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Precise Rates in Complete Moment Convergence for Negatively Associated Sequences

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Abstract

Let $\{X_n, n \geq 1\}$ be a negatively associated sequence of identically distributed random variables with mean zeros and positive finite variances. Set $S_n = \sum_{i=1}^n X_i$. Suppose that $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} \mathrm{Cov}(X_1, X_i) < \infty$. We prove that, if $EX_1^2(\log^+ |X_1|)^\delta < \infty$ for any $0 < \delta \leq 1$, then

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\left(|S_n| \geq \epsilon\sigma\sqrt{n \log n}\right) = \frac{E|N|^{2\delta+2}}{\delta},$$

where N is the standard normal random variable. We also prove that if S_n is replaced by $M_n = \max_{1\leq k\leq n} |S_k|$, then the precise rate still holds. Some results in Fu and Zhang (2007) are improved to the complete moment case.

Keywords: Precise rates, complete moment convergence, negatively associated, law of the logarithm.

1. Introduction

A finite sequence of random variables $\{X_i,\ 1 \leq i \leq n\}$ is said to be negatively associated(NA), if for every disjoint subsets A and B of $\{1, 2, \ldots, n\}$, we have $\mathrm{Cov}(f(X_i; i \in A), g(X_j; j \in B)) \leq 0$, whenever f on R^A and g on R^B are coordinatewise nondecreasing functions and the covariance exists. An infinite sequence of random variables is NA if every finite subsequence is NA.

The notion of NA was introduced by Alam and Saxena (1981). Joag-Dev and Proschan (1983) showed that many well known multivariate distributions possess the NA property. Some examples include: (a) the multinomial, (b) the convolution of unlike multinomials, (c) the multivariate hypergeometric distribution, (d) the Dirichlet, (e) the Dirichlet compound multinomial, (f) the negatively correlated normal distribution, (g) the permutation distribution, (h) the random sampling without replacement, and (i) the joint distribution of ranks. Because of its wide applications in multivariate statistical analysis and system reliability, the notion of NA has received considerable attention recently. We refer to Joag-Dev and Proschan (1983) for fundamental properties, Shao and Su (1999) for the law of the iterated logarithm, Shao (2000) for moment inequalities and the maximal inequalities of the partial sum, Liang (2000) for complete convergence, Kim *et al.* (2001) for the estimation of empirical distribution, Li and Zhang (2004) for complete moment convergence, Fu and Zhang (2007) for the precise rates of in the law of the logarithm and Ko (2009) for central limit theorem of a linear process based on the negatively associated process in a Hilbert space. Set $S_n = \sum_{i=1}^n X_i$ and denote $\log x = \ln(x \vee e)$. When $\{X_n,\ n \geq 1\}$ is a sequence of *i.i.d.* random variables Liu and Lin (2006) proved as follows: Let $\{X_n,\ n \geq 1\}$ be a sequence of *i.i.d.* random variables. Suppose that

$$EX_1 = 0, \quad 0 < EX_1^2 = \sigma^2 \quad \text{and} \quad EX_1^2 (\log^+ |X_1|)^\delta < \infty, \tag{1.1}$$

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for $0 < \delta \leq 1$. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\left(|S_n| \geq \epsilon \sqrt{n \log n}\right) = \frac{\sigma^{2+2\delta}}{\delta} E|N|^{2\delta+2}, \tag{1.2}$$

where N is the standard normal random variable.

Conversely, if (1.2) is true, then (1.1) holds. Let

$$\rho(n) =: \sup_{k\geq 1} \sup_{X\in L_2(\mathcal{F}_k^-)} \sup_{Y\in L_2(\mathcal{F}_{k+n}^+)} \frac{|EXY - EXEY|}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}},$$

where

$$\mathcal{F}_n^- = \sigma(X_i; 1 \leq i \leq n) \quad \text{and} \quad \mathcal{F}_n^+ = \sigma(X_i; i \geq n).$$

Then the sequence $\{X_n, n \geq 1\}$ is said to be ρ-mixing, if $\rho(n) \to 0$ *as* $n \to \infty$ (see *e.g.* Peligrad, 1987).

Zhao (2008) proved that (1.2) is true for ρ-mixing sequences under appropriate conditions as follows: Let $\{X_n, n \geq 1\}$ be a strictly stationary sequence of ρ-mixing random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Suppose that

$$\lim_{n\to\infty} \frac{ES_n^2}{n} = \sigma^2 > 0, \qquad \sum_{n=1}^{\infty} \rho^{\frac{2}{q}}(2^n) < \infty, \tag{1.3}$$

for $q > 2\delta + 2$ and $EX_1^2(\log^+ |X_1|)^\delta < \infty$ for any $0 < \delta \leq 1$. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\left(|S_n| \geq \epsilon\sigma \sqrt{n \log n}\right) = \frac{E|N|^{2\delta+2}}{\delta}. \tag{1.4}$$

Furthermore, Zhao (2008) proved that if S_n is replaced by $M_n = \max_{1\leq k\leq n} |S_k|$ then the precise rate still holds as follows: Let $\{X_n, n \leq 1\}$ be a strictly stationary sequence of ρ-mixing random variables satisfying above conditions. Then,

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} EM_n^2 I\left(|M_n| \geq \epsilon\sigma \sqrt{n \log n}\right) = \frac{2E|N|^{2\delta+2}}{\delta} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\delta+2}}. \tag{1.5}$$

The purpose of this paper is to show that, for negatively associated random variables (1.4) and (1.5) still hold under appropriate conditions.

2. Preliminaries

We introduce some preliminary results which are needed in proving the main results.

Lemma 1. **(Newman, 1984)** *Assume that* $\{X_n, n \geq 1\}$ *is a strictly stationary sequence of negatively associated random variables with* $EX_1 = 0$ *and* $EX_1^2 < \infty$. *If* $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) < \infty$, *then*

$$\frac{S_n}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1) \quad \text{as } n \to \infty, \tag{2.1}$$

where $\xrightarrow{\mathcal{D}}$ *indicates convergence in distribution, N a standard normal distribution.*

Lemma 2. **(Shao, 2000)** *Let $\{X_n,\ n \geq 1\}$ be a strictly stationary sequence of negatively associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. If $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) < \infty$, then $W_n \Rightarrow W$, where W is a standard Wiener measure, $W_n(t) = S_{[nt]}/\sigma\sqrt{n}, 0 \leq t \leq 1$, and "$\Rightarrow$" means weak convergence in $D[0, 1]$ with Skorohod topology. In particular,*

$$\frac{M_n}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} \sup_{0\leq t\leq 1} |W(t)|, \tag{2.2}$$

where $M_n = \max_{1\leq k\leq n} |S_k|$, $n \geq 1$ and $\{W(t); t \geq 0\}$ is a standard Wiener process.

Lemma 3. **(Shao, 2000)** *Let $\{Y_i,\ 1 \leq i \leq n\}$ be a sequence of NA random variables with mean zeros and finite variances. Denote $S_k = \sum_{i=1}^{k} Y_i, 1 \leq k \leq n$, $B_n = \sum_{i=1}^{n} EY_i^2$. Then, for any $u > 0,\ v > 0$.*

$$P\left(\max_{1\leq k\leq n} |S_k| \geq u\right) \leq 2P\left(\max_{1\leq k\leq n} |Y_k| \geq v\right) + 4\exp\left(-\frac{u^2}{8B_n}\right) + 4\left\{\frac{B_n}{4(uv + B_n)}\right\}^{\frac{u}{12v}}. \tag{2.3}$$

Proposition 1. **(Zhao, 2008)** *Suppose that N is a standard normal random variable. Then for any $0 < \delta \leq 1$,*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta+2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|N| \geq \epsilon\sqrt{\log n}\right) = \frac{E|N|^{2\delta+2}}{\delta + 1}. \tag{2.4}$$

Proof: For the proof see Theorem 1.3 in Huang and Zang (2005). □

Proposition 2. **(Zhao, 2008)** *Suppose that N is a standard normal random variable. Then for any $0 < \delta \leq 1$,*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right) = \frac{E|N|^{2\delta+2}}{\delta(\delta + 1)}. \tag{2.5}$$

Proof: See the proof of Proposition 5.1 in Liu and Lin (2006). □

The following result is similar to one of Proposition 3.2 in Fu and Zhang (2007).

Proposition 3. *Let $\{X_n; n \geq 1\}$ be a negatively associated sequence of identically distributed random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Suppose that $0 < \sigma^2 = EX_1^2 + 2\sum_{i=2}^{\infty} Cov(X_1, X_i) = 1$, and $EX_1^2(\log^+ |X_1|)^{\delta} < \infty$ for any $0 < \delta \leq 1$. Then*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta+2} \sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \left|P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| = 0. \tag{2.6}$$

Proof: Using the standard method, set $H(\epsilon) = [\exp(M/\epsilon^2)]$, where $M > 4,\ 0 < \epsilon < 1/4$. In fact, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \left|P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| \\
&= \sum_{n\leq H(\epsilon)} \frac{(\log n)^{\delta}}{n} \left|P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| \\
&\quad + \sum_{n>H(\epsilon)} \frac{(\log n)^{\delta}}{n} \left|P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) - P\left(|N| \geq \epsilon\sqrt{\log n}\right)\right| \\
&=: I + II.
\end{aligned} \tag{2.7}$$

By Lemma 3.4 in Fu and Zhang (2007) we have

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}I = 0. \tag{2.8}$$

Obviously, we have, for the second part II,

$$\begin{aligned} II &\leq \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|N| \geq \epsilon\sqrt{\log n}\right) + \sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) \\ &= III + IV. \end{aligned} \tag{2.9}$$

Notice that $H(\epsilon) - 1 \geq \sqrt{H(\epsilon)}$ for $M > 4$ and $0 < \epsilon < 1/4$, an easy calculation leads to

$$\begin{aligned} \epsilon^{2\delta+2}III &\leq \epsilon^{2\delta+2}\sum_{n>H(\epsilon)} \frac{(\log n)^\delta}{n} P\left(|N| \geq \epsilon\sqrt{\log n}\right) \\ &\leq C\int_{\sqrt{\frac{M}{2}}}^{\infty} y^{2\delta+1}P(|N| > y)\,dy \to 0 \quad \text{as } M \to \infty, \end{aligned} \tag{2.10}$$

uniformly with respect to $0 < \epsilon < 1/4$.

For IV, by Lemma 3 we have

$$\lim_{M\to\infty}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}IV \leq \lim_{M\to\infty}\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n>H(\epsilon)}\frac{(\log n)^\delta}{n}P\left(\max_{1\leq k\leq n}|S_k| \geq \epsilon\sqrt{n\log n}\right) = 0. \tag{2.11}$$

(See Lemma 3.7 in Fu and Zhang (2007).)

From (2.7)–(2.11), (2.6) follows. □

3. Results

Proposition 4. *Set $H(\epsilon) = [\exp(M/\epsilon^2)]$ and let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables. If $EX_1^2(\log^+|X_1|)^\delta < \infty$ for any $0 < \delta \leq 1$. Then we obtain*

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n\leq H(\epsilon)}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP(|S_n| \geq x)dx - \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right)dx\right| = 0. \tag{3.1}$$

Proof: Denote $\Delta_n = \sup_x|P(|S_n| \geq \sqrt{n}x) - P(|N| \geq x)|$. Assume that $x = (y+\epsilon)\sqrt{n\log n}$. By integral formula and transformation, it is enough to show that

$$\begin{aligned} &\sum_{n\leq H(\epsilon)} n^{-2}(\log n)^{\delta-1}\left|\int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP(|S_n| \geq x)\,dx - \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP\left(|N| \geq \frac{x}{\sqrt{n}}\right)dx\right| \\ &\leq C\sum_{n\leq H(\epsilon)} n^{-1}(\log n)^\delta\int_0^{\infty} 2(y+\epsilon)\left|P\left(|S_n| \geq (y+\epsilon)\sqrt{n\log n}\right)P\left(|N| \geq (y+\epsilon)\sqrt{\log n}\right)\right|dy \end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta}\Bigg[\int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 2(y+\epsilon)P\left\{|N| \ge (y+\epsilon)\sqrt{\log n}\right\}dy \\
&\quad + \int_{0}^{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}} 2(y+\epsilon)\left|P\left\{|S_n| \ge (y+\epsilon)\sqrt{n\log n}\right\} - P\left\{|N| \ge (y+\epsilon)\sqrt{\log n}\right\}\right|dy \\
&\quad + \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 2(y+\epsilon)P\left\{|S_n| \ge (y+\epsilon)\sqrt{n\log n}\right\}dy\Bigg] \\
&=: C \sum_{n\le H(\epsilon)} \frac{(\log n)\delta}{n}(\Lambda_1+\Lambda_2+\Lambda_3). \qquad (3.2)
\end{aligned}
$$

The estimates of Λ_1 and Λ_2 are similar to those of Proposition 5.2 in Liu and Lin (2006), so we omit them. It remains to estimate Λ_3, taking $\theta = \sqrt{1/EX_1^2}$, $u = (y+\epsilon)\sqrt{n\log n}$, $v = a(y+\epsilon)\sqrt{n\log n}$ and $a = 1/\{12(\delta+2)\}$ in Lemma 2.4, which yields

$$
\begin{aligned}
\sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta}\Lambda_3 &\le C \sum_{n\le H(\epsilon)} (\log n)^{\delta} \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 4(y+\epsilon)P\left\{|X_1| > a(y+\epsilon)\sqrt{n\log n}\right\}dy \\
&\quad + C \sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta} \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 8(y+\epsilon)\exp\left\{-\frac{\theta^2(y+\epsilon)^2\log n}{8}\right\}dy \\
&\quad + C \sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta} \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 8(y+\epsilon)\left\{\frac{n/\theta^2}{4a(y+\epsilon)^2 n\log n + n/\theta^2}\right\}^{\frac{1}{12a}}dy \\
&=: \Lambda_4+\Lambda_5+\Lambda_6. \qquad (3.3)
\end{aligned}
$$

Note that $\{\log H(\epsilon)\}^{\delta} = M^{\delta}/\epsilon^{2\delta}$ and $EX_1^2 I(|X_1| \ge \sqrt{n}) \to 0$ *as* $n \to \infty$. By Toeplitz's lemma, it follows that

$$
\begin{aligned}
\epsilon^{2\delta}\Lambda_4 &\le \epsilon^{2\delta} \sum_{n\le H(\epsilon)} C(\log n)^{\delta} E\left\{\int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 4(y+\epsilon)I\left(\frac{|X_1|}{a}\sqrt{n\log n} \ge y+\epsilon\right)\right\}dy \\
&\le C\epsilon^{2\delta} \sum_{n\le H(\epsilon)} \frac{E|X_1|^2 I\left(|X_1| \ge \sqrt{n}\right)(\log n)^{\delta-1}}{a^2 n} \\
&\le CM^{\delta}\left(\frac{1}{(\log H(\epsilon))^{\delta}}\right) \sum_{n\le H(\epsilon)} \frac{E|X_1|^2 I\left(|X_1| \ge \sqrt{n}\right)(\log n)^{\delta-1}}{a^2 n} \to 0 \quad \text{as } \epsilon \downarrow 0. \qquad (3.4)
\end{aligned}
$$

Observe that $\exp(-\theta^2/8\Delta_n^{1/2}) \to 0$ *as* $n \to \infty$. Using Toeplitz's lemma again, we have

$$
\begin{aligned}
\epsilon^{2\delta}\Lambda_5 &\le \epsilon^{2\delta} \sum_{n\le H(\epsilon)} Cn^{-1}(\log n)^{\delta-1}\left(\frac{32}{\theta^2}\right)\exp\left(-\frac{\theta^2}{8\Delta_n^{\frac{1}{2}}}\right) \\
&\le M^{\delta}\left(\frac{1}{(\log H(\epsilon))^{\delta}}\right)C \sum_{n\le H(\epsilon)} \frac{\exp\left(-\frac{\theta^2}{8\Delta_n^{\frac{1}{2}}}\right)(\log n)^{\delta-1}}{n} \to 0 \quad \text{as } \epsilon \downarrow 0. \qquad (3.5)
\end{aligned}
$$

Observe that $\Delta_n^{(\delta+1)/2} \to 0$ *as* $n \to \infty$. Using Toeplitz's lemma again, we have

$$\begin{aligned}
\epsilon^{2\delta}\Lambda_6 &\le \epsilon^{2\delta} C \sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta} \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 8(y+\epsilon)\left\{\frac{\theta^2 (y+\epsilon)^2 \log n}{3(\delta+2)}\right\}^{-(\delta+2)} dy \\
&\le \epsilon^{2\delta} C \sum_{n\le H(\epsilon)} n^{-1}(\log n)^{\delta} \left\{\frac{\theta^2}{3(\delta+2)}\right\}^{-(\delta+2)} \times \int_{1/\sqrt{\log n}\Delta_n^{\frac{1}{4}}}^{\infty} 8(y+\epsilon)\left\{(y+\epsilon)^2\log n\right\}^{-(\delta+2)} dy \\
&\le CM^{\delta}\left(\frac{1}{\log H(\epsilon)^{\delta}}\right) \sum_{n\le H(\epsilon)} \frac{\Delta_n^{\frac{\delta+1}{2}} (\log n)^{\delta-1}}{n} \to 0 \quad \text{as } \epsilon \downarrow 0. \qquad (3.6)
\end{aligned}$$

Combining (3.2)–(3.6), consequently, we obtain (3.1). □

Proposition 5. *Let* $\{X_n, n \ge 1\}$ *be a sequence of identically distributed NA random variables. If* $EX_1^2(\log^+|X_1|)^{\delta} < \infty$ *for any* $0 < \delta \le 1$. *Then we obtain, we have*

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP\left(|N| \ge \frac{x}{\sqrt{n}}\right)dx = 0, \qquad (3.7)$$

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP(|S_n| \ge x)dx = 0. \qquad (3.8)$$

Proof: The proof of (3.7) is quite routine, we omit it. Applying Lemma 2.4, the proof of (3.8) are exposed as follows: For $a = 1/\{12(\delta+2)\}$, we have □

$$\begin{aligned}
&\sum_{n>H(\epsilon)} n^{-2}(\log n)^{\delta-1} \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP\{|S_n| \ge x\}dx \\
&\le C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)P\left\{|S_n| \ge (x+\epsilon)\sqrt{n\log n}\right\}dx \\
&\le C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)\Bigg[\left\{2nP|X_1| > a(x+\epsilon)\sqrt{n\log n}\right\} \\
&\quad + 4\exp\left\{\frac{-\theta^2(x+\epsilon)^2\log n}{8}\right\} + 4\left\{4\theta^2 a(x+\epsilon)^2\log n\right\}^{-\frac{1}{12a}}\Bigg]dx \\
&=: C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)(II_1 + II_2 + II_3)dx. \qquad (3.9)
\end{aligned}$$

Recalling the moment condition, it suffices to prove that

$$\begin{aligned}
&\sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon)II_1 dx \\
&\le C \sum_{n>H(\epsilon)} (\log n)^{\delta} E \int_{\epsilon}^{\infty} 4xI\left(|X_1| \ge ax\sqrt{n\log n}\right)dx \\
&\le CE \int_{\epsilon}^{\infty} \frac{X_1^2}{x} \left|\log|X_1| - \log x\right|^{\delta-1} I\left(|X_1| \ge x\right) I\left(|X_1| \ge \sqrt{M}\right)dx
\end{aligned}$$

$$\leq CEX_1^2 \left|\log|X_1| - \log\epsilon\right|^{\delta} I\left(|X_1| \geq \sqrt{M}\right)$$
$$\leq CEX_1^2(\log|X_1|)^{\delta} + CEX_1^2|\log\epsilon|^{\delta} < \infty. \tag{3.10}$$

Hence, for II_1, we have

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon) II_1 dx \to 0.$$

Next, for II_2

$$\sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon) II_2 dx \leq C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_{\epsilon}^{\infty} 8x \exp\left(\frac{-\theta^2 x^2 \log n}{8}\right) dx$$
$$\leq C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \left(\frac{32}{\theta^2 \log n}\right) \exp\left(-\frac{\theta^2 \epsilon^2 \log n}{8}\right)$$
$$\leq C \sum_{n>H(\epsilon)} \left(\frac{32}{\theta^2}\right) n^{-1-\frac{\theta^2\epsilon^2}{8}} < \infty. \tag{3.11}$$

Hence

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon) II_2 dx \to 0 \quad \text{as } \epsilon \downarrow 0. \tag{3.12}$$

Finally, for II_3

$$\sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon) II_3 dx \leq C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_{\epsilon}^{\infty} 8x \left\{\frac{\theta^2 x^2 \log n}{3(\delta+2)}\right\}^{-(\delta+2)} dx$$
$$\leq C \sum_{n>H(\epsilon)} n^{-1}(\log n)^{-2} \left(\frac{4}{\delta+1}\right) \left\{\frac{\theta^2}{3(\delta+2)}\right\}^{-(\delta+2)} \epsilon^{-2\delta-2}. \tag{3.13}$$

Hence

$$\lim_{\epsilon\downarrow 0} \epsilon^{2\delta} \sum_{n>H(\epsilon)} n^{-1}(\log n)^{\delta} \int_0^{\infty} 2(x+\epsilon) II_3 dx \leq C\epsilon^{-2} \left(\frac{4}{\delta+1}\right) 3\left(\frac{\delta}{2}+1\right)^{-(\delta+2)} \sum_{n\geq H(\epsilon)} n^{-1}(\log n)^{-2}$$
$$\leq C\epsilon^{-2} \int_{H(\epsilon)} \frac{dx}{x(\log x)^2} \leq CM^{-1} \to 0 \quad \text{as } M \to \infty,$$

uniformly with respect to ϵ.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NA random variables. If $EX_1^2(\log^+|X_1|)^{\delta} < \infty$ for any $0 < \delta \leq 1$, then (1.4) holds.*

Proof: In fact, one can easily get

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} ES_n^2 I\left(|S_n| \geq \epsilon\sqrt{n\log n}\right)$$
$$= \epsilon^2 \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \geq \epsilon\sqrt{n\log n}\right) + \sum_{n=1}^{\infty} \frac{(\log n)^{\delta-1}}{n^2} \int_{\epsilon\sqrt{n\log n}}^{\infty} 2xP(|S_n| \geq x)\, dx$$
$$=: I_1 + I_2. \tag{3.14}$$

Consequently to verity Theorem 1 we only need to consider I_1 and I_2, respectively. From Propositions 1 and 3 we have

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}I_1 = \frac{E|N|^{2\delta+2}}{\delta+1}. \tag{3.15}$$

It follows from Propositions 4 and 5 that

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=1}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\left|\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP(|S_n|\geq x)\,dx-\int_{\epsilon\sqrt{\log n}}^{\infty}2xP\left(|N|\geq\frac{n}{\sqrt{x}}\right)dx\right|=0. \tag{3.16}$$

Hence, by Proposition 2 and (3.16) we also have

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}I_2 = \frac{E|N|^{2\delta+2}}{\delta(\delta+1)}, \tag{3.17}$$

which completes the proof together with (3.15).

Next we will show that if S_n is replaced by $M_n = \max_{1\leq k\leq n}|S_k|$, then Theorem 1 still holds. □

Proposition 6. **(Fu and Zhang, 2007)** *Suppose that* $\{W(t); t\geq 0\}$ *is a standard Wiener process(Browinian motion). Then, for any* $0<\delta\leq 1$,

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta+2}\sum_{n=1}^{\infty}\frac{(\log n)^{\delta}}{n}P\left(\sup_{0\leq\delta\leq 1}|W(s)|\geq\epsilon\sqrt{\log n}\right)=\frac{2E|N|^{2\delta+2}}{\delta+1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.$$

Proof: Refer to Huang and Zhang (2005). □

Proposition 7. **(Zhao, 2008)** *Suppose that* $\{W(t); t\geq 0\}$ *is a standard Wiener process. Then, for any* $0<\delta\leq 1$,

$$\lim_{\epsilon\downarrow 0}\epsilon^{2\delta}\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP\left(\sup_{0\leq t\leq 1}|W(t)|\geq\frac{x}{\sqrt{n}}\right)=\frac{2E|N|^{2\delta+2}}{\delta(\delta+1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2\delta+2}}.$$

Theorem 2. *Let* $\{X_n, n\geq 1\}$ *be a sequence of identically distributed NA random variables. If* $EX_1^2(\log^+|X_1|)^{\delta}<\infty$ *for any* $0<\delta\leq 1$, *then (1.5) holds.*

Proof: Note that Theorem 2 is the maximal version of Theorem 1. Hence, if we make some modification of the proof of Theorem 1, Theorem 2 will follow. As in (3.14), indeed, it suffices to study that

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}EM_n^2I\left(M_n\geq\epsilon\sqrt{n\log n}\right)\\&=\epsilon^2\sum_{n=2}^{\infty}\frac{(\log n)^{\delta}}{n}P\left(M_n\geq\epsilon\sqrt{n\log n}\right)+\sum_{n=2}^{\infty}\frac{(\log n)^{\delta-1}}{n^2}\int_{\epsilon\sqrt{n\log n}}^{\infty}2xP(M_n\geq x)\,dx\\&=: I_3+I_4.\end{aligned}$$

To pave the way for the proofs of I_3 of I_4 along the same lines as that of the proof of Theorem 1, together with Lemmas 2 and 3 and Propositions 6 and 7. □

4. Concluding Remark

It is of interest to show that the precise rate result in this kind of complete moment convergence also holds for moving average process. In the future study, we investigate the precise rate of convergence in complete moment of moving average processes based NA random variables by extending Theorems 1 and 2 to the moving average processes.

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