_t \leq N$. Therefore by the same argument as Theorem 2.1, we have a unique solution.

3. Crash time and its distribution

The crash models in Section 2 consist of downward jumps and their nondecreasing compensation with or without Brownian noise. Our first interest is to explore the behavior of the compensation part of the model. If there is no crash, the process will increase. It turns out that for the case of $f(x) = x^\alpha$, $\alpha > 0$, the process should explode in finite time. But then, since the jump rate tends to infinity, we conclude that the crash occurs before the time of explosion.

For $X_0 >$, consider the equation

$$X_t = X_0 + \int_0^t \int -\delta X_{s-} 1_{(0, X^\alpha_{s-})}(u) \tilde{\nu}(duds), \tag{3.1}$$

which is the case with $f(x) = x^\alpha$ in (2.2).

Theorem 3.1. *Let X_t be a solution of* (3.1). *Then conditional on no jumps, X_t is deterministic if X_0 is nonrandom, and is given by $X_t = (X_0^{-\alpha} - \alpha\delta t)^{\frac{1}{\alpha}}$. Therefore, the explosion occurs at $t_e = 1/(\alpha\delta X_0^\alpha)$.*

Proof. Until the first jump occurs, the jump part of the process vanishes. Therefore, the process consists of only the compensation part, and it takes the form of a deterministic curve satisfying

$$X_t = X_0 + \int_0^t \delta X_s^{\alpha+1} ds. \tag{3.2}$$

Solving this equation, with the nonrandom initial condition $X_0 > 0$, we have a unique blow up solution. More precisely, let $\tau = \inf\{t > 0 : X_t - X_{t-} \neq 0\}$ be the first jump time. Then, conditional on $t < \tau$, the solution of (3.2) is given by

$$X_t = \frac{1}{(X_0^{-\alpha} - \alpha\delta t)^{\frac{1}{\alpha}}}.$$

Clearly, the explosion occurs at $t = 1/(\alpha\delta X_0^\alpha)$. □

Remark 3.1. We can get Theorem 3.1 from Theorem 3.3 by letting $\sigma = 0$. The reason we state this case here is that we want to show its explosion time first.

For next case, we let $f(x) = (\ln x)^\alpha$, then, surprisingly, it turns out that $\alpha = 1$ is the borderline for processes of two different characters. Let us consider the following SDE:

$$X_t = X_0 + \int_0^t \int -\delta X_{s-} 1_{(X_{s-}>1)} 1_{(0,(\ln X_{s-})^\alpha)}(u) \tilde{\nu}(duds). \tag{3.3}$$

Note that this equation is just for the case $f(x) = (\ln x)^\alpha$ with a small modification term $1_{(X_{s-}>1)}$. The term $1_{(X_{s-}>1)}$ is necessary to keep $\ln X_{s-} > 0$. But it does not change our model much since, if the initial condition X_0 is much bigger than 1, even after the first jump, X_t is also bigger than 1. Our interest is to study the behavior of the process until the first jump. Now let X_t be the solution of (3.3), then we have the following theorem.

Theorem 3.2. *Assume $X_0 >> 1$. Conditional on no jumps, the solution of (3.3) X_t is deterministic if X_0 is nonrandom. Moreover, we get*

$$X_t = \begin{cases} \exp\left[(1-\alpha)\delta t + (\ln X_0)^{1-\alpha}\right]^{\frac{1}{1-\alpha}} & \text{if } \alpha \neq 1 \\ (X_0+1)\exp(\exp \delta t) - 1 & \text{if } \alpha = 1. \end{cases} \tag{3.4}$$

Proof. By Ito formula, we have

$$\begin{aligned} X_t = X_0 \exp[&\int_0^t \int \ln|1 - 1_{(X_{s-}>1)}\delta 1_{(0,(\ln X_{s-})^\alpha)}(u)|\tilde{\nu}(duds) \\ &+ \int_0^t \int (\ln|1 - 1_{(X_{s-}>1)}\delta 1_{(0,(\ln X_{s-})^\alpha)}(u)| \\ &+ 1_{(X_{s-}>1)}\delta 1_{(0,(\ln X_{s-})^\alpha)}(u))duds], \end{aligned}$$

which shows that $X_t > 0$ and (3.3) has a unique solution since it is a special case of (2.6) (see Theorem 2.1 and Remark 2.1).

Now to check the path properties of the process before the first jump, eliminate the jump part in (3.3) to get the equation

$$\begin{aligned} X_t &= X_0 + \int_0^t \int \left(\delta X_s 1_{(X_{s-}>1)} 1_{(0,(\ln X_{s-})^\alpha)}(u)\right) duds \\ &= X_0 + \int_0^t \delta 1_{(X_s>1)} X_s (\ln X_s)^\alpha ds. \end{aligned} \tag{3.5}$$

This is an ordinary differential equation, and, by a simple calculation with nonrandom initial condition $X_0 >> 1$, we get the following results. First, if $\alpha \neq 1$, then the solution of (3.5) is given by

$$\ln X_t = \left[(1-\alpha)\delta t + (\ln X_0)^{1-\alpha}\right]^{\frac{1}{1-\alpha}}.$$

Clearly, for $\alpha < 1$, the process never explodes in finite time, but grows super exponentially with order $\exp[((1-\alpha)\delta t)^{\frac{1}{1-\alpha}}]$. On the other hand, for $\alpha > 1$, it explodes at

$$t_e = \frac{(\ln X_0)^{-\alpha+1}}{\delta(\alpha - 1)}.$$

Finally, if $\alpha = 1$, the solution of (3.5) is $X_t = X_0 \exp(\exp \delta t)$.

As a result, if $\alpha > 1$, the process cannot remain longer than t_e, similarly to Theorem 3.1. However, if $\alpha \leq 1$, the process may stay longer. □

As explained above, we know from (3.4) that if $\alpha \leq 1$, then X_t never explodes, though it grows super exponentially. But for $\alpha > 1$, it explodes before

$$t_e = \frac{(\ln X_0)^{1-\alpha}}{\delta(\alpha - 1)}.$$

This phenomenon is called a phase transition in statistical physics and has been extensively studied by many authors for a long time. At this point, we do not have clear intuition as to why $\ln x$ is the borderline of the phase transition, but we have a similar result in a different model of Jeon [11].

Now we add a Brownian noise to (3.1), that is, consider the following SDE:

$$X_t = X_0 + \int_0^t \sigma X_s dB_s + \int_0^t \int -\delta X_{s-} 1_{(0,X_{s-}^{\alpha})}(u)\tilde{\nu}(duds). \tag{3.6}$$

Theorem 3.3. *Let X_t be a solution of* (3.6). *If $X_0 > 0$, then conditional on no jumps, X_t is given by*

$$X_t = \exp(-\frac{\sigma}{2}t + B_t)^{\sigma} \left(X_0^{-\alpha} - \alpha\delta \int_0^t [\exp(-\frac{\sigma s}{2} + B_s)]^{\alpha\sigma} ds \right)^{-\frac{1}{\alpha}}. \tag{3.7}$$

Proof. Let $\tau = \inf\{t > 0 : X_t - X_{t-} \neq 0\}$ be the first jump time. Again, before the first jump, the jump part vanishes. Then conditional on $t < \tau$, the process satisfies

$$X_t = X_0 + \int_0^t \sigma X_s dB_s + \int_0^t \delta X_s^{1+\alpha} ds.$$

To solve this SDE, apply Ito formula to $f(x) = x^{-\alpha}$ and change the form as follows:

$$dX_t^{-\alpha} = \left(-\alpha\delta + \frac{\sigma^2\alpha(\alpha+1)}{2} X_t^{-\alpha} \right) dt - \sigma\alpha X_t^{-\alpha} dB_t.$$

Note that the resulting SDE is linear in $X_t^{-\alpha}$, and the general solution of such an equation is well known. See, for example, Gihman and Skorohod ([9], p.36). By substituting all the coefficients in the general formula, we have

$$X_t = \exp(-\frac{\sigma}{2}t + B_t)^{\sigma} \left(X_0^{-\alpha} - \alpha\delta \int_0^t \left[\exp(-\frac{\sigma s}{2} + B_s) \right]^{\alpha\sigma} ds \right)^{-\frac{1}{\alpha}}$$

as desired. ☐

Note that the integral

$$\int_0^t [\exp(-\frac{\sigma s}{2} + B_s)]^{\alpha\sigma} ds$$

in (3.7) is finite almost surely, since $-\sigma s/2$ is the dominating factor. Indeed, for any $\epsilon > 0$,

$$\limsup_{s\to\infty} \frac{B_s}{\sqrt{2s \log\log s}} = 1,$$

almost surely, and therefore $B_s = o(s^{1/2+\epsilon})$, a.s. Hence, if X_0 is small enough, or for a fixed X_0, if δ is small or if σ is big enough, then, with positive probability,

$$X_0^{-\alpha} > \alpha\delta \int_0^t [\exp(-\frac{\sigma s}{2} + B_s)]^{\alpha\sigma} ds,$$

i.e., X_t does not explode in finite time. For such a case, with positive probability, $X_t \to 0$ exponentially fast, as $t \to \infty$. If X_t becomes small, then the jump intensity also becomes small and, therefore, there may not be any crash, i.e., the bubble lands softly. It is surprising that the noise may have a great influence on the sample paths of the price. The bubble may go into crash or may make a soft landing possibly depending on the size of the volatility of the noise.

An interesting question may be posed to this endogenous modeling: What is the distribution of the first crash time. The following two theorems provide answers. Let $\tau = \inf\{t > 0 : X_t - X_{t-} \neq 0\}$ be the first jump time. We get the distribution of τ as follows.

Theorem 3.4. *For the case of $f(x) = x^\alpha, \alpha > 0$, the distribution of the crash is given by*

$$P(\tau \le t) = 1 - \int_0^\infty e^{-v}\eta(dv),$$

where

$$\eta([0,v]) = P(\int_0^t Z_s^{\alpha\sigma} ds \le \frac{1 - e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta}) + P(\int_0^t Z_s^{\alpha\sigma} ds \ge \frac{1 + e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta})$$

and $Z_t \equiv \exp(-\frac{\sigma t}{2} + B_t)$.

Proof. Being X_t a continuous process before the first crash,

$$\begin{aligned} P(\tau > t) =& P(\int_0^t \int 1_{(0,X_{s-}^\alpha)}(u)\nu(duds) = 0) \\ =& P(\int_0^t \int 1_{(0,X_s^\alpha)}(u)\nu(duds) = 0). \end{aligned}$$

Let

$$N_t = \int_0^t \int 1_{(0,X_s^\alpha)}(u)\nu(duds).$$

Then ν and $\{X_s\}$ are independent and $N_t = \nu(A(t))$, where

$$A(t) = \{(u,s) : 0 \le u \le X_s^\alpha, 0 \le s \le t\}.$$

That is, N_t is a Poisson point process with intensity

$$\int_0^t \int 1_{A(s)}(u,s)duds = \int_0^t X_s^\alpha ds,$$

i.e., $N_t - \int_0^t X_s^\alpha ds$ is a jump martingale with jump size 1. Therefore if we let $\beta(t)$ be as $\int_0^{\beta(t)} X_s^\alpha ds = t$, then $N_{\beta(t)}$ is a Poisson process such that $N_{\beta(t)} - t$ is a martingale. Therefore, $V_t \equiv N_{\beta(t)}$ is a Poisson process with intensity t. Consequently,

$$\begin{aligned} P(\tau > t) &= P(N_r = 0 \quad \text{for all } r \le t) \\ &= P(N_{\beta(r)} = 0 \quad \text{for all } r \le \beta^{-1}(t)) \end{aligned}$$

$$\begin{aligned}
&= P(V_r = 0 \quad \text{for all } r \le \beta^{-1}(t)) \\
&= \int_0^\infty P(V_r = 0 \quad \text{for all } r \le \int_0^t X_s^\alpha ds | \int_0^t X_s^\alpha ds = v)\eta(dv) \\
&= \int_0^\infty e^{-v}\eta(dv),
\end{aligned}$$

where $\eta(\cdot) = P(\int_0^t X_s^\alpha ds \in \cdot)$ and X_t is given by (3.7). So

$$\begin{aligned}
P(\tau \le t) &= 1 - P(\tau > t) \\
&= 1 - \int_0^\infty e^{-v}\eta(dv).
\end{aligned}$$

We write $Z_t = \exp(-\frac{\sigma t}{2} + B_t)$ for notational simplicity. Then since

$$X_t^\alpha = -\frac{1}{\alpha\delta} \cdot \frac{\frac{d}{dt}(X_0^{-\alpha} - \alpha\delta \int_0^t Z_s^{\alpha\sigma} ds)}{X_0^{-\alpha} - \alpha\delta \int_0^t Z_s^{\alpha\sigma} ds}$$

we have

$$\int_0^t X_s^\alpha ds = \ln |1 - \alpha\delta X_0^\alpha \int_0^t Z_s^{\alpha\sigma} ds|^{-\frac{1}{\alpha\delta}}.$$

Hence

$$\begin{aligned}
\eta([0, v]) &= P(\int_0^t X_s^\alpha ds \le v) \\
&= P(|1 - \alpha\delta X_0^\alpha \int_0^t Z_s^{\alpha\sigma} ds| \ge e^{-v\alpha\delta}) \\
&= P(\int_0^t Z_s^{\alpha\sigma} ds \le \frac{1 - e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta}, \text{ or } \int_0^t Z_s^{\alpha\sigma} ds \ge \frac{1 + e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta}) \\
&= P(\int_0^t Z_s^{\alpha\sigma} ds \le \frac{1 - e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta}) + P(\int_0^t Z_s^{\alpha\sigma} ds \ge \frac{1 + e^{-v\alpha\delta}}{X_0^\alpha \alpha\delta}),
\end{aligned}$$

which completes the proof. □

Remark 3.2. For the case of $f(x) = x^\alpha, \alpha > 0$, without noise, the distribution of the crash is given by

$$P(\tau \le t) = 1 - |1 - \alpha\delta X_0^\alpha t|^{\frac{1}{\alpha\delta}} \quad \left(t < \frac{1}{\alpha\delta X_0^\alpha}\right). \tag{3.8}$$

This is a special case of Theorem 3.4, but it is worth to mention.

By the definition of $\beta(t)$ in the proof of Theorem 3.4,

$$\begin{aligned}
\beta^{-1}(t) &= \int_0^t X_s^\alpha ds \\
&= \int_0^t \frac{ds}{X_0^{-\alpha} - \alpha\delta s}
\end{aligned}$$

$$= \ln |1 - \alpha\delta X_0^\alpha t|^{-\frac{1}{\alpha\delta}}.$$

Therefore

$$P(\tau > t) = \exp[-\ln |1 - \alpha\delta X_0^\alpha t|^{-\frac{1}{\alpha\delta}}] = |1 - \alpha\delta X_0^\alpha t|^{\frac{1}{\alpha\delta}}$$

and (3.8) follows. One interesting result of this is that the distribution has a different form around $\alpha\delta = 1$. Indeed, the density function, the derivative of (3.8), blows up to infinity if $\alpha\delta > 1$, while it decreases to 0 if $\alpha\delta < 1$ as $t \to t_e$.

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