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Estimation of the Block and Basu model for system level life testing with censored data

Inho Jeong[1] · Kilho Cho[2] · Jangsik Cho[3]

[12]Department of Statistics, Kyungpook National University
[3]Department of Informational Statistics, Kyungsung University

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Abstract

We consider a life testing experiment in which several two component shared parallel system are put on test, and the test is terminated at a specified number of system failures. The bivariate data obtained from such a system level life testing can be classified into three classes: (1) the case of failed two components with known failure times, (2) the case of one censored component and the other failed component of which the failure time might be known or unknown, (3) the case of censored two components. In this thesis, the maximum likelihood estimators of parameters for Block and Basu bivariate exponential distribution under above censoring scheme are obtained. And the results of comparative studies are presented.

Keywords: Bivariate exponential distribution, Block and Basu model, shared parallel system.

1. Introduction

On the life testing of two-component system, the system works normally even though one of the two components has a failure. If each lifetime of the components is X and Y, it is generally assumed that X and Y are reciprocally dependent random variables. In two-component system, if one of the components has a failure, it influences the failure rate of the other. This is called shared parallel system. It might apply to the study of engine failures in two-engine planes, or to the performance of a person's eyes, ears, kidneys, or other paired organs. That is, the lifetimes of each component of shared parallel system are interdependent. The lifetimes of shared parallel system which consists of two components have a bivariate exponential distribution (BVE) and this BVE has been studied a lot so far. An interesting model based on the exponential distribution has been used by Freund (1961). However, the distribution does not have exponential marginals. Marshall and Olkin (1967) suggested BVE which has the exponential marginals and the loss of memory property (LMP).

[1] Graduate student, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea.
[2] Corresponding author: Professor, Department of Statistics, Kyungpook National University, Daegu 702-701, Korea. E-mail: khcho@knu.ac.kr
[3] Professor, Department of Informational Statistics, Kyungsung University, Busan, 608-736, Korea.

The BVE enjoys some properties that have useful physical interpretations but, unfortunately, are inappropriate in situations where the two components cannot fail simultaneously with positive probability. An absolutely continuous bivariate exponential distribution (ACBVE) would be more appropriate in the aforementioned situations. The distribution of Block and Basu (1974) is absolutely continuous and retains the LMP of the BVE, but it does not have exponential marginals. Block and Basu proved, however, that the only absolutely continuous bivariate distribution with exponential marginals and the LMP is a bivariate distribution with independent exponential marginals. Using Freund's derivation Block and Basu (1974) derived the density of two variables (X, Y) as follows

$$f(x,y) = \begin{cases} \dfrac{\lambda_1\lambda(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2} & e^{-\lambda_1 x-(\lambda_2+\lambda_{12})y} \quad \text{if } x<y \\ \dfrac{\lambda_2\lambda(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2} & e^{-(\lambda_1+\lambda_{12})x-\lambda_2 y} \quad \text{if } x>y \end{cases}, \tag{1.1}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

Hanagal and Kale (1991) and Weier (1981) also used Freund's derivation for their studies. They applied it to the only cases that have complete lifetime data. However, if all components do not have any failures within planned time, the number of the data will be decreased. Also if the time is extended, the cost for the test will be increased. Therefore it is hard to get complete lifetime data.

There have been many studies using the data from unfinished test. Hong (1998) studied the Freund model for system level life testing using type II censored data, and Cho and Kim (2003) studied the model with type I censored data. Hwang *et al.* (2007) studied the Block and Basu model with type I censored data. In this thesis, the testing is terminated when a fixed portion of the systems has a failure. Then using the obtained data until up to that time we obtain the maximum likelihood estimators (MLEs) for the parameters of the model. The facts which would be emphasized at this point are that the testing unit is a system, but the lifetime data obtained come from components. The lifetimes of each component can be observed when the system is failure. But in case of the censored system with one failed component, the lifetime of the failed component is classified in two cases which can be observed or cannot be observed.

2. MLEs of system parameters

2.1. Data description

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be the random vectors having the bivariate exponential distribution in equation (1.1), and $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be their observations. Provided the z_r is terminated point in the life testing when the fixed r of n have failure, (x_i, y_i), the lifetimes of the observed *ith* components, can be divided by

$$(x_i, y_i) = \begin{cases} (x_i, y_i), & \max(x_i, y_i) < z_r \\ (x_i, z_r), & x_i < z_r < y_i \\ (z_r, y_i), & y_i < z_r < x_i \\ (z_r, z_r), & z_r < \min(x_i, y_i) \end{cases}.$$

And let $I(\bullet)$ be an indicate function, $n_j (j = 1, \cdots, 5)$ be defined as follows;

$$n_1 = \sum_{i=1}^{n} I(x_i < y_i < z_r), \quad n_2 = \sum_{i=1}^{n} I(y_i < x_i < z_r), \quad n_3 = \sum_{i=1}^{n} I(x_i < z_r < y_i),$$
$$n_4 = \sum_{i=1}^{n} I(y_i < z_r < x_i), \quad n_5 = \sum_{i=1}^{n} I(\min(x_i, y_i) > z_r)$$

and let D_j be the set of systems which satisfy the conditions of $n_j \quad (j = 1, \cdots, 5)$. (Hwang *et al.*, 2007)

2.2. Maximum likelihood estimators

Among n systems which the life testing is terminated, there are the systems that only one component has a failure. Such the systems are functionally normal but can be classified into three cases depending on the structure of a system ① the case of a failed component with known failure time, ② the case of a failed component with known failed fact but unknown failure time, ③ the case of a component with unknown failed fact. In this paper, ① and ② are handled.

2.2.1. The case of one failed component with known failure time

In the life testing above we can obtain five types of data and the likelihood function in each case.

(a) The case of $0 < x < y < z_r$

The failure rates of two components A and B in a two-components system have respectively the parameters λ_1 and λ_2 until from 0 to x. But if the first failure of the component A occurs at x, the failure rate of the component B changes over λ_{12} until from x to y. Consequently the density is

$$\frac{\lambda_1 \lambda(\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} e^{-\lambda_1 x - (\lambda_2 + \lambda_{12}) y},$$

and the likelihood function for the failed data of observed components in the number of n_1 systems is

$$L_a = \left(\frac{\lambda_1 \lambda(\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \right)^{n_1} e^{-\lambda_1 \sum_{i \in D_1} x_i - (\lambda_2 + \lambda_{12}) \sum_{i \in D_1} y_i}. \tag{2.1}$$

(b) The case of $0 < y < x < z_r$

In the similar way with the case (a), the likelihood function for the failed data of observed components in the number of n_2 systems is

$$L_b = \left(\frac{\lambda_2 \lambda(\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \right)^{n_2} e^{-(\lambda_1 + \lambda_{12}) \sum_{i \in D_2} x_i - \lambda_2 \sum_{i \in D_2} y_i}. \tag{2.2}$$

(c) The case of $0 < x < z_r < y$

The likelihood function for the failed data of observed components in the number of n_3 systems in which the component A has a failure at first and the component B has not yet been failed until the terminated point z_r is

$$L_c = \left(\frac{\lambda_1 \lambda}{\lambda_1 + \lambda_2}\right)^{n_3} e^{-\lambda_1 \sum_{i \in D_3} x_i - (\lambda_2 + \lambda_{12}) n_3 z_r}. \tag{2.3}$$

(d) The case of $0 < y < z_r < x$

In the same way with the case (c), the likelihood function for the failed data of the observed components in the number of n_4 systems is

$$L_d = \left(\frac{\lambda_2 \lambda}{\lambda_1 + \lambda_2}\right)^{n_4} e^{-(\lambda_1 + \lambda_{12}) n_4 z_r - \lambda_2 \sum_{i \in D_4} y_i}. \tag{2.4}$$

(e) The case of $0 < z_r < \min(x_i, y_i)$

The likelihood function for data of the observed components in the number of n_5 systems in which two components A and B all have not yet been failed until the terminated point z_r is

$$L_e = e^{-\lambda n_5 z_r}. \tag{2.5}$$

The logarithm likelihood function for n systems level life testing under type Ⅱ censored data is the sum of their logarithms of equations (2.1), (2.2), (2.3), (2.4) and (2.5).

$$\begin{aligned}
\ln L(\lambda_1, \lambda_2, \lambda_{12}) =& \ln L_a + \ln L_b + \ln L_c + \ln L_d + \ln L_e \\
=&(n - n_5)\ln(\lambda_1 + \lambda_2 + \lambda_{12}) + (n_1 + n_3)\ln\lambda_1 \\
&+ (n_2 + n_4)\ln\lambda_2 - (n - n_5)\ln(\lambda_1 + \lambda_2) \\
&+ n_1 \ln(\lambda_2 + \lambda_{12}) + n_2 \ln(\lambda_1 + \lambda_{12}) \\
&- \lambda_1\Big(\sum_{i \in D_1} x_i + \sum_{i \in D_2} x_i + \sum_{i \in D_3} x_i + n_4 z_r + n_5 z_r\Big) \\
&- \lambda_2\Big(\sum_{i \in D_1} y_i + \sum_{i \in D_2} y_i + \sum_{i \in D_4} y_i + n_3 z_r + n_5 z_r\Big) \\
&- \lambda_{12}\Big(\sum_{i \in D_1} y_i + \sum_{i \in D_2} x_i + n_3 z_r + n_4 z_r + n_5 z_r\Big).
\end{aligned} \tag{2.6}$$

The maximum likelihood estimators of the parameters from equation (2.6) are (Hwang *et al.*, 2007)

$$\widehat{\lambda} = \frac{n - n_5}{\sum_{i \in D_1} x_i + \sum_{i \in D_2} y_i + \sum_{i \in D_3} x_i + \sum_{i \in D_4} y_i + n_5 z_r}, \tag{2.7}$$

$$\widehat{\lambda_1} = \widehat{\lambda} - \frac{n_1}{\sum_{i\in D_1} y_i - \sum_{i\in D_1} x_i + n_3 z_r - \sum_{i\in D_3} x_i}, \tag{2.8}$$

$$\widehat{\lambda_2} = \widehat{\lambda} - \frac{n_2}{\sum_{i\in D_2} x_i - \sum_{i\in D_2} y_i + n_4 z_r - \sum_{i\in D_4} y_i}, \tag{2.9}$$

$$\widehat{\lambda_{12}} = \widehat{\lambda} - \widehat{\lambda_1} - \widehat{\lambda_2}. \tag{2.10}$$

2.2.2. The case of one failed component with unknown failure time

A failed fact of one component until the terminated point in the life testing is known, but the estimated results might be different in accordance with methods that handle the lifetime of the failed component in the case that it is impossible to measure the failure time by some reason. In this thesis, in the case of the failed component A with known failed fact but unknown failure time, we put the lifetime of the component A on $p_1 z_r$, replace x in equation (2.3) with $p_1 z_r$, and derive equation (2.11). Similarly, in the case of the failed component B, we replace y in equation (2.4) with $p_2 z_r$, and derive equation (2.12). Here $p_i (0 \le p_i \le 1, i = 1, 2)$ means the given probability.

$$L_{c'} = \left(\frac{\lambda_1 \lambda}{\lambda_1 + \lambda_2}\right)^{n_3} e^{-\lambda_1 p_1 n_3 z_r - (\lambda_2 + \lambda_{12}) n_3 z_r}. \tag{2.11}$$

$$L_{d'} = \left(\frac{\lambda_2 \lambda}{\lambda_1 + \lambda_2}\right)^{n_4} e^{-(\lambda_1 + \lambda_{12}) n_4 z_r - \lambda_2 p_2 n_4 z_r}. \tag{2.12}$$

Therefore, the logarithm likelihood function is sum of equations (2.1), (2.2), (2.5), (2.11) and (2.12). That is

$$\begin{aligned}
\ln L(\lambda_1, \lambda_2, \lambda_{12}) =& \ln L_a + \ln L_b + \ln L_{c'} + \ln L_{d'} + \ln L_e \\
=& (n - n_5)\ln(\lambda_1 + \lambda_2 + \lambda_{12}) + (n_1 + n_3)\ln \lambda_1 \\
& + (n_2 + n_4)\ln \lambda_2 - (n - n_5)\ln(\lambda_1 + \lambda_2) \\
& + n_1 \ln(\lambda_2 + \lambda_{12}) + n_2 \ln(\lambda_1 + \lambda_{12}) \\
& - \lambda_1 (\sum_{i\in D_1} x_i + \sum_{i\in D_2} x_i + p_1 n_3 z_r + n_4 z_r + n_5 z_r) \\
& - \lambda_2 (\sum_{i\in D_1} y_i + \sum_{i\in D_2} y_i + p_2 n_4 z_r + n_3 z_r + n_5 z_r) \\
& - \lambda_{12} (\sum_{i\in D_1} y_i + \sum_{i\in D_2} x_i + n_3 z_r + n_4 z_r + n_5 z_r).
\end{aligned} \tag{2.13}$$

Hence the maximum likelihood estimators of the parameters from equation (2.13) are

(Hwang *et al.*, 2007)

$$\widehat{\lambda} = \frac{n - n_5}{\sum_{i \in D_1} x_i + \sum_{i \in D_2} y_i + p_1 n_3 z_r + p_2 n_4 z_r + n_5 z_r}, \tag{2.14}$$

$$\widehat{\lambda_1} = \widehat{\lambda} - \frac{n_1}{\sum_{i \in D_1} y_i - \sum_{i \in D_1} x_i + n_3 z_r - p_1 n_3 z_r}, \tag{2.15}$$

$$\widehat{\lambda_2} = \widehat{\lambda} - \frac{n_2}{\sum_{i \in D_2} x_i - \sum_{i \in D_2} y_i + n_4 z_r - p_2 n_4 z_r}, \tag{2.16}$$

$$\widehat{\lambda_{12}} = \widehat{\lambda} - \widehat{\lambda_1} - \widehat{\lambda_2}. \tag{2.17}$$

3. Comparison of estimators

The method that Friday and Patil (1977) suggested to generate data having a bivariate exponential distribution of Block and Basu model is used. Friday and Patil suggested the transform equations to variables X_1 and X_2 having a bivariate exponential distribution from two independent variables Y_1 and Y_2 having the standard exponential distribution. The equations are given by

$$X_1 = \begin{cases} \{Y_1 + \lambda_{12}\lambda^{-1}Y_2\}(\lambda_{12} + \lambda_1)^{-1}, & \lambda_2 Y_1 > \lambda_1 Y_2 \\ (\lambda_{1+}\lambda_2)(\lambda\lambda_1)^{-1}Y_1, & \lambda_2 Y_1 < \lambda_1 Y_2 \end{cases}, \tag{3.1}$$

$$X_2 = \begin{cases} (\lambda_1 + \lambda_2)(\lambda\lambda_2)^{-1}Y_2, & \lambda_2 Y_1 > \lambda_1 Y_2 \\ \lambda_{12}\lambda^{-1}Y_1 + Y_2(\lambda_{12} + \lambda_2)^{-1}, & \lambda_2 Y_1 < \lambda_1 Y_2 \end{cases}. \tag{3.2}$$

We can generate samples having a bivariate exponential distribution of the Block and Basu model from the equations. When we generate a bivariate data of the sample size 20 having equation (1.1) by method of Friday and Patil for $\lambda_1 = 1.7, \lambda_2 = 1.7, \lambda_{12} = 3.2$, they are $(.5714*, \underline{.04505})$, $(.3131*, .3750*)$, $(.4466, .2728*)$, $(.1038.0326)$, $(.2534, .0963)$, $(\underline{.0413}, .2549*)$, $(.1780, .2417)$, $(.1397, .0079)$, $(.0811, .1324)$, $(.2005.1130)$, $(.3841*, .4065*)$, $(.7222*, .2824*)$, $(.4616*, \underline{.1206})$, $(.5590*, \underline{.2178})$, $(.2301, .0615)$, $(.0728, .1830)$, $(.0691, .0729)$, $(\underline{.0311}, .6831*)$, $(\underline{.1431}, .2541*)$, $(.0052, .1133)$. Provided that $r = 10$, the values of indicator $[*]$ are censored data and are replaced with 0.253 as $z_{10} = 0.253$ in the process of calculating. And the underlined values are the case of unknown failure time and are replaced with $0.253p_i \quad (0 \le p_i \le 1, i = 1, 2)$. The tables and figures show the relative efficiencies of the total absolute biases and the total mean square errors (MSEs) for the parameters between the case of one failed component with known failure time and unknown failure time. The values are results that are calculated through 10,000 times repetition as the changes of sample size n and the fixed r for $\lambda_1 = 1.7, \lambda_2 = 1.7, \lambda_{12} = 3.2$ and varied $p = p_1 = p_2 = 0.2, 0.25, \cdots, 0.7$. From the tables and figures, when p is about 0.45 in the case of one failed component with unknown failure time, we can know that the results are good regardless of n and r in the aspects of the relative efficiencies of the total absolute biases and the total MSEs. Table 3.1, Figure 3.1 and Figure 3.2 are results in the cases of $n = 30, r = 15$ and $r = 20$.

Table 3.1 Relative efficiencies of total absolute biases and total MSEs : n=30

p	$r=15$		$r=20$	
	$RE(T.ABias)$	$RE(TotalMSE)$	$RE(T.ABias)$	$RE(TotalMSE)$
.20	0.1629	0.5082	0.1582	0.5164
.25	0.212	0.6657	0.2076	0.7204
.30	0.3086	0.8958	0.2861	0.9332
.35	0.5107	1.1416	0.5389	1.2191
.40	1.4852	1.279	1.8917	1.3511
.45	1.9001	1.3555	1.1168	1.3873
.50	0.538	1.2253	0.4201	1.2303
.55	0.2989	0.96385	0.2653	1.0224
.60	0.2159	0.7348	0.1911	0.792
.65	0.1609	0.5361	0.1478	0.603
.07	0.1282	0.3899	0.1189	0.4478

$RE(TotalAbsoluteBias) = \dfrac{Bias(*)}{Bias(p)}, RE(TotalMSE) = \dfrac{MSE(*)}{MSE(p)}$

$*$: The case of one failed component with known failure time.

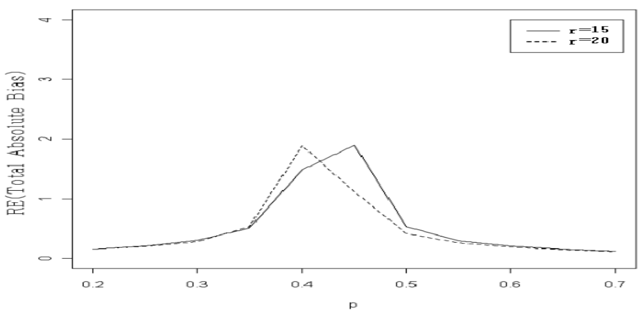

Figure 3.1 Relative efficiency of total absolute bias as varied $p(r = 15 and 20 : n = 30)$

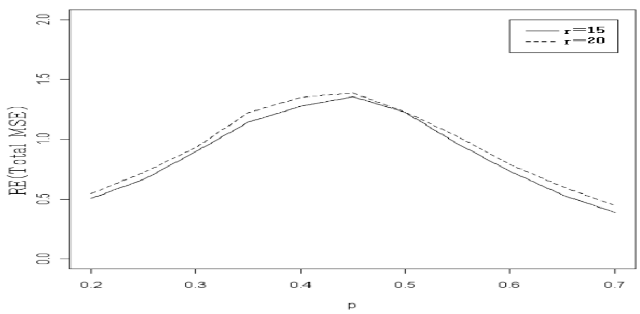

Figure 3.2 Relative efficiency of total MSE as varied $p(r = 15 and 20 : n = 30)$

4. Discussion

In this thesis, the bivariate lifetime data that follow the Block and Basu model for the system level life testing are obtained by type II censored data. And they are divided by the cases of one failed component with known failure time or unknown failure time. We also compare two cases through the simulations. In the case that a failed fact of one failed component is known but the failure time is unknown, if we assume that the failure time of the component is happened at about 40% of the terminated point in the life testing, we can know that the results are good in aspects of the relative efficiencies of the total absolute biases and the total MSEs.

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