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EXISTENCE OF SOLUTIONS FOR IMPULSIVE NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

B. SELVARAJ, M. MALLIKA ARJUNAN†, AND V. KAVITHA

DEPT. OF MATH., KARUNYA UNIV., KARUNYA NAGAR, COIMBATORE- 641 114, TAMIL NADU, INDIA.
E-mail address: selvaraj@karunya.edu, {arjunphd07,kavi_velubagyam}@yahoo.co.in

ABSTRACT. In this article, we study the existence and uniqueness of mild and classical solutions for a nonlinear impulsive differential equation with nonlocal conditions

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \le t \le T_0, \quad t \ne t_i, \\ u(0) + g(u) &= u_0, \\ \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, \dots p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0, \end{aligned}$$

in a Banach space X, where A is the infinitesimal generator of a C_0 semigroup, g constitutes a nonlocal conditions, and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ represents an impulsive conditions.

1. INTRODUCTION

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison with the duration of the processes. Consequently, it is natural to assume that these perturbations act instantaneously, that is in the form of impulses. For more details on this theory and applications, see the monographs of Bainov and Simeonov [2], Lakshmikantham et al. [9], and Samoilenko and Perestyuk [14], where numerous properties of their solutions are studied and detailed bibliographies are given.

The starting point of this paper is the works in papers [1, 11, 12]. Especially, the authors in [12] investigated the existence and uniqueness of mild and classical solutions for an impulsive first order system

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad 0 \le t \le T_0, \quad t \ne t_i, \\ u(0)) &= u_0, \\ \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, \dots p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0. \end{aligned}$$

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† Corresponding author.

by using semigroup theory and Schauder's fixed point theorem. And in [11], authors studied existence and uniqueness of mild and classical solutions for the following impulsive system

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \quad 0 \le t \le K, \quad t \ne t_i, \\ u(0) + g(u) &= u_0, \\ \Delta u(t_i) &= I_i(u(t_i)), \quad i = 1, 2, \dots p, \quad 0 < t_1 < t_2 < \cdots < t_p < K. \end{aligned}$$

by using the Banach contraction principle and Schauder's fixed point theorem.

Motivated by the above mentioned works [11, 12], the main purpose of this paper is to prove the existence and uniqueness of mild and classical solutions for the following first order impulsive system

$$u'(t) = Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \le t \le T_0, \quad t \ne t_i, \tag{1.1}$$

$$u(0) + g(u) = u_0, \tag{1.2}$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \dots p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0. \tag{1.3}$$

in a Banach space X, where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)|t \ge 0\}$, $f \in C([0, T_0] \times X \times X \times X, X)$, $g \in \mathcal{PC}([0, T_0], X)$,

$$\begin{aligned} Tu(t) &= \int_0^t K(t, s)u(s)\mathrm{d}s, \ K \in C[D, R^+], \\ Su(t) &= \int_0^{T_0} H(t, s)u(s)\mathrm{d}s, \ H \in C[D_0, R^+], \end{aligned}$$

where $D = \{(t, s) \in R^2 : 0 \le s \le t \le T_0\}$, $D_0 = \{(t, s) \in R^2 : 0 \le t, s \le T_0\}$ and $\mathcal{PC}([0, T_0], X)$ consist of a functions u that are a map from $[0, T_0]$ into X, such that $u(t)$ is continuous at $t \ne t_i$ and left continuous at $t = t_i$, and the right limit $u(t_i^+)$ exists for $i = 1, 2, \dots p$.

Evidently $\mathcal{PC}([0, T_0], X)$ is a Banach space with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{t \in [0, T_0]} \|u(t)\|.$$

The nonlocal Cauchy problem was considered by Byszewski [4] and the importance of nonlocal conditions in different fields has been discussed in [4] and [6] and the references therein. For example, in [6] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x) = \sum_{k=0}^{n} c_k x(t_k),$$

where $c_k, \ k = 0, 1, \dots, n$ are given constants and $0 < t_0 < t_1 < \cdots < t_n < a$. In this case the above equations allows the additional measurements at $t = t_k, \ k = 0, 1, \dots, n$. In the past several years theorem about existence and uniqueness of differential, impulsive differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by many authors [3, 5, 7, 8, 10].

In the present paper, we discuss the existence and uniqueness for the impulsive problem (1.1)-(1.3). Our approach here is based on the semigroup theory [13] and fixed point theorem.

2. Existence results

In this section, first we define the concept of mild and classical solutions for the problem (1.1)-(1.3).

Definition 1. *A function $u(\cdot) \in \mathcal{PC}([0,T_0],X)$ is a mild solution of equations (1.1)-(1.3) if it satisfies*

$$u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)),\ 0 \le t \le T_0.$$

Definition 2. *A classical solution of equations (1.1)-(1.3) is a function $u(\cdot)$ in $\mathcal{PC}([0,T_0],X) \cap C^1([0,T_0]\backslash\{t_1,t_2,\ldots,t_p\},X)$, $u(t) \in D(A)$(the domain of A) for $t \in [0,T_0]\backslash\{t_1,t_2,\ldots,t_p\}$, which satisfies equations (1.1)-(1.3) on $[0,T_0]$.*

The mild and classical solutions of (1.1)-(1.3) will be established under different conditions of the functions f, g, I_i and the semigroup $T(\cdot)$.

2.1. **Lipschitz conditions.** Let $B(X)$ be the Banach space of all linear and bounded operators on X. Define

$$M = \sup_{t\in[0,T_0]} \|T(t)\|_{B(X)}, \tag{2.1}$$

which is a finite number.

Now we list out the following hypotheses:

(H1) $f : [0,T_0] \times X \times X \times X \to X,\ g : \mathcal{PC}([0,T_0],X) \to X$ and $I_i : X \to X,\ i = 1,2,\ldots p$ are continuous and there exists constants $L_1, L_2, L_3 > 0,\ G > 0,\ h_i > 0,\ i = 1,2,\ldots p$, such that

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \le L_1\|x_1-y_1\| + L_2\|x_2-y_2\| + L_3\|x_3-y_3\|,$$
$$t \in [0,T_0],\ x_i,y_i \in X,\ i = 1,2,3.$$
$$\|g(u) - g(v)\| \le G\|u-v\|, \quad u,v \in \mathcal{PC}([0,T_0],X),$$
$$\|I_i(x) - I_i(y)\| \le h_i\|u-v\|, \quad x,y \in X.$$

(H2) Denote $L = \max\{L_1,L_2,L_3\}$, $\quad K^* = \sup_{t\in[0,T_0]} \int_0^t |K(t,s)|dt < \infty$, and

$$H^* = \sup_{t\in[0,T_0]} \int_0^{T_0} |H(t,s)|dt < \infty.$$

(H3) The constants L, G, K^*, H^* satisfy the inequality

$$M\Big[G + LT_0(1 + K^* + H^*) + \sum_{i=1}^{p} h_i\Big] < 1.$$

Theorem 2.1. *Assume that the hypotheses (H1)-(H3) are satisfied. Then for every $u_0 \in X$, for $t \in [0, T_0]$ the equation*

$$\begin{aligned} u(t) = T(t)[u_0 - g(u)] &+ \int_0^t T(t-s) f(s, u(s), Tu(s), Su(s)) ds \\ &+ \sum_{0<t_i<t} T(t - t_i) I_i(u(t_i)) \end{aligned}$$

has a unique mild solution.

Proof. Let $u_0 \in X$ be fixed. Define an operator F on $\mathcal{PC}([0, T_0], X)$ by

$$\begin{aligned} (Fu)(t) = T(t)[u_0 - g(u)] &+ \int_0^t T(t-s) f(s, u(s), Tu(s), Su(s)) ds \\ &+ \sum_{0<t_i<t} T(t - t_i) I_i(u(t_i)), \quad 0 \le t \le T_0. \end{aligned}$$

Then it is clear that $F : \mathcal{PC}([0, T_0], X) \to \mathcal{PC}([0, T_0], X)$. Now we show that F is contraction. For any $u, v \in \mathcal{PC}([0, T_0], X)$, we have

$$\begin{aligned} \|(Fu)(t) - (Fv)(t)\| \le \|T(t)[g(u) - g(v)]\| &+ \int_0^t \|T(t-s)\|_{B(X)} \|f(s, u(s), Tu(s), Su(s)) \\ &- f(s, v(s), Tv(s), Sv(s))\| ds \\ &+ \sum_{0<t_i<t} \|T(t - t_i)\|_{B(X)} \|I_i(u(t_i)) - I_i(v(t_i))\| \end{aligned}$$

Using the hypothesis (H1) and equation (2.1), we have

$$\begin{aligned} \|(Fu)(t) - (Fv)(t)\| \le MG\|u - v\|_{\mathcal{PC}} + M\Big[\int_0^t L_1\|u - v\| + L_2\|Tu - Tv\| \\ + L_3\|Su - Sv\|\Big] ds + M\|u - v\|_{\mathcal{PC}} \sum_{i=1}^{p} h_i. \end{aligned} \tag{2.2}$$

Now,

$$\int_0^t L_2\|Tu - Tv\| \mathrm{d}s \le L_2 \int_0^t \int_0^s \|K(s, \tau)\| \|u(\tau) - v(\tau)\| \mathrm{d}\tau \mathrm{d}s$$

$$\begin{aligned}
&\le L_2 \int_0^t \|u(s)-v(s)\| \int_0^s \|K(s,\tau)\| \mathrm{d}\tau \mathrm{d}s \\
&\le L_2 \|u(t)-v(t)\| \int_0^t K^* \mathrm{d}s \\
&\le L_2 \|u-v\|_{\mathcal{PC}} K^* T_0
\end{aligned} \tag{2.3}$$

Similarly,

$$\int_0^t L_3 \|Su - Sv\| \mathrm{d}s \le L_3 \|u-v\|_{\mathcal{PC}} H^* T_0. \tag{2.4}$$

Substitute the equations (2.3) and (2.4) into the equation (2.2), we have

$$\begin{aligned}
\|(Fu)(t)-(Fv)(t)\| &\le MG\|u-v\|_{\mathcal{PC}} + M\Big[L_1 T_0 \|u-v\|_{\mathcal{PC}} + L_2 \|u-v\|_{\mathcal{PC}} K^* T_0 \\
&\qquad + L_3 \|u-v\|_{\mathcal{PC}} H^* T_0 + \|u-v\|_{\mathcal{PC}} \sum_{i=1}^{p} h_i\Big] \\
&\le M\Big[G + L_1 T_0 + L_2 K^* T_0 + L_3 H^* T_0 + \sum_{i=1}^{p} h_i\Big] \|u-v\|_{\mathcal{PC}}.
\end{aligned}$$

Using the definition of L, we have

$$\|(Fu)(t)-(Fv)(t)\| \le M\Big[G + LT_0(1+K^*+H^*) + \sum_{i=1}^{p} h_i\Big] \|u-v\|_{\mathcal{PC}}$$

From hypothesis (H3), we have

$$\|Fu - Fv\|_{\mathcal{PC}} \le \|u-v\|_{\mathcal{PC}}, \quad u, v \in \mathcal{PC}([0,T_0], X).$$

Therefore, F is a contraction operator on $\mathcal{PC}([0,T_0],X)$. Thus F has a unique fixed point, which gives rise to a unique mild solution. This completes the proof. □

Remark 1. *If $S = 0$ in equation (1.1) and assume that the hypotheses (H1)-(H2) are satisfied (with simple modifications) and $M[G + LT_0(1+K^*) + \sum_{i=1}^{p} h_i] < 1$, then equation (1.1)-(1.3) has a unique mild solution.*

2.2. **g is compact.** It should be pointed out that a compact operator is a continuous operator which maps a bounded set into a precompact set.

Now we assume the following hypotheses:

- (H4) f is continuous and maps a bounded set into a bounded set.
- (H5) $g : \mathcal{PC}([0,T_0],X) \to X$ and $I_i : X \to X,\ i = 1,2,\dots p$, are compact operators, and $T(\cdot)$ is also compact.

(H6) For each $u_0 \in X$, there exists a constant $r > 0$ such that

$$M(\|u_0\| + \sup_{\varphi \in Y_r} \|g(\varphi)\| + T_0 \sup_{s\in[0,T_0],\varphi\in Y_r} \|f(s,\varphi(s),T\varphi(s),S\varphi(s)\| + \sup_{\varphi\in Y_r} \sum_{i=1}^{p} \|I_i(\varphi(t_i))\|) \leq r,$$

where $Y_r = \{\varphi \in \mathcal{PC}([0,T_0],X) : \|\varphi(t)\| \leq r \text{ for } t \in [0,T_0]\}$.

Under these hypotheses, we can prove the following result.

Theorem 2.2. *Let (H4)-(H6) be satisfied. Then for every $u_0 \in X$, for $t \in [0,T_0]$ the equation*

$$u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))$$

has at least a mild solution.

Proof. Let $u_0 \in X$ be fixed. Define an operator F on $\mathcal{PC}([0,T_0],X)$ by

$$(Fu)(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)).$$

The operator F is continuous from Y_r to Y_r. In order to use Schauder's second fixed point theorem to obtain a fixed point and hence a mild solution, we need to prove that F is a compact operator. For this reason, we split $(Fu)(t)$ as $(F_1u)(t) + (F_2u)(t)$. That is $(Fu)(t) = (F_1u)(t) + (F_2u)(t)$, where

$$(F_1u)(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds, \quad 0 \leq t \leq T_0,$$
$$(F_2u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad 0 \leq t \leq T_0.$$

Now, we show that F_1 and F_2 are compact operators. First, we prove that F_2 is a compact operator. The operator

$$(F_2u)(t) = \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)) = \begin{cases} 0, & t \in [0,t_1], \\ T(t-t_1)I_1(u(t_1)), & t \in (t_1,t_2], \\ \vdots \\ \sum_{i=1}^{p} T(t-t_i)I_i(u(t_i)), & t \in (t_p,T_0], \end{cases}$$

and that the interval $[0, T_0]$ is divided into finite subintervals by $t_i,\ i = 1, 2, \dots, p$, so that we only need to prove that

$$W = \{T(\cdot - t_1)I_1(u(t_1)) : \cdot \in [t_1, t_2],\ u \in Y_r\}$$

is precompact in $C([t_1, t_2], X)$, as the cases for other subintervals are the same. From hypothesis (H5), we see that for each $t \in [t_1, t_2]$, the set $\{T(t - t_1)I_1(u(t_1)) : u \in Y_r\}$ is precompact in X. Next, for $t_1 \le s < t \le t_2$, we have, using the semigroup property,

$$\begin{aligned} \|T(t - t_1)I_1(u(t_1)) - T(s - t_1)I_1(u(t_1))\| &= \|T(s - t_1)[T(t - s) - T(0)]I_1(u(t_1))\| \\ &\le M\|[T(t - s) - T(0)]I_1(u(t_1))\| \end{aligned} \tag{2.5}$$

Thus, the functions in W are equicontinuous due to compactness of I_1 and the strong continuity of $T(\cdot)$. From the Arzela-Ascoli theorem, we deduce that F_2 is a compact operator.

Similarly we can prove compactness of F_1. That is, for each $t \in [0, T_0]$, the set $\{T(t)[u_0 - g(u)] : u \in Y_r\}$ is precompact in X, since g is compact. Also, for each $t \in (0, T_0]$ and $\epsilon \in (0, t)$,

$$\begin{aligned} &\left\{\int_0^{t-\epsilon} T(t - s)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r\right\} \\ &= \left\{T(\epsilon)\int_0^{t-\epsilon} T(t - s - \epsilon)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r\right\} \end{aligned}$$

is precompact in X, since $T(\cdot)$ is compact. Then, as

$$\int_0^{t-\epsilon} T(t - s)f(s, u(s), Tu(s), Su(s))ds \to \int_0^{t} T(t - s)f(s, u(s), Tu(s), Su(s))ds \quad \text{as } \epsilon \to 0.$$

We can conclude that $\left\{\int_0^t T(t - s)f(s, u(s), Tu(s), Su(s))ds : u \in Y_r\right\}$ is precompact in X using the total boundedness. Therefore, for each $t \in [0, T_0]$, $\{(F_1u)(t) : u \in Y_r\}$ is precompact in X.

Next, we show that the equicontinuity of $Q = \{(F_1u)(\cdot) : \cdot \in [0, T_0],\ u \in Y_r\}$. By using the idea of equation (2.5), we can prove the equicontinuity of $\{T(\cdot)[u_0 - g(u)] : \cdot \in [0, T_0],\ u \in$

$Y_r\}$. For the second term in Q, we let $0 \le s_1 < s_2 \le T_0$ and obtain

$$\begin{aligned}
&\left\| \int_0^{s_2} T(s_2 - s) f(s, u(s), Tu(s), Su(s)) ds - \int_0^{s_1} T(s_1 - s) f(s, u(s), Tu(s), Su(s)) ds \right\| \\
&= \| \int_0^{s_1} [T(s_2 - s) - T(s_1 - s)] f(s, u(s), Tu(s), Su(s)) ds \\
&\quad + \int_{s_1}^{s_2} T(s_2 - s) f(s, u(s), Tu(s), Su(s)) ds \| \\
&\le \int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \, \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\quad + M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\| ds \qquad (2.6)
\end{aligned}$$

If $s_1 = 0$, then the right-hand side of (2.6) can be made small when s_2 is small independently of $u \in Y_r$. If $s_1 > 0$, then we can find a small number $\eta > 0$ so that if $s_1 \le \eta$, then the right-hand side of (2.6) can be estimated as

$$\begin{aligned}
&\int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \, \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\quad + M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\le 2\eta M \max\{\|f(s, u(s), Tu(s), Su(s))\| : u \in Y_r, s \in [0, T_0]\} \\
&\quad + M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\| ds,
\end{aligned}$$

which can be made small when $s_2 - s_1$ is small independently of $u \in Y_r$.

If $s_1 > \eta$, then the right-hand side of (2.6) can be estimated as

$$\begin{aligned}
&\int_0^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \, \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\quad + M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\le \int_0^{s_1 - \eta} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \, \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\quad + \int_{s_1 - \eta}^{s_1} \|T(s_2 - s) - T(s_1 - s)\|_{L(X)} \, \|f(s, u(s), Tu(s), Su(s))\| ds \\
&\quad + M \int_{s_1}^{s_2} \|f(s, u(s), Tu(s), Su(s))\| ds
\end{aligned}$$

$$\leq \int_0^{s_1-\eta} \|T(s_2-s)-T(s_1-s)\|_{L(X)} \; \|f(s,u(s),Tu(s),Su(s))\|ds$$
$$+ 2\eta M \max\{\|f(s,u(s),Tu(s),Su(s))\| : u \in Y_r, s \in [0,T_0]\}$$
$$+ M \int_{s_1}^{s_2} \|f(s,u(s),Tu(s),Su(s))\|ds$$

Now, as $T(\cdot)$ is compact, $T(t)$ is operator norm continuous for $t > 0$. Thus $T(t)$ is operator norm continuous uniformly for $t \in [\eta, T_0]$. Therefore, $\|T(s_2-s)-T(s_1-s)\|_{L(X)}$ and hence $\int_0^{s_1-\eta} \|T(s_2-s)-T(s_1-s)\|_{L(X)}\|f(s,u(s),Tu(s),Su(s))\|ds$ can be made small when $s_2 - s_1$ is small independently of $u \in Y_r$. Thus the function in Q are equicontinuous. Therefore, F_1 is a compact operator by the Arzela-Ascoli theorem, and hence F is also a compact operator. Now, Schauder's second fixed point theorem implies that F has a fixed point, which gives rise to a mild solution. This completes the proof. □

2.3. **g is not Lipschitz and not compact.** Here, we will prove mild solutions under the following hypotheses:

(H7) The function f is continuous and there exists a constant $L > 0$ such that
$$\|f(t,x)-f(t,y)\| \leq L\|x-y\|, \quad t \in [0,T_0], \quad x,y \in X.$$

(H8) The function $I_i : X \to X, i = 1,2,\ldots,p$, are compact operators, and $T(\cdot)$ is also compact.

(H9) For each $u_0 \in X$, there exists a constant $r > 0$ such that
$$M(\|u_0\| + \sup_{\varphi \in Y_r}\|g(\varphi)\| + T_0 \sup_{s\in[0,T_0],\varphi\in Y_r} \|f(s,\varphi(s),T\varphi(s),S\varphi(s)\|$$
$$+ \sup_{\varphi\in Y_r} \sum_{i=1}^{p} \|I_i(\varphi(t_i))\|) \leq r.$$

(H10) The function $g : \mathcal{PC}([0,T_0],X) \to X$ is continuous, maps Y_r into a bounded set, and there is a $\delta = \delta(r) \in (0,t_1)$ such that $g(\varphi) = g(\ )$ for any $\varphi, \ \in Y_r$ with $\varphi(s) = \ (s), s \in [\delta, T_0]$.

Theorem 2.3. *Let (H7)-(H10) be satisfied. Then for every $u_0 \in X$, for $t \in [0,T_0]$ the equation*

$$u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds$$
$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i))$$

has at least a mild solution.

Proof. For $\delta = \delta(r) \in (0,t_1)$, set $Y(\delta) = \mathcal{PC}([\delta,T_0],X) =$ restrictions of functions in $\mathcal{PC}([0,T_0],X)$ on $[\delta,T_0]$, $Y_r(\delta) = \{\varphi \in Y(\delta); \|\varphi(t)\| \leq r$ for $t \in [\delta,T_0]\}$. For $u \in Y_r(\delta)$

fixed, we define a mapping F_u on Y_r by

$$(F_u\varphi)(t) = T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi(s),T\varphi(s),S\varphi(s))ds$$
$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [0,T_0],$$

where

$$\tilde{u}(t) = \begin{cases} u(t), & \text{if } t \in [\delta, T_0], \\ u(\delta), & \text{if } t \in [0,\delta]. \end{cases}$$

By hypothesis (H9), the mapping F_u maps Y_r into itself. Moreover, by hypothesis (H7) we deduce inductively that for $m \in N$,

$$\|(F_u^m\varphi)(t) - (F_u^m \quad)(t)\| \leq \frac{[MLt(1+K^*+H^*)]^m}{m!} \sup_{s\in[0,t]} \|\varphi(s) - \quad (s)\|,$$
$$t \in [0,T_0],\ \varphi, \quad \in Y_r,\ m = 1,2,\ldots.$$

Hence, we infer that for m large enough, the mapping F_u^m is a contractive mapping. Thus, by a well-known extension of the Banach contraction mapping principle, F_u has a unique fixed point $\varphi_u \in Y_r$, i.e.,

$$\varphi_u(t) = T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi_u(s),T\varphi_u(s),S\varphi_u(s))ds$$
$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [0,T_0].$$

Based on this fact, we define a mapping $\mathcal{F}$ from $Y_r(\delta)$ into itself by

$$\begin{aligned}(\mathcal{F}u)(t) &= \varphi_u(t), \quad t \in [\delta, T_0] \\ &= T(t)[u_0 - g(\tilde{u})] + \int_0^t T(t-s)f(s,\varphi_u(s),T\varphi_u(s),S\varphi_u(s))ds \\ &\quad + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)), \quad t \in [\delta,T_0].\end{aligned}$$

Similarly to the above and [10], we can use compactness and equicontinuity and then apply the Arzela-Ascoli theorem to prove that $\mathcal{F}$ is a compact operator. Therefore, we can use Schauder's second fixed point theorem to conclude that $\mathcal{F}$ has a fixed point $u_* \in Y_r(\delta)$. We set $u = \varphi_{u_*}$. Then

$$u(t) = T(t)[u_0 - g(\tilde{u}_*)] + \int_0^t T(t-s)f(s,u(s),Tu(s),Su(s))ds$$
$$+ \sum_{0<t_i<t} T(t-t_i)I_i(u_*(t_i)), \quad t \in [0,T_0]. \tag{2.7}$$

But $g(\tilde{u}_*) = g(u)$ and $u_*(t_i) = u(t_i)$, since $u_*(t) = (\mathcal{F}u_*)(t) = \varphi_{u_*}(t) = u(t), t \in [\delta, T_0]$, by the definition of $\mathcal{F}$. This concludes, together with (2.7), that $u(t)$ is a solution of (1.1)-(1.3). This completes the proof. □

2.4. **Classical solutions.** Now, we recall the following result.

Lemma 1. [1] *Assume that $u_0 \in D(A)$, $q_i \in D(A)$, $i = 1, 2, \ldots, p$. and that $f \in C^1([0, T_0] \times X \times X \times X, X)$. Then the impulsive equation*

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 < t < T_0,\ t \neq t_i, \\ u(0) &= u_0, \\ \Delta u(t_i) &= q_i, \quad i = 1, 2, 3, \ldots, p. \end{aligned}$$

has a unique classical solution $u(\cdot)$ which, for $t \in [0, T_0]$, satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s), Tu(s), Su(s))\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)q_i.$$

Now, we make the following hypothesis:

(H11) There exists a constant $L > 0$ such that

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|, \quad t \in [0, T_0],\ x, y \in X.$$

Theorem 2.4. *Let (H11) be satisfied and $u(\cdot)$ be a mild solution of (1.1)-(1.3). Assume that $u_0 \in D(A),\ I_i(u(t_i)) \in D(A),\ i = 1, 2, \ldots, p$, and that $f \in C^1([0, T_0] \times X \times X \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of (1.1)-(1.3).*

Proof. Let $u(\cdot)$ be the mild solution. Let $q_i = I_i(u(t_i)),\ i = 1, 2, \ldots, p$. Then from Lemma 2.1,

$$\begin{aligned} v'(t) &= Av(t) + f(t, v(t), Tv(t), Sv(t)), \quad 0 < t < T_0,\ t \neq t_i, \\ v(0) &= u(0) = u_0 - g(u) \\ \Delta v(t_i) &= q_i, \quad i = 1, 2, 3, \ldots, p. \end{aligned}$$

has a unique classical solution $v(\cdot)$ which satisfies for $t \in [0, T_0]$,

$$v(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s, v(s), Tv(s), Sv(s))\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)).$$

Since $u(\cdot)$ is the mild solution of (1.1)-(1.3), for $t \in [0, T_0]$,

$$u(t) = T(t)[u_0 - g(u)] + \int_0^t T(t-s)f(s, u(s), Tu(s), Su(s))\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)I_i(u(t_i)).$$

Thus, we get

$$v(t) - u(t) = \int_0^t T(t-s)[f(s, v(s), Tv(s), Sv(s)) - f(s, u(s), Tu(s), Su(s))]ds,$$

which gives, by hypothesis (H11) and an application of Gronwall's inequality,

$$\|v - u\|_{\mathcal{PC}} = 0.$$

This implies that $u(\cdot)$ gives rise to a classical solution. This completes the proof. $\square$

Remark 2. *If $S = 0$ in equation (1.1) amd assume that the hypothesis (H11) be satisfied and $u(\cdot)$ be a mild solution of (1.1)-(1.3). Assume that $u_0 \in D(A),\ I_i(u(t_i)) \in D(A),\ i = 1, 2, \ldots, p$, and that $f \in C^1([0, T_0] \times X \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of (1.1)-(1.3).*

Now, we consider the simple case on g. If $g = 0$ in (1.2), then the problem (1.1)-(1.3) is reduced to initial value problem

$$u'(t) = Au(t) + f(t, u(t), Tu(t), Su(t)), \quad 0 \le t \le T_0, \quad t \ne t_i, \tag{1.4}$$

$$u(0) = u_0, \tag{1.5}$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots p, \quad 0 < t_1 < t_2 < \cdots < t_p < T_0. \tag{1.6}$$

where A, f, I_i are defined as in problem (1.1)-(1.3).

Now, we prove the existence and uniqueness of mild and classical solution for the problem (1.4)-(1.6).

For this reason, we list the following hypotheses:

(H1′) $f : [0, T_0] \times X \times X \times X \to X$, and $I_i : X \to X,\ i = 1, 2, \ldots p$ are continuous and there exists constants $L_1, L_2, L_3 > 0,\ h_i > 0,\ i = 1, 2, \ldots p$, such that

$$\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \le L_1\|x_1 - y_1\| + L_2\|x_2 - y_2\| + L_3\|x_3 - y_3\|,$$
$$t \in [0, T_0],\ x_i, y_i \in X,\ i = 1, 2, 3.$$
$$\|I_i(x) - I_i(y)\| \le h_i\|x - y\|, \quad x, y \in X.$$

(H3′) The constants L, K^*, H^* satisfy the inequality

$$M\Big[LT_0(1 + K^* + H^*) + \sum_{i=1}^{p} h_i\Big] < 1.$$

Now, we state the following theorems without proof. The proof is similar to Theorem 2.1 and Theorem 2.4.

Theorem 2.5. *Assume that the hypotheses (H1′), (H2) and (H3′) are satisfied. Then for every $u_0 \in X$, for $t \in [0, T_0]$ the equation*

$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s, u(s), Tu(s), Su(s))ds$$
$$+ \sum_{0<t_i<t} T(t - t_i) I_i(u(t_i))$$

has a unique mild solution.

Theorem 2.6. *Let (H11) be satisfied and $u(\cdot)$ be a mild solution of (1.4)-(1.6). Assume that $u_0 \in D(A),\ I_i(u(t_i)) \in D(A),\ i = 1, 2, \ldots, p$, and that $f \in C^1([0, T_0] \times X \times X \times X, X)$. Then $u(\cdot)$ gives rise to a unique classical solution of (1.4)-(1.6).*

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