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Estimations in a Skewed Double Weibull Distribution

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Abstract

We obtain a skewed double Weibull distribution by a double Weibull distribution, and evaluate its coefficient of skewness. And we obtain the approximate maximum likelihood estimator(AML) and moment estimator of skew parameter in the skewed double Weibull distribution, and hence compare simulated mean squared errors(MSE) of those estimators. We compare simulated MSE of two proposed reliability estimators in two independent skewed double Weibull distributions each with different skew parameters. Finally we introduce a skewed double Weibull distribution generated by a uniform kernel.

Keywords: Approximate ML, double Weibull distribution, reliability, skew parameter.

1. Introduction

Many authors have studied estimation and characterization in a double Weibull distribution with a shape parameter δ and a scale parameter β, whose distribution was used widely in engineering applications in Johnson *et al.* (1995). Especially if $\delta = 1$, then the double Weibull distribution is a Laplace distribution.

Azzalini (1985) studied a class of distributions which includes the normal ones. Azzalini and Capitanio (1999) studied the multivariate skewed normal distribution. Woo (2007) studied reliability in half-triangle distributions and skewness of a skewed distribution. Son and Woo (2007a,b) studied AML in a skewed Laplace distribution and a skewed double Rayleigh distribution. Ali *et al.* (2008) also studied skewed double distributions generated by a double gamma kernel. Balakrishnan and Cohen (1991) proposed a method of finding AML of parameter in several distributions. Han and Kang (2006) studied AML of parameters in several distributions with censored samples.

Let X and Y be independently identical distributed continuous random variables with the probability density function(pdf) $f(x) = F'(x)$ which is symmetric about zero. Then for $\forall\alpha \in R^1$, $1/2 = P\{X - \alpha Y \le 0\} = \int_{-\infty}^{\infty} f(t)F(\alpha t)dt$.

And hence, a skewed density is as given in Azzalini (1985) by:

$$f(z;\alpha) \equiv 2f(z)F(\alpha z). \tag{1.1}$$

The density $f(z;\alpha)$ becomes skewed density of a random variable Z and parameter α is skew parameter of the skewed distribution. Especially if $\alpha = 0$, $f(z;0)$ becomes the original symmetric density.

In this paper, we obtain a skewed double Weibull distribution from a double Weibull distribution, and also evaluate its coefficient of skewness. And we obtain AML and moment estimator of skew parameter in the skewed double Weibull distribution. We compare simulated MSE of two proposed reliability estimators in independent skewed double Weibull distributions with two different skew parameters. And as we introduce a skewed double Weibull generated by a uniform kernel, we observe the skewness by evaluating its coefficient of skewness.

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2. A Skewed Double Weibull Distribution

The following density and cdf of double Weibull random variable are as given in Johnson *et al.* (1995) by:

$$f(x) = \frac{\delta}{2\beta}|x|^{\delta-1}e^{-\frac{|x|^\delta}{\beta}}, \quad -\infty < x < \infty, \tag{2.1}$$

$$F(x) = \frac{1}{2} + \mathrm{sgn}(x)\frac{1}{2}\left(1 - e^{-\frac{|x|^\delta}{\beta}}\right), \quad -\infty < x < \infty. \tag{2.2}$$

From the density (2.1) and the *cdf* (2.2) of a double Weibull random variable and a skewed density (1.1), a skew-symmetric double Weibull density is given as:

$$f(z;\alpha) = \frac{\delta}{2\beta}|z|^{\delta-1}e^{-\frac{|z|^\delta}{\beta}}\left\{1 + \mathrm{sgn}(\alpha z)\left(1 - e^{-\frac{|\alpha z|^\delta}{\beta}}\right)\right\}, \quad z \in R^1. \tag{2.3}$$

From the density (2.1) and the cdf (2.2), the cdf of the skewed double Weibull random variable Z is given by: For $\alpha > 0$,

$$F(z;\alpha) = \left(1 - e^{-\frac{|z|^\delta}{\beta}}\right)I_{(0,\infty)}(z) + \frac{1}{2(1+\alpha^\delta)}\exp\left(-\frac{1+\alpha^\delta}{\beta}|z|^\delta\right), \quad z \in R^1, \tag{2.4}$$

where $I_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$

Remark 1. If $\alpha < 0$, then $F(z;\alpha) = 1 - F(-z;-\alpha)$ from Lemma 2(b) in Ali and Woo (2006).

2.1. Skewness of skewed double Weibull distribution

From the density (2.3) and formula 3.381(4) in Gradshteyn and Ryzhik (1965, p.317), we obtain k^{th} moment of the skewed double Weibull random variable Z: For given $\alpha > 0$,

$$E\left(Z^k;\alpha\right) = \beta^{\frac{k}{\delta}} \cdot \Gamma\left(\frac{k}{\delta}+1\right)\left\{\frac{(-1)^k-1}{2}\left(1+\alpha^\delta\right)^{-1-\frac{k}{\delta}} + 1\right\}, \quad k = 1,2,\ldots, \tag{2.5}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is a gamma function of $x > 0$.

Especially if $k = 1$ and 2, then

$$E(Z) = \beta^{\frac{1}{\delta}} \cdot \Gamma\left(1 + \frac{1}{\delta}\right)\left\{1 - \left(1+\alpha^\delta\right)^{-1-\frac{1}{\delta}}\right\} \quad \text{and} \quad E\left(Z^2\right) = \beta^{\frac{2}{\delta}} \cdot \Gamma\left(1 + \frac{2}{\delta}\right). \tag{2.6}$$

From moment in (2.5) and $E(Z^k;\alpha) = (-1)^k E(Z^k;-\alpha)$ for $\alpha < 0$ in Ali and Woo (2006), Table 1 in the Appendix provides mean, variance, and coefficient of skewness of the skewed density (2.3) for $\beta = 1$.

When $\delta = 1$ and 2, we had already observed the skewness of skewed Laplace density and skewed Rayleigh density, respectively in Son and Woo (2007a, 2007b). And hence other δ values are chosen. From Table 1 in the Appendix, we observe the following:

Fact 1. For the density (2.3) with $\beta = 1$, (a) when $\delta = 1/3$, the density (2.3) is skewed to the right when $\alpha > 0$, and the density is skewed to the left when $\alpha < 0$. (b) when $\delta = 3, 4$ and 6, the density is skewed to the left when $\alpha > 0$ and the density is skewed to the right when $\alpha < 0$.

2.2. Estimation of skew parameter

Here we consider estimation of a skew parameter $\alpha(>0)$ in the skewed double Weibull density (2.3) with known β and δ.

Assume $Z_1, Z_2, \ldots, Z_n$ be *i.i.d.* random variables each with the density (2.3) having two known scale β and shape δ. Then by method of finding AML of parameter in a distribution in Balakrishnan and Cohen (1991), from log-likelihood function of α, we obtain AML $\hat{\alpha}$ of skew parameter α as the following:

For $Z_1 = z_1, Z_2 = z_2, \ldots, Z_n = z_n$, the likelihood function is given by:

$$f(\alpha; z_1, \ldots, z_n) = \left(\frac{\delta}{2\beta}\right)^n \cdot \left(\prod_{i=1}^{n} |z_i|^{\delta-1}\right) \cdot e^{-\sum_{i=1}^{n} \frac{|z_i|^\delta}{\beta}} \cdot \prod_{i=1}^{n} \left\{1 + \mathrm{sgn}(\alpha z_i)\left(1 - e^{-\frac{|\alpha z_i|^\delta}{\beta}}\right)\right\}$$

and hence

$$\frac{d\ln f(\alpha; z_1, \ldots, z_n)}{d\alpha} = \sum_{i=1}^{n} \frac{\mathrm{sgn}(\alpha z_i)\, |\alpha z_i|^{\delta-1}\, \delta \cdot |z_i| \cdot e^{-\frac{|\alpha z_i|^\delta}{\beta}}}{1 + \mathrm{sgn}(\alpha z_i)\left(1 - e^{-\frac{|\alpha z_i|^\delta}{\beta}}\right)}.$$

As taking first two terms of Taylor's series for $g(\alpha) \equiv g(\alpha|z)$ below, the AML of α is obtained by:

$$\hat{\alpha}(c) = c - \sum_{i=1}^{n} g(c|Z_i) \Big/ \sum_{i=1}^{n} g'(c|Z_i), \quad c \text{ is any real number,} \tag{2.7}$$

where

$$g(c|z_i) = \frac{\mathrm{sgn}(cz_i)\, |cz_i|^{\delta-1}\, \delta \cdot |z_i| \cdot e^{-\frac{|cz_i|^\delta}{\beta}}}{1 + \mathrm{sgn}(cz_i)\left(1 - e^{-\frac{|cz_i|^\delta}{\beta}}\right)} \quad \text{and} \quad g'(c|z_i) = \frac{h(c|z_i)}{\left\{1 + \mathrm{sgn}(cz_i)\left(1 - e^{-\frac{|cz_i|^\delta}{\beta}}\right)\right\}^2}$$

the function $h(c|z)$ is defined by:

$$h(c|z_i) = \frac{\delta\, |cz_i|^{\delta-2}\, z_i^2}{\beta} \cdot e^{-\frac{|cz_i|^\delta}{\beta}} \times \left[(\delta - 1)\left\{1 + \mathrm{sgn}(cz_i) - e^{-\frac{|cz_i|^\delta}{\beta}}\right\} - \frac{\delta\, |cz_i|^\delta}{\beta}\{1 + \mathrm{sgn}(cz_i)\}\right].$$

And MME $\tilde{\alpha}$ of skew parameter $\alpha > 0$ is given by:

$$\tilde{\alpha} \equiv \tilde{\alpha}(Z) = \left[\left\{1 - \left(\sum_{i=1}^{n} \frac{Z_i}{n}\right) \Big/ \left(\beta^{\frac{1}{\delta}} \Gamma\left(1 + \frac{1}{\delta}\right)\right)\right\}^{-\frac{\delta}{1+\delta}} - 1\right]^{\frac{1}{\delta}} \tag{2.8}$$

provided if bases of each power are positive.

Remark 2. As AML $\hat{\alpha}(\alpha)$ performs better in the sense of simulated MSE than AML $\hat{\alpha}(c)$ for $c \neq \alpha$ (Son and Woo, 2007a), we simulate MSE of AML $\hat{\alpha}(\alpha)$ for given true value α.

We first consider the following process of generating distribution numbers to simulate MSE's of two estimators, AML and MME of skew parameter α by the *cdf* (2.4) with two known scale β and shape δ:

Process of generating distribution numbers

(I) We choose random numbers u from a uniform distribution over (0,1).

(II) For given $0 < u < 1$, by incremental search method in Deitel *et al.* (2003), we choose number $y = G^{-1}(u)$, which $G(y)$ is the cdf of a skewed Laplace random variable in Son and Woo (2007a).

(III) $z = \mathrm{sgn}(y)|y|^{1/\delta}$ is generating number of skewed double Weibull random variable, where $\mathrm{sgn}(y) = \begin{cases} 1, & \text{if } y > 0, \\ -1, & \text{if } y < 0, \end{cases}$ and number of simulations is 10,000.

From above process of generating distribution numbers, we simulate MSE's of AML and MME of skew parameter α when the density (2.3) has two known scale $\beta = 1$ and shape $\delta = 1/3, 1/2, 3, 4, 5$. Table 2 in the Appendix provides simulated MSE's of AML $\hat{\alpha}$ and MME $\tilde{\alpha}$ of skew parameter α when $n = 10(10)30$, $\alpha = c = 0.25, 0.5, 1, 2, 4$. And hence from Table 2, we observe the following:

Fact 2. Assume the density (2.3) has known scale $\beta = 1$ and shape $\delta = 1/3, 1/2, 3, 4, 5$. If a true value $\alpha = 0.25, 0.5, 1, 2, 4$, then AML of α performs better than MME of α in the sense of MSE.

Remark 3. (Application) AML $\hat{\alpha}(c)$ in (2.7) can be applied to real data problem as we use MME $\tilde{\alpha}$ instead of c in $\hat{\alpha}(c)$.

2.3. Estimation of reliability

In this Section, we obtain reliability $P(W < Z)$, when Z and W are independent skewed double Weibull random variables which they have the density (2.3) with parameters $(\delta, \beta_1, \alpha_1)$ and $(\delta, \beta_2, \alpha_2)$, respectively.

From the density (2.3) and the cdf (2.4), we obtain the following:

Fact 3. When Z and W are independent skewed double Weibull random variables which they have the density (2.3) with parameter $(\delta, \beta_1, \alpha_1)$ and $(\delta, \beta_2, \alpha_2)$, respectively, then for $\alpha_i > 0$, $i = 1, 2$, $\rho \equiv P(W < Z) = R(\alpha_1, \alpha_2)$ is given by:

$$P(W<Z) = 1 - \frac{1}{1+\beta_1/\beta_2} - \frac{1}{2\left(1+\alpha_1^\delta\right)} + \frac{1}{2\left(1+\beta_1/\beta_2+\alpha_1^\delta\right)} + \frac{1}{2\left(1+\alpha_2^\delta\right)\left\{1+(1+\alpha_2^\delta)\beta_1/\beta_2\right\}}. \quad (2.9)$$

Remark 4. For $\alpha_i < 0\,(i = 1, 2)$, or other cases, we can obtain reliability $P(W < Z)$ by the similar manner as in $\alpha_i > 0$. Especially if Z and W are identical random variables with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, then it is obvious that reliability is 1/2.

Assume $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_m$ be two independent samples each with the density $f(z; \alpha_1)$ and $f(w; \alpha_2)$ in (2.3) with known $\beta_i\,(i = 1, 2)$, and common known δ respectively. Then from AML $\hat{\alpha}$ and MME $\tilde{\alpha}$ of α, the following estimators of $\rho \equiv R(\alpha_1, \alpha_2) = P(W < Z)$ are defined by:

$$\hat{\rho} \equiv 1 - \frac{1}{1+\beta_1/\beta_2} - \frac{1}{2\left(1+\hat{\alpha}_1^\delta\right)} + \frac{1}{2\left(1+\hat{\alpha}_1^\delta+\beta_1/\beta_2\right)} + \frac{1}{2\left(1+\hat{\alpha}_2^\delta\right)\left(1+\beta_1/\beta_2+\beta_1\hat{\alpha}_2^\delta/\beta_2\right)} \quad \text{and}$$

$$\tilde{\rho} \equiv 1 - \frac{1}{1+\beta_1/\beta_2} - \frac{1}{2\left(1+\tilde{\alpha}_1^\delta\right)} + \frac{1}{2\left(1+\tilde{\alpha}_1^\delta+\beta_1/\beta_2\right)} + \frac{1}{2\left(1+\tilde{\alpha}_2^\delta\right)\left(1+\beta_1/\beta_2+\beta_1\tilde{\alpha}_2^\delta/\beta_2\right)}, \quad (2.10)$$

where AML $\hat{\alpha}_i\,(i = 1,2)$ of α_1 and α_2 are given by:

$$\hat{\alpha}_1 = c - \sum_{i=1}^{n} g(c|Z_i) \Big/ \sum_{i=1}^{n} g'(c|Z_i), \qquad \hat{\alpha}_2 = c - \sum_{i=1}^{m} g(c|W_i) \Big/ \sum_{i=1}^{m} g'(c|W_i)$$

and MME $\tilde{\alpha}_i\,(i = 1,2)$ of α_1 and α_2 are given by:

$$\widetilde{\alpha_1} \equiv \widetilde{\alpha_1}(Z) = \left[\left\{1 - \left(\sum_{i=1}^{n}\frac{Z_i}{n}\right)\Big/\left(\beta^{\frac{1}{\delta}}\Gamma\left(1+\frac{1}{\delta}\right)\right)\right\}^{-\frac{\delta}{1+\delta}} - 1\right]^{\frac{1}{\delta}} \quad \text{and}$$

$$\widetilde{\alpha_2} \equiv \widetilde{\alpha_2}(W) = \left[\left\{1 - \left(\sum_{i=1}^{m}\frac{W_i}{m}\right)\Big/\left(\beta^{\frac{1}{\delta}}\Gamma\left(1+\frac{1}{\delta}\right)\right)\right\}^{-\frac{\delta}{1+\delta}} - 1\right]^{\frac{1}{\delta}}$$

provided if bases of each power in $\widetilde{\alpha_1}$ and $\widetilde{\alpha_2}$ are positive.

By the process of generating distribution numbers as in Section 2.1, we simulate MSE's of two estimators (2.10) when the density (2.3) has two known $\beta = 1$ and shape $\delta = 1/3, 0.5, 3, 4, 5$. Table 3 in the Appendix provides simulated MSE's of two estimators $\hat{\rho}$ and $\tilde{\rho}$ in a skewed double Weibull with common known scale $\beta = 1$ and shape $\delta = 1/3, 0.5, 3, 4, 5$, when $n(m) = 10(10)30$, $(\alpha_1, \alpha_2) = (0.25, 4), (0.25, 0.5), (0.5, 2), (2, 4)$ and $c_i = \alpha_i$ for each $i = 1, 2$. And from Table 3, we observe the following:

Fact 4. Assume $Z_1, Z_2, \ldots, Z_n$ and $W_1, W_2, \ldots, W_m$ are two independent samples each with the density $f(z;\alpha_1)$ and $f(w;\alpha_2)$ which have common known scale $\beta_i = 1\,(i = 1,2)$ and shape $\delta = 1/3, 0.5, 3, 4, 5$. When $(\alpha_1, \alpha_2) = (0.25, 4), (0.25, 0.5), (0.5, 2), (2, 4)$, estimator $\hat{\rho} = R(\hat{\alpha}_1, \hat{\alpha}_2)$ performs better than another estimator $\tilde{\rho} = R(\tilde{\alpha}_1, \tilde{\alpha}_2)$ in the sense of MSE.

3. A Skewed Double Weibull Distribution Generated by a Uniform Kernel

Let X and Y be two independently identical distributed continuous random variables with the density $f(x) = F'(x)$ of X and the density $g(x) = G'(x)$ of Y, which they are symmetric about origin. Then a skewed density generated by cdf $G(x)$ kernel is as given in Ali *et al.* (2008) by:

$$f(z;\alpha) \equiv 2f(z)G(\alpha z). \tag{3.1}$$

If $f(x) = \delta/(2\beta)|x|^{\delta-1}e^{-|x|^{\delta}/\beta}$, $-\infty < x < \infty$ is a double Weibull density, and $G(x) = (x+1)/2$, if $|x| \le 1$ is a uniform distribution, then from (3.1),

$$f(z;\alpha) = \begin{cases} \dfrac{\delta}{2\beta}|z|^{\delta-1}e^{-\frac{|z|^{\delta}}{\beta}}(\alpha z + 1), & \text{if } |\alpha z| \le 1, \\[2mm] \dfrac{\delta}{\beta}|z|^{\delta-1}e^{-\frac{|z|^{\delta}}{\beta}}, & \text{if } \alpha z > 1, \end{cases} \tag{3.2}$$

which is a skewed double Weibull density generated by a uniform kernel, whose random variable is denoted by Z.

From the density (3.2) and formulas 3.381(1)&(3) in Gradshteyn and Ryzhik (1965, p.317) we obtain k^{th} moment as follows:

For $\alpha > 0$,

$$
\begin{aligned}
E\left(Z^k;\alpha\right) &= \beta^{\frac{k}{\delta}}\left[\Gamma\left(\frac{k}{\delta}+1,\frac{1}{\beta\alpha^{\delta}}\right)+\frac{1+(-1)^k}{2}\gamma\left(\frac{k}{\delta}+1,\frac{1}{\beta\alpha^{\delta}}\right)\right] \\
&\quad + \frac{1-(-1)^k}{2}\alpha\beta^{\frac{k+1}{\delta}}\cdot\gamma\left(\frac{k+1}{\delta}+1,\frac{1}{\beta\alpha^{\delta}}\right),
\end{aligned}
\tag{3.3}
$$

where $\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt$, $a > 0$ in Gradshteyn and Ryzhik (1965).

From the density (3.3) and formula 3.381(1) in Gradshteyn and Ryzhik (1965, p.317), the *cdf* $F(z;\alpha)$ of Z is given by:

For $\alpha > 0$,

$$
F(z;\alpha) = \begin{cases}
e^{-\frac{(-z)^{\delta}}{\beta}}\dfrac{(1+\alpha z)}{2} - \dfrac{\alpha\beta^{\frac{1}{\delta}}}{2\delta}\left[\gamma\left(\dfrac{1}{\delta},\dfrac{1}{\beta\alpha^{\delta}}\right)-\gamma\left\{\dfrac{1}{\delta},\dfrac{(-z)^{\delta}}{\beta}\right\}\right], & \text{if } -\dfrac{1}{\alpha} \le z < 0, \\
1 - \dfrac{\left(e^{-\frac{z^{\delta}}{\beta}}+\alpha z e^{-\frac{z^{\delta}}{\beta}}\right)}{2} + \dfrac{\alpha\beta^{\frac{1}{\delta}}}{2\delta}\left\{\gamma\left(\dfrac{1}{\delta},\dfrac{z^{\delta}}{\beta}\right)-\gamma\left(\dfrac{1}{\delta},\dfrac{1}{\beta\alpha^{\delta}}\right)\right\}, & \text{if } 0 < z, \le \dfrac{1}{\alpha} \\
1 - e^{-\frac{z^{\delta}}{\beta}}, & \text{if } z \ge \dfrac{1}{\alpha}.
\end{cases}
\tag{3.4}
$$

From integrals 8.350(1)&(2) in Gradshteyn and Ryzhik (1965, p.940) and k^{th} moment (3.3), Table 4 in the Appendix provides mean, variance, and coefficient of skewness of a skewed double Weibull generated a uniform kernel, which it has the density (3.2) for $\beta = 1$:

When $\delta = 1$ and 2, we already observed skewness of a skewed Laplace density and skewed Rayleigh density in Son and Woo (2007a, 2007b). And hence other δ values are chosen. From Table 4 in the Appendix and $E(Z^k;\alpha) = (-1)^k E(Z^k;-\alpha)$ for $\alpha < 0$ in Ali and Woo (2006), we observe the following:

Fact 5. Assume the density (3.2) has scale parameter $\beta = 1$, shape parameter δ and skew parameter α.

(a) When $\delta = 1/3$, the density is skewed to the right when $\alpha > 0$, and the density is skewed to the left when $\alpha < 0$.

(b) When $\delta = 3$, the density is skewed to the right when $\alpha = 1, 2, 4$ and $-1/4$, $-1/2$, but the density is skewed to the left else where.

(c) When $\delta = 4$, the density is skewed to the right when $\alpha = 2, 4$ and $-1/4$, $-1/2$, -1, but the density is skewed to the left else where.

(d) When $\delta = 6$, the density is skewed to the right when $\alpha = 2$ and $-1/4$, $-1/2$ -1, -4, but the density is skewed to the left else where.

Remark 5. From Fact 1 and Fact 5, the trend of skewness in the skewed double Weibull distribution generated by a uniform kernel is different from that in the skewed double Weibull distribution.

Remark 6. Throughout this paper, especially if $\delta = 1$ and 2, then properties in skewed double Weibull distribution are an extension of those in skewed Laplace distribution and skewed double Rayleigh distribution in Son and Woo (2007a, 2007b).

Appendix

Table 1: Mean, variance, and skewness of the skewed double Weibull density (2.3) with $\beta = 1$ and skew parameter α (signs preserve its order for each row).

δ	α	mean	variance	skewness
	±0.25	±5.14995	693.478	±19.1264
	±0.5	±5.42037	690.620	±19.3087
1/3	±1	±5.62500	688.359	±19.4201
	±2	±5.76997	686.707	±19.4883
	±4	±5.86613	685.589	±19.5299
	±0.25	±0.01827	0.90241	∓0.02209
	±0.5	±0.12978	0.88590	∓0.16457
3	±1	±0.53860	0.61265	∓0.82615
	±2	±0.84528	0.18825	∓1.14675
	±4	±0.88956	0.11142	∓0.04091
	±0.25	±0.00406	0.88621	∓0.00655
	±0.5	±0.06615	0.88185	∓0.09996
4	±1	±0.52531	0.61028	∓0.96672
	±2	±0.88014	0.11157	∓1.71199
	±4	±0.90552	0.06626	∓0.20440
	±0.25	±0.00026	0.89298	∓0.00045
	±0.5	±0.01663	0.89270	∓0.02866
6	±1	±0.51447	0.62830	∓1.07021
	±2	±0.92060	0.04547	∓2.19317
	±4	±0.92766	0.03242	∓0.39612

Table 2: MSE of AML and MME of skew parameter α in a skewed double Weibull density (2.3) with known scale $\beta = 1$ and shape δ.

(A) $\delta = 5$

	(i) $\alpha = 0.25 = c$			(ii) $\alpha = 0.5 = c$	
n	AML	MME	n	AML	MME
10	0.006771	0.053749	10	0.003107	0.054696
20	0.003772	0.046394	20	0.001658	0.045726
30	0.002635	0.041218	30	0.001192	0.043926

	(iii) $\alpha = 1.0 = c$			(iv) $\alpha = 2 = c$	
n	AML	MME	n	AML	MME
10	0.006370	0.020525	10	0.000006	0.023676
20	0.000634	0.014513	20	0.000005	0.019629
30	0.000277	0.013251	30	0.000004	0.017157

	(v) $\alpha = 4 = c$	
n	AML	MME
10	0.000003	0.021967
20	0.000002	0.017575
30	0.000001	0.016592

(B) $\delta = 4$

	(i) $\alpha = 0.25 = c$			(ii) $\alpha = 0.5 = c$	
n	AML	MME	n	AML	MME
10	0.001221	0.046760	10	0.006484	0.047276
20	0.000678	0.038275	20	0.003349	0.036980
30	0.000475	0.033227	30	0.002401	0.034322

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	(iii) $\alpha = 1.0 = c$	
n	AML	MME
10	0.000427	0.057248
20	0.000394	0.023509
30	0.000100	0.013437

	(iv) $\alpha = 2 = c$	
n	AML	MME
10	0.000004	0.032575
20	0.000003	0.032502
30	0.000002	0.028881

	(v) $\alpha = 4 = c$	
n	AML	MME
10	0.000003	0.033350
20	0.000002	0.031899
30	0.000001	0.029640

(C) $\delta = 3$

	(i) $\alpha = 0.25 = c$	
n	AML	MME
10	0.002957	0.040014
20	0.001640	0.030621
30	0.001189	0.025000

	(ii) $\alpha = 0.5 = c$	
n	AML	MME
10	0.009519	0.033829
20	0.008022	0.027329
30	0.003162	0.023793

	(iii) $\alpha = 1.0 = c$	
n	AML	MME
10	0.001713	0.005601
20	0.000516	0.003947
30	0.000153	0.002987

	(iv) $\alpha = 2 = c$	
n	AML	MME
10	0.000009	0.002004
20	0.000007	0.001916
30	0.000006	0.001211

	(v) $\alpha = 4 = c$	
n	AML	MME
10	0.000003	0.000500
20	0.000002	0.000480
30	0.000001	0.000452

(D) $\delta = 1/2$

	(i) $\alpha = 0.25 = c$	
n	AML	MME
10	0.002710	0.019387
20	0.001748	0.017982
30	0.001309	0.016737

	(ii) $\alpha = 0.5 = c$	
n	AML	MME
10	0.000320	0.059684
20	0.000256	0.049513
30	0.000156	0.037287

	(iii) $\alpha = 1.0 = c$	
n	AML	MME
10	0.003056	0.050379
20	0.002149	0.046224
30	0.001083	0.038653

	(iv) $\alpha = 2 = c$	
n	AML	MME
10	0.007862	0.027115
20	0.004359	0.024210
30	0.003705	0.021002

	(v) $\alpha = 4 = c$	
n	AML	MME
10	0.000039	0.031720
20	0.000017	0.027977
30	0.000003	0.019831

(E) $\delta = 1/3$

	(i) $\alpha = 0.25 = c$	
n	AML	MME
10	0.002635	0.057286
20	0.002420	0.057224
30	0.002346	0.056628

	(ii) $\alpha = 0.5 = c$	
n	AML	MME
10	0.000378	0.053929
20	0.000185	0.053891
30	0.000108	0.053797

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(iii) $\alpha = 1.0 = c$

n	AML	MME
10	0.000286	0.067759
20	0.000162	0.040209
30	0.000140	0.037522

(iv) $\alpha = 2 = c$

n	AML	MME
10	0.001645	0.005069
20	0.001103	0.005012
30	0.000980	0.004880

(v) $\alpha = 4 = c$

n	AML	MME
10	0.000707	0.004913
20	0.000291	0.004864
30	0.000150	0.004787

Table 3: MSE of two reliability estimators $\hat{\rho}$ and $\tilde{\rho}$ in two skewed double Weibull density (2.3) with common known scale $\beta = 1$ and shape δ.

(A) $\delta = 5$

(i) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.167014$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000008	0.008659
	20	0.000007	0.008633
	30	0.000006	0.008310
20	10	0.000006	0.005706
	20	0.000005	0.005697
	30	0.000004	0.004518
30	10	0.000003	0.005333
	20	0.000002	0.004444
	30	0.000001	0.004166

(ii) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 0.5 = c_2, \rho = 0.486913$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000079	0.002390
	20	0.000034	0.001836
	30	0.000026	0.001718
20	10	0.000077	0.002236
	20	0.000031	0.001809
	30	0.000025	0.001501
30	10	0.000071	0.002132
	20	0.000029	0.001661
	30	0.000024	0.001453

(iii) $\alpha_1 = 0.5 = c_1$, $\alpha_2 = 2 = c_2, \rho = 0.179214$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000049	0.005324
	20	0.000046	0.005276
	30	0.000040	0.005092
20	10	0.000028	0.003020
	20	0.000025	0.003014
	30	0.000024	0.002119
30	10	0.000018	0.002826
	20	0.000017	0.002130
	30	0.000016	0.002000

(iv) $\alpha_1 = 2 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.346721$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000007	0.000324
	20	0.000006	0.000309
	30	0.000005	0.000307
20	10	0.000006	0.005706
	20	0.000005	0.005697
	30	0.000004	0.004518
30	10	0.000003	0.005333
	20	0.000002	0.004444
	30	0.000001	0.004166

(B) $\delta = 4$

(i) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.168160$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000007	0.005385
	20	0.000006	0.005360
	30	0.000005	0.005105
20	10	0.000005	0.005020
	20	0.000004	0.004674
	30	0.000003	0.004220
30	10	0.000003	0.003793
	20	0.000002	0.003738
	30	0.000001	0.003468

(ii) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 0.5 = c_2, \rho = 0.475468$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000372	0.012667
	20	0.000172	0.010052
	30	0.000121	0.008695
20	10	0.000265	0.012215
	20	0.000163	0.009695
	30	0.000116	0.008667
30	10	0.000129	0.011708
	20	0.000114	0.009650
	30	0.000108	0.008001

(iii) $\alpha_1 = 0.5 = c_1$, $\alpha_2 = 2 = c_2, \rho = 0.192338$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000210	0.003910
	20	0.000202	0.003874
	30	0.000198	0.003740

(iv) $\alpha_1 = 2 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.356535$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000007	0.000180
	20	0.000006	0.000173
	30	0.000005	0.000161

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	10	0.000117	0.002059		10	0.000006	0.000179
20	20	0.000114	0.002061	20	20	0.000005	0.000165
	30	0.000089	0.001394		30	0.000004	0.000150
	10	0.000114	0.001914		10	0.000004	0.000153
30	20	0.000086	0.001387	30	20	0.000003	0.000149
	30	0.000055	0.001288		30	0.000002	0.000146

(C) $\delta = 3$

(i) $\alpha_1 = 0.25 = c_1, \alpha_2 = 4 = c_2, \rho = 0.172828$				(ii) $\alpha_1 = 0.25 = c_1, \alpha_2 = 0.5 = c_2, \rho = 0.457043$			
n	m	$\hat{\rho}$	$\tilde{\rho}$	n	m	$\hat{\rho}$	$\tilde{\rho}$
	10	0.000016	0.012010		10	0.001666	0.002045
10	20	0.000015	0.009665	10	20	0.000773	0.001403
	30	0.000014	0.009040		30	0.000554	0.001268
	10	0.000008	0.007621		10	0.000963	0.001699
20	20	0.000007	0.005485	20	20	0.000766	0.001097
	30	0.000006	0.005157		30	0.000531	0.000968
	10	0.000007	0.006086		10	0.000779	0.001495
30	20	0.000006	0.003959	30	20	0.000730	0.000924
	30	0.000005	0.003668		30	0.000429	0.000776

(iii) $\alpha_1 = 0.5 = c_1, \alpha_2 = 2 = c_2, \rho = 0.218957$				(iv) $\alpha_1 = 2 = c_1, \alpha_2 = 4 = c_2, \rho = 0.368883$			
n	m	$\hat{\rho}$	$\tilde{\rho}$	n	m	$\hat{\rho}$	$\tilde{\rho}$
	10	0.000958	0.011775		10	0.000005	0.000175
10	20	0.000788	0.009310	10	20	0.000004	0.000169
	30	0.000768	0.008089		30	0.000003	0.000158
	10	0.000808	0.008830		10	0.000004	0.000117
20	20	0.000741	0.006440	20	20	0.000003	0.000112
	30	0.000680	0.005149		30	0.000002	0.000110
	10	0.000743	0.005519		10	0.000003	0.000113
30	20	0.000660	0.004133	30	20	0.000002	0.000111
	30	0.000622	0.003803		30	0.000001	0.000109

(D) $\delta = 1/2$

(i) $\alpha_1 = 0.25 = c_1, \alpha_2 = 4 = c_2, \rho = 0.352348$				(ii) $\alpha_1 = 0.25 = c_1, \alpha_2 = 0.5 = c_2, \rho = 0.443698$			
n	m	$\hat{\rho}$	$\tilde{\rho}$	n	m	$\hat{\rho}$	$\tilde{\rho}$
	10	0.000075	0.003295		10	0.000071	0.000920
10	20	0.000068	0.003272	10	20	0.000047	0.000901
	30	0.000067	0.003100		30	0.000039	0.000827
	10	0.000066	0.003212		10	0.000063	0.000879
20	20	0.000065	0.003113	20	20	0.000029	0.000769
	30	0.000064	0.003086		30	0.000027	0.000722
	10	0.000065	0.002903		10	0.000041	0.000730
30	20	0.000063	0.002860	30	20	0.000024	0.000613
	30	0.000059	0.002730		30	0.000021	0.000609

(iii) $\alpha_1 = 0.5 = c_1, \alpha_2 = 2 = c_2, \rho = 0.403995$				(iv) $\alpha_1 = 2 = c_1, \alpha_2 = 4 = c_2, \rho = 0.419401$			
n	m	$\hat{\rho}$	$\tilde{\rho}$	n	m	$\hat{\rho}$	$\tilde{\rho}$
	10	0.000052	0.001454		10	0.000007	0.001008
10	20	0.000031	0.001317	10	20	0.000006	0.000909
	30	0.000030	0.001287		30	0.000005	0.000864
	10	0.000028	0.001364		10	0.000006	0.000850
20	20	0.000026	0.001178	20	20	0.000005	0.000840
	30	0.000024	0.001162		30	0.000004	0.000803
	10	0.000025	0.001224		10	0.000004	0.000807
30	20	0.000023	0.001153	30	20	0.000003	0.000703
	30	0.000020	0.001104		30	0.000002	0.000502

(E) $\delta = 1/3$

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(i) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.386279$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000021	0.004001
	20	0.000019	0.003960
	30	0.000018	0.003876
20	10	0.000019	0.003967
	20	0.000018	0.003952
	30	0.000017	0.003843
30	10	0.000018	0.003901
	20	0.000017	0.003836
	30	0.000016	0.003834

(ii) $\alpha_1 = 0.25 = c_1$, $\alpha_2 = 0.5 = c_2, \rho = 0.448985$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000032	0.001685
	20	0.000020	0.001638
	30	0.000019	0.001571
20	10	0.000963	0.001699
	20	0.000766	0.001097
	30	0.000531	0.000968
30	10	0.000779	0.001495
	20	0.000730	0.000924
	30	0.000429	0.000776

(iii) $\alpha_1 = 0.5 = c_1$, $\alpha_2 = 2 = c_2, \rho = 0.422015$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000958	0.011775
	20	0.000788	0.009310
	30	0.000768	0.008089
20	10	0.000808	0.008830
	20	0.000741	0.006440
	30	0.000680	0.005149
30	10	0.000743	0.005519
	20	0.000660	0.004133
	30	0.000622	0.003803

(iv) $\alpha_1 = 2 = c_1$, $\alpha_2 = 4 = c_2, \rho = 0.431049$

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	10	0.000005	0.000175
	20	0.000004	0.000169
	30	0.000003	0.000158
20	10	0.000004	0.000117
	20	0.000003	0.000112
	30	0.000002	0.000110
30	10	0.000003	0.000113
	20	0.000002	0.000111
	30	0.000001	0.000109

Table 4: Mean, variance, and coefficient of skewness of the skewed double Weibull density generated by a uniform distribution which it has the density (2.14) with $\beta = 1$ and skew parameter α (signs preserve its order for each row).

δ	α	mean	variance	skewness
1/3	±0.25	±5.65311	688.042	±19.4500
	±0.5	±5.88240	685.397	±19.5678
	±1	±6.00000	684.000	±19.5849
	±2	±6.05304	683.361	±19.6066
	±4	±6.07175	683.134	±19.6143
3	±0.25	∓0.22324	0.85291	∓0.42191
	±0.5	∓0.44684	0.70308	∓0.89936
	±1	±0.89298	0.10534	±0.16775
	±2	±0.93665	0.02544	±28.0950
	±4	±0.90169	0.08985	±0.90647
4	±0.25	±0.22660	0.83488	∓0.45804
	±0.5	±0.45320	0.68084	∓0.99545
	±1	±0.90640	0.06467	∓0.08823
	±2	±0.93055	0.02031	±21.0371
	±4	±0.90874	0.06042	±0.26948
6	±0.25	±0.23194	0.83918	∓0.48759
	±0.5	±0.46386	0.67781	∓1.07506
	±1	±0.92772	0.03232	∓0.37269
	±2	±0.93436	0.01995	±5.63199
	±4	±0.92788	0.03202	∓0.30847

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