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NEW INTERIOR POINT METHODS FOR SOLVING $P_*(\kappa)$ LINEAR COMPLEMENTARITY PROBLEMS

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ABSTRACT. In this paper we propose new primal-dual interior point algorithms for $P_*(\kappa)$ linear complementarity problems based on a new class of kernel functions which contains the kernel function in [8] as a special case. We show that the iteration bounds are $\mathcal{O}((1+2\kappa)n^{\frac{9}{14}}\log\frac{n\mu^0}{\epsilon})$ for large-update and $\mathcal{O}((1+2\kappa)\sqrt{n}\log\frac{n\mu^0}{\epsilon})$ for small-update methods, respectively. This iteration complexity for large-update methods improves the iteration complexity with a factor $n^{\frac{5}{14}}$ when compared with the method based on the classical logarithmic kernel function. For small-update, the iteration complexity is the best known bound for such methods.

1. INTRODUCTION

In this paper, we consider the linear complementarity problem(LCP) as follows:

$$\begin{cases} s = Mx + q, \\ xs = 0, \\ x \geq 0,\ s \geq 0, \end{cases} \tag{1.1}$$

where $x, s, q \in \mathbf{R}^n$ and $M \in \mathbf{R}^{n\times n}$ is a $P_*(\kappa)$ matrix.

Primal-dual interior point method(IPM) is one of the most efficient numerical methods for various optimization problems.([14]) Even though significant research has been devoted to this topic, the influence on nonlinear programming theory and practice has to be studied. Linear complementarity problems(LCPs) are one of the fundamental problems in mathematical programming and have many applications in science, economics, and engineering.([7])

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Most of polynomial-time interior point algorithms are based on the logarithmic kernel function. Peng et al.([10] - [12]) proposed a new variant of IPMs based on self-regular kernel functions for linear optimization(LO) problems and extended to semidefinite optimization problems and second order cone optimization problems. They improved the complexity result for large-update methods up to $O(\sqrt{n}\log n \log \frac{n}{\epsilon})$ based on a specific self-regular kernel function. This is the best complexity result for such methods. Bai et al.([3], [4]) proposed new primal-dual interior point methods(IPMs) for LO problems based on eligible kernel functions and the scheme for analyzing the algorithm based on four conditions on the kernel function.([4]) They simplified the analysis and obtained the best known complexity result for a specific eligible kernel function.([4]) Cho([5]) and Cho et al.([6]) extended these algorithms to $P_*(\kappa)$ LCPs and obtained the similar complexity results. Recently, Amini et al.([2]) introduced a generalized version of the kernel function in [4] and extended the algorithm in [4] to $P_*(\kappa)$ LCPs based on this kernel function. They obtained $\mathcal{O}\big((1+2\kappa)\sqrt{n}\log n \log(\log n)\log \frac{n}{\epsilon}\big)$ for large-update methods.

Motivated by their works, we propose new primal-dual IPMs for $P_*(\kappa)$ LCPs based on a new class of kernel functions which contains the kernel function in [8] as a special case. We obtained $\mathcal{O}\big((1+2\kappa)n^{\frac{p+2}{2(p+1)}}\log \frac{n\mu^0}{\epsilon}\big)$ and $\mathcal{O}((1+2\kappa)\sqrt{n}\ \log \frac{n\mu^0}{\epsilon})$ iteration complexity for large-update and small-update IPMs, respectively. Taking $p=\frac{5}{2}$, $\tau=\mathcal{O}(n)$, and $\theta=\Theta(1)$, we get the $\mathcal{O}\big((1+2\kappa)n^{\frac{9}{14}}\log \frac{n\mu^0}{\epsilon}\big)$ iteration complexity for large-update method which improves the iteration complexity with a factor $n^{\frac{9}{14}}$ when compared with the method based the classical logarithmic kernel function. For small-update methods this is the best known complexity result so far.

The paper is organized as follows. In Section 2 we give a generic IPM and recall some basic concepts for LCP. In Section 3 we propose a new interior point algorithm based a new class of kernel functions and show its properties which are essential to the complexity analysis. In Section 4 we compute the iteration bound for the algorithm based on kernel function.

We use the following notations throughout the paper. $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x, s \in \mathbf{R}^n$, x_{min} and xs denote the smallest component of the vector x and the componentwise product of the vectors x and s, respectively. e denotes the n-dimensional vector of ones. For any $\mu > 0$, we define $v := \sqrt{xs/\mu}$, $v^{-1} := \sqrt{\mu e/(xs)}$ whose i-th components are $\sqrt{x_i s_i/\mu}$ and $\sqrt{\mu/(x_i s_i)}$, respectively. We denote X the diagonal matrix from a vector x, i.e., $X = \mathrm{diag}(x)$. I denotes the index set, e.g. $I = \{1, 2, \cdots, n\}$. For $f(x)$, $g(x) : \mathbf{R}_{++} \to \mathbf{R}_{++}$, $f(x) = \mathcal{O}(g(x))$ if $f(x) \le c_1 g(x)$ for some positive constant c_1 and $f(x) = \Theta(g(x))$ if $c_2 g(x) \le f(x) \le c_3 g(x)$ for some positive constants c_2 and c_3.

2. Preliminaries

In this section we introduce some basic concepts and a generic primal-dual interior point algorithm.

Definition 2.1. ([9]) *Let* $\kappa \geq 0$. $P_*(\kappa)$ *is the class of matrices M satisfying*

$$(1+4\kappa)\sum_{i\in I_+(\xi)}\xi_i[M\xi]_i+\sum_{i\in I_-(\xi)}\xi_i[M\xi]_i\geq 0,\ \xi\in\mathbf{R}^n,$$

where $[M\xi]_i$ *denotes the* i*-th component of the vector* $M\xi$ *and*

$$I_+(\xi)=\{i\in I\ :\ \xi_i[M\xi]_i\geq 0\},\ I_-(\xi)=\{i\in I\ :\ \xi_i[M\xi]_i<0\}.$$

Lemma 2.2. ([9]) *Let* $M\in\mathbf{R}^{n\times n}$ *be a* $P_*(\kappa)$ *matrix and* $x,s\in\mathbf{R}^n_{++}$. *Then for all* $c\in\mathbf{R}^n$ *the system*

$$\begin{cases}-M\Delta x+\Delta s=0,\\ S\Delta x+X\Delta s=c\end{cases}$$

has a unique solution $(\Delta x,\Delta s)$.

The basic idea of primal-dual IPMs is to replace the second equation in (1.1) by the parameterized equation $xs=\mu e$ with $\mu>0$ as follows:

$$\begin{cases}s=Mx+q,\\ xs=\mu e,\\ x>0,s>0.\end{cases}\tag{2.1}$$

Without loss of generality, we assume that (1.1) is strictly feasible, i.e., there exists a (x^0,s^0) such that $s^0=Mx^0+q,\ x^0>0,\ s^0>0$. For this, the reader refer to [9]. Since M is a $P_*(\kappa)$ matrix and (1.1) is strictly feasible, the system (2.1) has a unique solution for each $\mu>0$.([9]) We denote the solution $(x(\mu),s(\mu))$, $\mu>0$, which is called the μ-center. The set of μ-centers ($\mu>0$) is the central path of (1.1). IPMs follow the central path approximately and approach the solution of (1.1) as μ goes to zero.
For given $(x,s):=(x^0,s^0)$ by applying Newton method to the system (2.1) we have the following Newton system:

$$\begin{cases}-M\Delta x+\Delta s=0,\\ S\Delta x+X\Delta s=\mu e-xs.\end{cases}\tag{2.2}$$

By Lemma 2.2, the system (2.2) has a unique search direction $(\Delta x,\Delta s)$. By taking a step along the search direction $(\Delta x,\Delta s)$, one constructs a new iterate (x_+,s_+), where

$$x_+=x+\alpha\Delta x,\ s_+=s+\alpha\Delta s,$$

for some $\alpha\geq 0$. For the motivation of the new algorithm we define the scaled vectors:

$$v:=\sqrt{\frac{xs}{\mu}},\ d:=\sqrt{\frac{x}{s}},\ d_x:=\frac{v\Delta x}{x},\ d_s:=\frac{v\Delta s}{s}.\tag{2.3}$$

Using (2.3), we can rewrite the system (2.2) as follows:

$$\begin{cases}-\bar{M}d_x+d_s=0,\\ d_x+d_s=v^{-1}-v,\end{cases}\tag{2.4}$$

where $\bar{M} := DMD$ and $D := \mathrm{diag}(d)$. Note that the right side of the second equation in (2.4) equals the negative gradient of the logarithmic barrier function $\Psi_l(v)$, i.e., $d_x + d_s = -\nabla\Psi_l(v)$,

$$\Psi_l(v) := \sum_{i=1}^{n} \psi_l(v_i),\ \psi_l(t) = \frac{t^2 - 1}{2} - \log\ t,\ t > 0. \tag{2.5}$$

We call ψ_l the classical logarithmic kernel function of $\Psi_l(v)$.

The generic interior point algorithm works as follows. Assume that we are given a strictly feasible point (x, s) which is in a τ-neighborhood of the given μ-center. Then we decrease μ to $\mu_+ = (1-\theta)\mu$, for some fixed $\theta \in (0, 1)$ and solve the Newton system (2.2) to obtain the unique search direction. The positivity condition of a new iterate is ensured with the right choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, s_+) that is in a τ-neighborhood of the μ_+-center and then we let $\mu := \mu_+$ and $(x, s) := (x_+, s_+)$. Then μ is again reduced by the factor $1-\theta$ and we solve the Newton system targeting at the new μ_+-center, and so on. This process is repeated until μ is small enough, say until $n\mu < \varepsilon$.

Generic Primal-Dual Algorithm

Input:
  A threshold parameter $\tau \geq 1$;
  an accuracy parameter $\varepsilon > 0$;
  a fixed barrier update parameter $\theta,\ 0 < \theta < 1$;
  (x^0, s^0) and $\mu^0 > 0$ such that $\Psi_l(x^0, s^0, \mu^0) \leq \tau$.
begin
  $x := x^0; s := s^0; \mu := \mu^0;$
  while $n\mu > \epsilon$ do
  begin
    $\mu := (1-\theta)\mu;$
    while $\Psi_l(v) > \tau$ do
    begin
      Solve the system (2.2) for Δx and Δs;
      Determine a step size α;
      $x := x + \alpha\Delta x;$
      $s := s + \alpha\Delta s;$
      $v := \sqrt{\frac{xs}{\mu}};$
    end
  end
end

When $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, we call the algorithm a large-update method. Taking $\tau = \mathcal{O}(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we call the algorithm a small-update method.

3. The kernel function

In this section we define a new class of kernel functions and give its properties which are essential to our analysis. We call $\psi : \mathbf{R}_{++} \to \mathbf{R}_{+}$ a kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\begin{aligned} &\psi'(1) = \psi(1) = 0, \\ &\psi''(t) > 0,\ t > 0, \\ &\lim_{t\to 0^+} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty. \end{aligned} \tag{3.1}$$

Now we consider a new class of kernel function $\psi(t)$ as follows:

$$\psi(t) := 8t^2 - 11t + 1 + \frac{2}{t^p} - (5-2p)\ \log\ t,\ \frac{7}{15} \le p \le \frac{5}{2},\ t > 0. \tag{3.2}$$

For $\psi(t)$ we have the first three derivatives as follows:

$$\begin{aligned} \psi'(t) &= 16t - 11 - 2pt^{-p-1} - (5-2p)t^{-1}, \\ \psi''(t) &= 16 + 2p(p+1)t^{-p-2} + (5-2p)t^{-2}, \\ \psi'''(t) &= -2p(p+1)(p+2)t^{-p-3} - 2(5-2p)t^{-3}. \end{aligned} \tag{3.3}$$

From (3.1) and (3.3), $\psi(t)$ is clearly a kernel function and

$$\psi''(t) > 16,\ \frac{7}{15} \le p \le \frac{5}{2},\ t > 0. \tag{3.4}$$

In this paper, we replace the function $\Psi_l(v)$ in (2.5) with the function $\Psi(v)$ as follows:

$$d_x + d_s = -\nabla\Psi(v), \tag{3.5}$$

where $\Psi(v) = \sum_{i=1}^{n} \psi(v_i)$, $\psi(t)$ is defined in (3.2) and assume that $\tau \ge 1$. Hence the new search direction $(\Delta x, \Delta s)$ is obtained by solving the following modified Newton-system:

$$\begin{cases} -M\Delta x + \Delta s = 0, \\ S\Delta x + X\Delta s = -\mu v \nabla\Psi(v). \end{cases} \tag{3.6}$$

Since $-\mu v\nabla\Psi(v) = -\mu v\left(16v - 11 - 2pv^{-p-1} - (5-2p)v^{-1}\right)$, the second equation in (3.6) can be written as

$$S\Delta x + X\Delta s = -16xs + 11\sqrt{\mu xs} + 2p\sqrt{\mu^{2+p}(xs)^{-p}} + (5-2p)\mu.$$

Since $\Psi(v)$ is strictly convex and minimal at $v = e$, we have

$$\Psi(v) = 0 \ \Leftrightarrow\ v = e \Leftrightarrow x = x(\mu),\ s = s(\mu).$$

We use $\Psi(v)$ as the proximity function. Also, we define norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}||\nabla\Psi(v)|| = \frac{1}{2}||d_x + d_s||. \tag{3.7}$$

In the following we give technical properties of $\psi(t)$ which are essential to our analysis.

Lemma 3.1. *Let $\psi(t)$ be as defined in* (3.2). *Then we have the following properties:*
(i) *$\psi(t)$ is exponentially convex, for all $t > 0$,*
(ii) *$\psi''(t)$ is monotonically decreasing, for all $t > 0$,*
(iii) *$t\psi''(t) - \psi'(t) > 0$, for all $t > 0$,*
(iv) *$\psi^{''}(t)\psi^{'}(\beta t) - \beta\psi^{'}(t)\psi^{''}(\beta t) > 0$, for all $t > 1$ and $\beta > 1$.*

Proof: For (i): By Lemma 2.1.2 in [12], it suffices to show that $\psi(t)$ satisfies $t\psi''(t) + \psi'(t) \geq 0$ for all $t > 0$. Using (3.3), we have for $\frac{7}{15} \leq p \leq \frac{5}{2}$

$$t\psi''(t) + \psi'(t) = 32t - 11 + 2p^2 t^{-p-1}.$$

Let $g(p,t) = 32t - 11 + 2p^2 t^{-p-1}$. Then $g_t(p,t) = 32 - 2p^2(p+1)t^{-p-2}$ and $g_{tt}(p,t) = 2p^2(p+1)(p+2)t^{-p-3} > 0$, $t > 0$. Letting $g_t(p,t) = 0$, we have $t = (\frac{p^2(p+1)}{16})^{\frac{1}{p+2}}$. Since $g(p,t)$ is strictly convex in t, $g(p,t)$ has a global minimum at $t^* = (\frac{p^2(p+1)}{16})^{\frac{1}{p+2}}$, i.e., $g(p,t^*) \leq g(p,t)$, for $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t > 0$. For $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t := t^*$, we have

$$\begin{aligned} g_p(p,t^*) &= 2p(t^*)^{-p-1}(2 - p\log\ t^*) \\ &= 2p\left(\frac{p^2(p+1)}{16}\right)^{-\frac{p+1}{p+2}}\left(2 - \frac{p}{p+2}\log\Big(\frac{p^2(p+1)}{16}\Big)\right) > 0, \end{aligned}$$

since $\log(\frac{p^2(p+1)}{16}) < 1$ for $\frac{7}{15} \leq p \leq \frac{5}{2}$. This implies that $g(\frac{7}{15}, t^*) \leq g(p,t^*)$, for $\frac{7}{15} \leq p \leq \frac{5}{2}$. Hence $g(\frac{7}{15}, t^*)$ is the smallest value for $\frac{7}{15} \leq p \leq \frac{5}{2}$ and $t > 0$. Since $g(\frac{7}{15}, t^*) > 0.0111$, we have the result.
For (ii): From (3.3), it is clear.
For (iii): Using (3.3), we have

$$t\psi''(t) - \psi'(t) = 11 + 2p(p+2)t^{-p-1} + 2(5-2p)t^{-1} > 0.$$

For (iv): By Lemma 2.4 in [4], it suffices to show that $\psi(t)$ satisfies Lemma 3.1 (ii) and (iii). This completes the proof. □

Lemma 3.2. *For $\psi(t)$ we have*
(i) $8(t-1)^2 \ \leq\ \psi(t)\ \leq\ \frac{1}{32}(\psi'(t))^2$, $t > 0$,
(ii) $\psi(t)\ \leq\ \frac{21+2p^2}{2}(t-1)^2$, $t \geq 1$.

Proof: For (i): Using the first condition of (3.1) and (3.4), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \geq 16 \int_1^t \int_1^\xi d\zeta d\xi = 8(t-1)^2,$$

which proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned} \psi(t) &= \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \leq \frac{1}{16}\int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta) d\zeta d\xi \\ &= \frac{1}{16}\int_1^t \psi''(\xi)\psi'(\xi) d\xi = \frac{1}{16}\int_1^t \psi'(\xi) d\psi'(\xi) = \frac{1}{32}(\psi'(t))^2. \end{aligned}$$

For (ii): Using Taylor's theorem, $\psi(1) = \psi^{'}(1) = 0$, $\psi^{'''} < 0$, and $\psi^{''}(1) = 21 + 2p^2$, we have

$$\begin{aligned}\psi(t) &= \psi(1) + \psi^{'}(1)(t-1) + \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &= \frac{1}{2}\psi^{''}(1)(t-1)^2 + \frac{1}{3!}\psi^{'''}(\xi)(\xi-1)^3 \\ &< \frac{21+2p^2}{2}(t-1)^2,\end{aligned}$$

for some ξ, $1 \le \xi \le t$. This completes the proof. □

Remark 3.3. *Define $\psi_b(t) = -11t + 9 + \frac{2}{t^p} - (5-2p)\log t$. Then $\psi(t) := 8t^2 - 8 + \psi_b(t)$. $\psi_b'(t) = -11 - \frac{2p}{t^{p+1}} - \frac{5-2p}{t}$ and $\psi_b''(t) = \frac{2p(p+1)}{t^{p+2}} + \frac{5-2p}{t^2} > 0$. Hence, $\psi'(t)$ is monotonically increasing with respect to t.*

Lemma 3.4. *Let $\varrho : [0,\infty) \to [1,\infty)$ be the inverse function of $\psi(t)$, for $t \ge 1$, ρ and $\underline{\rho} : [0,\infty) \to (0,1]$, the inverse functions of $-\frac{1}{2}\psi'(t)$ and $-\psi_b{}'(t)$, for $0 < t \le 1$, respectively. Then we have*
(i) $\sqrt{\frac{r}{8}+1} \le \varrho(r) \le 1 + \sqrt{\frac{r}{8}},\ r \ge 0,$
(ii) $\rho(z) \ge \underline{\rho}(16+2z),\ z \ge 0,$
(iii) $\underline{\rho}(u) = (\frac{2p}{u-16+2p})^{\frac{1}{p+1}},\ u \ge 16.$

Proof: For (i): Letting $r = \psi(t)$ for $t \ge 1$, we have $\varrho(r) = t$. By the definition of $\psi(t)$, $r = 8t^2 - 11t + 1 + \frac{2}{t^p} - (5-2p)\ \log\ t$. This implies

$$8t^2 = r + 11t - 1 - \frac{2}{t^p} + (5-2p)\ \log\ t \ge r + 8.$$

Hence we have

$$t = \varrho(r) \ge \sqrt{\frac{r}{8}+1}.$$

Using Lemma 3.2 (i), we have $r = \psi(t) \ge 8(t-1)^2$. Then we have

$$t = \varrho(r) \le 1 + \sqrt{\frac{r}{8}}.$$

For (ii): Let $z = -\frac{1}{2}\psi'(t)$, for $0 < t \le 1$. By Remark 3.3, we have

$$-2z = \psi'(t) = 16t + \psi_b{}'(t).$$

This implies that

$$-\psi_b{}'(t) = 16t + 2z \le 16 + 2z.$$

Using Remark 3.3 and $t = \rho(z)$, we have

$$t = \rho(z) \ge \underline{\rho}(16+2z).$$

For (iii): Letting $u = -\psi_b{}'(t)$, for $0 < t \leq 1$, we have $\underline{\rho}(u) = t$. By the definition of $-\psi_b{}'(t)$, we have $u = 11 + \frac{2p}{t^{p+1}} + \frac{5-2p}{t} \geq 16$, $0 < t \leq 1$. This implies

$$\frac{2p}{t^{p+1}} = u - 11 - \frac{5-2p}{t} \leq u - 16 + 2p.$$

Hence we have

$$t = \underline{\rho}(u) \geq \left(\frac{2p}{u - 16 + 2p}\right)^{\frac{1}{p+1}}, \ u \geq 16.$$

□

Corollary 3.5. *Let ρ be as defined in Lemma 3.4. Then we have*

$$\rho(z) \geq \left(\frac{p}{z+p}\right)^{\frac{1}{p+1}}, \ z \geq 0.$$

Proof: Using Lemma 3.3 (ii) and (iii), we have

$$\rho(z) \geq \underline{\rho}(16 + 2z) = \left(\frac{p}{z+p}\right)^{\frac{1}{p+1}}.$$

This completes the proof. □

Lemma 3.6. (Theorem 3.2 in [3]) *Let ϱ be as defined in Lemma 3.4. Then we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right), \ v \in \mathbf{R}^n_{++}, \ \beta \geq 1.$$

In the following we obtain an estimate for the effect of a μ-update on the value of $\Psi(v)$.

Theorem 3.7. *Let $0 \leq \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Psi(v) \leq \tau$, then we have*

$$\Psi(v_+) \leq \frac{21 + 2p^2}{2(1-\theta)}\left(\sqrt{n}\theta + \sqrt{\frac{\tau}{8}}\right)^2.$$

Proof: Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(v)}{n}\right) \geq 1$, we have $\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Lemma 3.6 with $\beta = \frac{1}{\sqrt{1-\theta}}$, Lemma 3.2 (ii), Lemma 3.4 (i), and $\Psi(v) \leq \tau$, we have

$$\begin{aligned}
\Psi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right)\right) \leq \frac{(21+2p^2)n}{2}\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2 \\
&= \frac{(21+2p^2)n}{2}\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right) - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2
\end{aligned}$$

$$\leq \quad \frac{(21+2p^2)n}{2}\left(\frac{1+\sqrt{\frac{\tau}{8n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2$$

$$\leq \quad \frac{(21+2p^2)n}{2}\left(\frac{\theta+\sqrt{\frac{\tau}{8n}}}{\sqrt{1-\theta}}\right)^2 = \frac{21+2p^2}{2(1-\theta)}\left(\sqrt{n}\theta+\sqrt{\frac{\tau}{8}}\right)^2,$$

where the last inequality holds from $1-\sqrt{1-\theta}=\frac{\theta}{1+\sqrt{1-\theta}}\leq\theta$, $0\leq\theta<1$. This completes the proof. □

Denote

$$\tilde{\Psi}_0 := \frac{21+2p^2}{2(1-\theta)}\left(\sqrt{n}\theta+\sqrt{\frac{\tau}{8}}\right)^2. \tag{3.8}$$

Then $\tilde{\Psi}_0$ is an upper bound for $\Psi(v)$ during the process of the algorithm.

Remark 3.8. *For large-update method with $\tau=\mathcal{O}(n)$ and $\theta=\Theta(1)$, $\tilde{\Psi}_0=\mathcal{O}(n)$ and for small-update method with $\tau=\mathcal{O}(1)$ and $\theta=\Theta(\frac{1}{\sqrt{n}})$, $\tilde{\Psi}_0=\mathcal{O}(1)$.*

4. Complexity result

In this section we compute a feasible step size and the decrease of the proximity function during an inner iteration and give the complexity results of the algorithm. For fixed μ if we take a step size α, then we have new iterates $x_+=x+\alpha\Delta x,\ s_+=s+\alpha\Delta s$. Using (2.3), we have

$$x_+ = x\left(e+\alpha\frac{\Delta x}{x}\right) = x\left(e+\alpha\frac{d_x}{v}\right) = \frac{x}{v}(v+\alpha d_x)$$

and

$$s_+ = s\left(e+\alpha\frac{\Delta s}{s}\right) = s\left(e+\alpha\frac{d_s}{v}\right) = \frac{s}{v}(v+\alpha d_s).$$

Thus we have

$$v_+ = \sqrt{\frac{x_+s_+}{\mu}} = \sqrt{(v+\alpha d_x)(v+\alpha d_s)}.$$

Define for $\alpha>0$,

$$f(\alpha) = \Psi(v_+)-\Psi(v).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. Using Lemma 3.1 (i), we have

$$\Psi(v_+) = \Psi\big(\sqrt{(v+\alpha d_x)(v+\alpha d_s)}\big) \leq \frac{1}{2}(\Psi(v+\alpha d_x)+\Psi(v+\alpha d_s)).$$

Hence we have $f(\alpha)\leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v+\alpha d_x)+\Psi(v+\alpha d_s))-\Psi(v).$$

Obviously, we have

$$f(0)=f_1(0)=0.$$

By taking the derivative of $f_1(\alpha)$ with respect to α, we have

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}(\psi'(v_i + \alpha[d_x]_i)[d_x]_i + \psi'(v_i + \alpha[d_s]_i)[d_s]_i),$$

where $[d_x]_i$ and $[d_s]_i$ denote the i-th components of the vectors d_x and d_s, respectively. Using (3.5) and (3.7), we have

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2.$$

Differentiating $f_1'(\alpha)$ with respect to α, we have

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i + \alpha[d_x]_i)[d_x]_i^2 + \psi''(v_i + \alpha[d_s]_i)[d_s]_i^2\right).$$

Since $f_1''(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in α unless $d_x = d_s = 0$. Since **M** is a $P_*(\kappa)$ matrix and $M\Delta x = \Delta s$ from (3.6), for $\Delta x \in \mathbf{R}^n$,

$$(1 + 4\kappa)\sum_{i\in I_+}[\Delta x]_i[\Delta s]_i + \sum_{i\in I_-}[\Delta x]_i[\Delta s]_i \geq 0,$$

where $I_+ = \{i \in I \ : \ [\Delta x]_i[\Delta s]_i \geq 0\},\ I_- = I - I_+$. Since $d_x d_s = \frac{v^2\Delta x\Delta s}{xs} = \frac{\Delta x\Delta s}{\mu}$ and $\mu > 0$, we have

$$(1 + 4\kappa)\sum_{i\in I_+}[d_x]_i[d_s]_i + \sum_{i\in I_-}[d_x]_i[d_s]_i \geq 0.$$

Lemma 4.1. *Let $\delta(v)$ be as defined in* (3.7). *Then we have*

$$\delta(v) \geq 2\sqrt{2\Psi(v)}.$$

Proof: Using (3.7) and Lemma 3.2 (i), we have

$$\delta(v)^2 = \frac{1}{4}||\nabla\Psi(v)||^2 = \frac{1}{4}\sum_{i=1}^{n}(\psi'(v_i))^2 \geq \frac{1}{4}\sum_{i=1}^{n}32\psi(v_i) = 8\Psi(v).$$

Hence we have $\delta(v) \geq 2\sqrt{2\Psi(v)}$. □

Remark 4.2. *Using Lemma 4.1 and the assumption $\Psi(v) \geq \tau \geq 1$, we have*

$$\delta(v) \geq 2\sqrt{2\Psi(v)} \geq 2\sqrt{2}. \tag{4.1}$$

For notational convenience we denote $\delta := \delta(v)$ and $\Psi := \Psi(v)$.

Lemma 4.3. (Modification of lemma 4.4 in [6]) $f_1'(\alpha) \leq 0$ *if α is satisfying*

$$-\psi'(v_{min} - 2\alpha\delta\sqrt{1 + 2\kappa}) + \psi'(v_{min}) \leq \frac{2\delta}{\sqrt{1 + 2\kappa}}. \tag{4.2}$$

Lemma 4.4. (Modification of lemma 4.5 in [6]) *Let ρ be as defined in Lemma 3.4. Then the largest step size α satisfying* (4.2) *is given by*

$$\hat{\alpha} := \frac{1}{2\delta\sqrt{1+2\kappa}}\Big(\rho(\delta) - \rho\big(\,(1+\frac{1}{\sqrt{1+2\kappa}}\,)\delta\big)\Big).$$

Lemma 4.5. (Modification of lemma 4.6 in [6]) *Let ρ and $\hat{\alpha}$ be as defined in Lemma 4.4. Then*

$$\hat{\alpha} \geq \frac{1}{(1+2\kappa)\psi''\Big(\rho\big(\,(1+\frac{1}{\sqrt{1+2\kappa}}\,)\delta\big)\Big)}.$$

Define

$$\bar{\alpha} := \frac{1}{(1+2\kappa)\psi''\Big(\rho\big(\,(1+\frac{1}{\sqrt{1+2\kappa}}\,)\delta\big)\Big)}. \tag{4.3}$$

Then we have $\bar{\alpha} \leq \hat{\alpha}$.

Lemma 4.6. (Lemma 1.3.3 in [12]) *Suppose that $h(t)$ is twice differentiable convex function with*

$$h(0) = 0,\ h'(0) < 0,$$

$h(t)$ attains its (global) minimum at $t^* > 0$, and $h''(t)$ is increasing with respect to t. Then for any $t \in [0, t^*]$,

$$h(t) \leq \frac{th'(0)}{2}.$$

Lemma 4.7. (Modification of lemma 4.8 in [6]) *If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Theorem 4.8. *Let $\bar{\alpha}$ be as defined in* (4.3). *Then for $a = 1 + \frac{1}{\sqrt{1+2\kappa}}$ and $\kappa \geq 0$, we have*

$$f(\bar{\alpha}) \leq -\frac{2^{\frac{3p}{2(p+1)}}}{(1+2\kappa)L(p,a)}\Psi^{\frac{p}{2p+2}},$$

where $L(p,a) := 4\sqrt{2} + 2p(p+1)\Big(\frac{1}{2\sqrt{2}} + \frac{a}{p}\Big)^{\frac{p+2}{p+1}} + (5-2p)\Big(\frac{1}{2\sqrt{2}} + \frac{a}{p}\Big)^{\frac{2}{p+1}}$.

Proof: Using Corollary 3.5, we have

$$\rho(a\delta) \geq \left(\frac{p}{a\delta+p}\right)^{\frac{1}{p+1}}. \tag{4.4}$$

Using Lemma 4.7, (4.3), (4.4), and Lemma 3.1 (ii), we obtain

$$f(\bar{\alpha}) \leq -\bar{\alpha}\delta^2 = -\frac{\delta^2}{(1+2\kappa)\psi''\big(\rho(a\delta)\big)} \leq -\frac{\delta^2}{(1+2\kappa)\psi''\Big(\big(\frac{p}{a\delta+p}\big)^{\frac{1}{p+1}}\Big)}. \tag{4.5}$$

Using (3.3) and (4.1), we have

$$\begin{aligned}
&\psi''\Big(\big(\frac{p}{a\delta+p}\big)^{\frac{1}{p+1}}\Big)\\
= \;& 16+2p(p+1)\Big(1+\frac{a\delta}{p}\Big)^{\frac{p+2}{p+1}}+(5-2p)\Big(1+\frac{a\delta}{p}\Big)^{\frac{2}{p+1}}\\
\le \;& 4\sqrt{2}\delta+2p(p+1)\Big(\frac{\delta}{2\sqrt{2}}+\frac{a\delta}{p}\Big)^{\frac{p+2}{p+1}}+(5-2p)\Big(\frac{\delta}{2\sqrt{2}}+\frac{a\delta}{p}\Big)^{\frac{2}{p+1}}\\
= \;& 4\sqrt{2}\delta+2p(p+1)\Big(\frac{1}{2\sqrt{2}}+\frac{a}{p}\Big)^{\frac{p+2}{p+1}}\delta^{\frac{p+2}{p+1}}+(5-2p)\Big(\frac{1}{2\sqrt{2}}+\frac{a}{p}\Big)^{\frac{2}{p+1}}\delta^{\frac{2}{p+1}}\\
\le \;& L(p,a)\delta^{\frac{p+2}{p+1}}, \qquad (4.6)
\end{aligned}$$

where

$$L(p,a):=4\sqrt{2}+2p(p+1)\Big(\frac{1}{2\sqrt{2}}+\frac{a}{p}\Big)^{\frac{p+2}{p+1}}+(5-2p)\Big(\frac{1}{2\sqrt{2}}+\frac{a}{p}\Big)^{\frac{2}{p+1}}.$$

Using (4.5), (4.6), and Lemma 4.1, we have

$$\begin{aligned}
f(\bar{\alpha}) \;&\le\; -\frac{1}{(1+2\kappa)L(p,a)}\delta^{\frac{p}{p+1}}\le -\frac{1}{(1+2\kappa)L(p,a)}(2\sqrt{2\Psi})^{\frac{p}{p+1}}\\
&=\; -\frac{2^{\frac{3p}{2(p+1)}}}{(1+2\kappa)L(p,a)}\Psi^{\frac{p}{2p+2}}.
\end{aligned}$$

This completes the proof. □

Lemma 4.9. (Lemma 1.3.2 in [12]) *Let $t_0, t_1, \cdots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1}\le t_k-\gamma t_k^{1-\lambda},\; k=0,1,\cdots,K-1,$$

where $\gamma>0$ and $0<\lambda\le 1$. Then $K \le \left\lfloor \frac{t_0^\lambda}{\gamma\lambda} \right\rfloor$.

We define the value of $\Psi(v)$ after the μ-update as Ψ_0 and the subsequent values in the same outer iteration are denoted as Ψ_k, $k=1,2,\cdots$. Then we have

$$\Psi_0 \le \tilde{\Psi}_0, \qquad (4.7)$$

where $\tilde{\Psi}_0$ is defined in (3.8). Let K denote the total number of inner iterations per outer iteration. Then we have

$$\Psi_{K-1}>\tau,\; 0\le \Psi_K\le\tau.$$

Lemma 4.10. *Let $\tilde{\Psi}_0$ be as defined in (3.8) and K the total number of inner iterations in the outer iteration. Then we have for $\frac{7}{15}\le p\le\frac{5}{2}$ and $a=1+\frac{1}{\sqrt{1+2\kappa}}$*

$$K\le (1+2\kappa)\tilde{L}(p,a)\tilde{\Psi}_0^{\frac{p+2}{2(p+1)}},$$

where $\tilde{L}(p,a) := \frac{(p+1)2^{\frac{2-p}{2(p+1)}}}{p+2} L(p,a)$.

Proof: Using Theorem 4.8, Lemma 4.9 with $\gamma := \frac{2^{\frac{3p}{2(p+1)}}}{(1+2\kappa)L(p,a)}$ and $\lambda := \frac{p+2}{2(p+1)}$, and (4.7), we have

$$K \leq \frac{(1+2\kappa)L(p,a)}{2^{\frac{3p}{2(p+1)}}} \cdot \frac{2(p+1)}{p+2} \Psi_0^{\frac{p+2}{2(p+1)}} \leq (1+2\kappa)\tilde{L}(p,a)\tilde{\Psi}_0^{\frac{p+2}{2(p+1)}},$$

where

$$\tilde{L}(p,a) := \frac{(p+1)2^{\frac{2-p}{2(p+1)}}}{p+2} L(p,a). \tag{4.8}$$

This completes the proof. $\square$

Theorem 4.11. *Let a $P_*(\kappa)$ LCP be given. If $\tau \geq 1$, the total number of iterations to have an approximate solution with $n\mu \leq \epsilon$ is bounded by*

$$\left\lceil \frac{(1+2\kappa)\tilde{L}(p,a)}{\theta} \tilde{\Psi}_0^{\frac{p+2}{2(p+1)}} \log \frac{n\mu^0}{\epsilon} \right\rceil,$$

where $\tilde{\Psi}_0$ as defined in (3.8), $\tilde{L}(p,a)$ in (4.8), $\frac{7}{15} \leq p \leq \frac{5}{2}$, and $0 < \theta < 1$.

Proof: If the central path parameter μ has the initial value $\mu^0 > 0$ and is updated by multiplying $1-\theta$ with $0 < \theta < 1$, then after at most

$$\left\lceil \frac{1}{\theta} \log \frac{n\mu^0}{\epsilon} \right\rceil$$

iterations we have $n\mu \leq \epsilon$.([13]) For the total number of iterations, we multiply the number of inner iterations by that of outer iterations. Hence the total number of iterations is bounded by

$$\left\lceil \frac{(1+2\kappa)\tilde{L}(p,a)}{\theta} \tilde{\Psi}_0^{\frac{p+2}{2(p+1)}} \log \frac{n\mu^0}{\epsilon} \right\rceil.$$

This completes the proof. $\square$

Remark 4.12. *By Remark 3.8, for large-update methods by taking $\tau = \mathcal{O}(n)$, $\theta = \Theta(1)$, and $p = \frac{5}{2}$, the algorithm has $\mathcal{O}((1+2\kappa)n^{\frac{9}{14}} \log \frac{n\mu^0}{\epsilon})$ iteration complexity which improves the iteration complexity with a factor $n^{\frac{5}{14}}$ when compared with the method based on the classical logarithmic kernel function. For small-update methods by taking $\tau = \mathcal{O}(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have $\mathcal{O}((1+2\kappa)\sqrt{n} \log \frac{n\mu^0}{\epsilon})$ iteration complexity which is the best known complexity result for such methods.*

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