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A PRIORI ERROR ESTIMATES OF A DISCONTINUOUS GALERKIN METHOD FOR LINEAR SOBOLEV EQUATIONS*

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ABSTRACT. A discontinuous Galerkin method with interior penalty terms is presented for linear Sobolev equation. On appropriate finite element spaces, we apply a symmetric interior penalty Galerkin method to formulate semidiscrete approximate solutions. To deal with a damping term $\nabla \cdot (\nabla u_t)$ included in Sobolev equations, which is the distinct character compared to parabolic differential equations, we choose special test functions. A priori error estimate for the semidiscrete time scheme is analyzed and an optimal $L^\infty(L^2)$ error estimation is derived

1. INTRODUCTION

Discontinuous Galerkin methods using interior penalties have been used very widely for solving various types of differential equations, including computational fluid problems. By virture of the potential of error control and mesh adaptation and the local mass conservation, DG methods are preferred over the standard Galerkin method.

Since Baker [4] firstly introduced the interior penalty method with nonconforming elements for elliptic equations, discontinuous Galerkin methods with interior penalties for elliptic and parabolic equations have been developed by several authors [1, 5, 14]. They generalized the Nitsche method in [6] which treated the Dirichlet boundary condition by introducing the penalty terms on the boundary.

New applications of disconticontinuous Galerkin methods with interior penalties to nonlinear parabolic equations are considered in [9, 10, 11]. The authors in [9, 10, 11] developed

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elementwise conservative DG methods and derived a priori and a posteriori error estimates in higher dimensions.

The purpose of this paper is to consider the discontinuous Galerkin approximations of Sobolev differential equations with one time derivative appearing in the highest space derivative term. Sobolev equations are used to study the consolidation of clay, heat conduction, homogeneous fluid flow in fissured material, shear in second order fluids and other physical models.

In [12, 13], the authors constructed semidiscrete DG approximations and fully discrete DG approciamtions and obtained the optimal $L^\infty(H^1)$ error estimates for the nonlinear Sobolev equations. In this paper we construct semidiscrete DG approximations and analyze a priori optimal $L^\infty(L^2)$ error estimaties for the linear Sobolev equations. In section 2 we introduce a model problem and some assumptions. In section 3 several notations and preliminaries are described and discontiuous Galerkin seimidiscrete scheme is formulated. Finally an optimal a priori $L^\infty(L^2)$ error estimate is analyzed in section 4.

2. Model problems and assumptions

Consider the following linear Sobolev equation

$$u_t - \nabla \cdot (\nabla u + \nabla u_t) = f(x,u) \ \text{ in } \Omega \times (0,T], \tag{2.1}$$

with the boundary condition

$$(\nabla u + \nabla u_t) \cdot n = 0 \ \text{ on } \partial\Omega \times (0,T], \tag{2.2}$$

and the initial condition

$$u(x,0) = u_0(x) \ \text{ in } \Omega, \tag{2.3}$$

where Ω is a bounded convex domain in $\mathbb{R}^d$, $d = 2,3$ and n is the unit outward nomal vector to $\partial\Omega$.

We assume that the following conditions are satisfied.

1. f is uniformly Lipschitz continuous with respect to its second variable.
2. The model problem has a unique solution satisfying the following regularity conditions:

$$u \in L^\infty((0,T), H^s(\Omega)), \ \ u_t \in L^2((0,T), H^s(\Omega))$$

for $s \geq 2$.

3. NOTATIONS AND DISCONTINUOUS GALERKIN APPROXIMATIONS

Let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a regular quasi-uniform subdivision of Ω, where E_j is a triangle or a quadilateral if $d = 2$ and E_j is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = \text{diam}(E_j)$ be the diameter of E_j and $h = \max_{1\le j\le N_h} h_j$. The regularity means that there exists a constant $\rho > 0$ such that each E_j contains a ball of radius ρh_j. The quasiuniformity reguirement is that there is a constant $\gamma > 0$ such that

$$\frac{h}{h_j} \le \gamma, \ \ j = 1, \cdots, N_h$$

These quasi-uniformity and regularity assumptions are reguired for driving error estimates in terms of the degree of polynomials.

We denote the set of all edges of the elements by $\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{M_h}\}$ where $e_k \subset \Omega$, for $1 \le k \le P_h$, $e_k \subset \partial\Omega$ for $P_h + 1 \le k \le M_h$. n_k is unit outward nomal vector to E_i if $e_k = \partial E_i \cap \partial E_j$ for $i < j$ and $1 \le k \le P_h$ and $n_k = n$, $P_h + 1 \le k \le M_h$.

For an $s \ge 0$ and a domain $E \subset \mathbb{R}^d$, the usual norm of Sobolev space $H^s(E)$ is denoted by $\|\cdot\|_{s,E}$, and the usual seminorm is denoted by $|\cdot|_{s,E}$. If $E = \Omega$ we write $\|\cdot\|_s$, $|\cdot|_s$ instead of $\|\cdot\|_{s,\Omega}$, $|\cdot|_{s,\Omega}$ and if $s = 0$ we use $\|\cdot\|$ instead of $\|\cdot\|_0$.

For $s \ge 0$ and a given subdivision $\mathcal{E}_h$, we define the following space

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), j = 1, \cdots, N_h\}.$$

Now, for $\phi \in H^s(\mathcal{E}_h)$, $s > \frac{1}{2}$, we define the following average function $\{\phi\}$ and jump function $[\phi]$,

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_k} + \frac{1}{2}(\phi|_{E_j})|_{e_k}, \ \ \forall x \in e_k, \ \ 1 \le k \le P_h$$
$$[\phi] = (\phi|_{E_i})|_{e_k} - (\phi|_{E_j})|_{e_k}, \ \ \forall x \in e_k, \ \ 1 \le k \le P_h,$$

where $e_k = \partial E_i \cap \partial E_j$, $i < j$.

The usual L^2 inner product, for the functions ϕ, $\psi \in L^2(E)$, is denoted by $(\phi, \psi)_E$. If $E = \Omega$ we use (ϕ, ψ) instead of $(\cdot, \cdot)_\Omega$.

We define the following broken norms associated with $H^s(\mathcal{E}_h)$ for $s \ge 2$,

$$|||\phi|||_0^2 = \sum_{j=1}^{N_h} \|\phi\|_{0,E_j}^2$$

$$|||\phi|||_1^2 = \sum_{j=1}^{N_h} (\|\phi\|_{1,E_j}^2 + h_j^2|\phi|_{2,E_j}^2) + J^\sigma(\phi, \phi)$$

$$|||\phi|||_2^2 = \sum_{j=1}^{N_h} \|\phi\|_{2,E_j}^2$$

where $J^\sigma(\phi,\psi) = \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [\phi][\psi]ds$ is an interior penalty term and σ is a discrete positive function that takes the constant value σ_k on the edge e_k and is bounded below by $\sigma_0 > 0$ and above by σ^*.

Let r be a positive integer. The finite element space is taken by

$$D_r(\mathcal{E}_h) = \prod_{j=1}^{N_h} P_r(E_j)$$

where $P_r(E_j)$ denotes the set of all polynomials of total degree not greater than r on E_j. Throughout this paper the symbol C indicates a generic positive constant independent of h and is not necessarily the same in any two places. The following hp approximation properties are proved in [2, 3].

Lemma 3.1. *Let $E_j \in \mathcal{E}_h$, and $u \in H^s(E_j)$. There are a constant C independent of u r and h, and $\widehat{u} \in P_r(E_j)$ such that for any $0 \le q \le s$,*

$$\|u-\widehat{u}\|_{q,E_j} \le C\frac{h_j^{\mu-q}}{r^{s-q}}\|u\|_{s,E_j} \quad s \ge 0$$

$$\|u-\widehat{u}\|_{0,e_j} \le C\frac{h_j^{\mu-1/2}}{r^{s-1/2}}\|u\|_{s,E_j} \quad s > \frac{1}{2}$$

$$\|u-\widehat{u}\|_{1,e_j} \le C\frac{h_j^{\mu-3/2}}{r^{s-3/2}}\|u\|_{s,E_j} \quad s > \frac{3}{2}$$

where $\mu = \min(r+1,s)$ and e_j is an edge or a face of E_j.

The following Lemma states the trace inequalities whose proofs are given in [1].

Lemma 3.2. *For each $E_j \in \mathcal{E}_h$, there exists a positive constant C depending only on γ and ρ such that the following two trace inequalities hold:*

$$\|\phi\|_{0,e_j}^2 \le C\left(\frac{1}{h_j}|\phi|_{0,E_j}^2 + h_j|\phi|_{1,E_j}^2\right), \quad \forall\phi \in H^1(E_j),$$

$$\|\nabla\phi\cdot\eta_j\|_{0,e_j} \le C\left(\frac{1}{h_j}|\phi|_{1,E_j}^2 + h_j|\phi|_{2,E_j}^2\right), \quad \forall\phi \in H^2(E_j),$$

where e_j is an edge or a face of E_j and η_j is the unit outward normal vector to e_j.

We define a bilinear functional A on $H^2(\mathcal{E}_h) \times H^2(\mathcal{E}_h)$ by

$$A(\phi,\psi) = \sum_{k=1}^{N_h}(\nabla\phi,\nabla\psi)_{E_k} - \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\phi\cdot n_k\}[\psi]ds - \sum_{k=1}^{P_h}\int_{e_k}\{\nabla\psi\cdot n_k\}[\phi]ds + J^\sigma(\phi,\psi).$$

From (2.1), u satisfies the following weak formulation

$$(u_t, v) + A(u, v) + A(u_t, v) = (f(u), v), \ \ \forall v \in H^s(\mathcal{E}_h). \tag{3.1}$$

Now we formulate a semidiscrete DG approximation to (3.1) as follows: Find $U(\cdot, t) \in D_r(\mathcal{E}_h)$ satisfying

$$\begin{cases} (U_t, v) + A(U, v) + A(U_t, v) = (f(U), v), \ \ \forall D_r(\mathcal{E}_h), \\ U(\cdot, 0) = U_0 \end{cases} \tag{3.2}$$

where U_0 is an appropriate projection of the initial condition $u_0(x)$ onto $D_r(\mathcal{E}_h)$. For example, we can choose U_0 as $\widetilde{u}(x, 0)$ to be defined later.

Define $A_\lambda(\phi, \psi) = A(\phi, \psi) + \lambda(\phi, \psi)$, with $\lambda > 0$. Then we obtain the following lemmas which can be proved easily by using Lemma 3.2 and the definition of $\|\!| \cdot |\!\|_1$.

Lemma 3.3. *For $\lambda > 0$, there exists a constant C independent of h satisfying*

$$|A_\lambda(\phi, \psi)| \le C |\!|\!| \phi |\!|\!|_1 |\!|\!| \psi |\!|\!|_1, \ \ \forall \phi, \psi \in H^2(\mathcal{E}_h).$$

Lemma 3.4. *For a sufficiently lage σ and $\lambda > 0$, there exists a positive constant β such that*

$$A_\lambda(v, v) \ge \beta |\!|\!| v |\!|\!|_1^2, \ \ \forall v \in D_r(\mathcal{E}_h).$$

Proof. For an arbitrary small constant $\delta > 0$, we have, by Lemma 3.2

$$\begin{aligned} A_\lambda(v, v) = & \sum_{j=1}^{N_h} (\nabla v, \nabla v)_{E_j} - 2 \sum_{k=1}^{P_h} \int_{e_k} \{\nabla v \cdot n_k\}[v] + J^\sigma(v, v) + \lambda(v, v) \\ \ge & |\!|\!| \nabla v |\!|\!|_0^2 - \delta \sum_{k=1}^{P_h} |e_k| \|\{\nabla v\}\|_{0,e_k}^2 - \frac{\delta^{-1}}{\sigma_0} \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \|[v]\|_{0,e_k}^2 + J^\sigma(v, v) + \lambda \|v\|^2 \\ \ge & |\!|\!| \nabla v |\!|\!|_0^2 - C\delta \sum_{j=1}^{N_h} h_j (h_j^{-1} \|\nabla v\|_{0,E_j}^2 + h_j \|\nabla^2 v\|_{0,E_j}^2) + \left(1 - \frac{\delta^{-1}}{\sigma_0}\right) J^\sigma(v, v) \\ & + \lambda \|v\|^2 \\ \ge & |\!|\!| \nabla v |\!|\!|_0^2 - C\delta |\!|\!| \nabla v |\!|\!|_0^2 + \left(1 - \frac{\delta^{-1}}{\sigma_0}\right) J^\sigma(v, v) + \lambda \|v\|^2 \\ = & \left(\frac{1}{2} - C\delta\right) |\!|\!| \nabla v |\!|\!|_0^2 + \frac{1}{2} |\!|\!| \nabla v |\!|\!|_0^2 + \left(1 - \frac{c\delta^{-1}}{\sigma_0}\right) J^\sigma(v, v) + \lambda \|v\|^2 \\ \ge & \left(\frac{1}{2} - C\delta\right) |\!|\!| \nabla v |\!|\!|_0^2 + C \sum_{j=1}^{N_h} h_j^2 \|\nabla^2 v\|_{0,E_j}^2 + \left(1 - \frac{C\delta^{-1}}{\sigma_0}\right) J^\sigma(v, v) + \lambda \|v\| \\ \ge & \beta |\!|\!| v |\!|\!|_1^2. \end{aligned}$$

By Lemma 3.3 and Lemma 3.4, if $\lambda > 0$ there exists $\widetilde{u} \in D_r(\mathcal{E}_h)$ satisfying

$$A_\lambda(u - \widetilde{u}, \chi) = 0, \ \ \forall \chi \in D_r(\mathcal{E}_h).$$

Now we state the following Lemma which is essential for the proof of the optimal convergence of the semidiscrete approximation in the norm $L^\infty(L^2)$. The proof can be found in [8].

Lemma 3.5. *For $\lambda \geq 0$, we let $t \in [0, T]$ be fixed and suppose that there exists $\phi \in H^2(\mathcal{E}_h)$ satisfying*

$$A_\lambda(\phi, v) = F(v), \ \ \forall v \in D_r(\mathcal{E}_h),$$

where $F : H^2(\mathcal{E}_h) \to \mathbb{R}$ is a linear map. If there exist M_1, $M_2 > 0$ satisfying

$$|F(\psi)| \leq M_1 |||\psi|||_1, \ \ \psi \in H^2(\mathcal{E}_h)$$
$$|F(\psi)| \leq M_2 \|\psi\|_2, \ \ \psi \in H^2(\Omega) \cap H_0^1(\Omega),$$

then we have the following estimation

$$\|\phi\| \leq C(|||\phi|||_1 + M_1)h + M_2.$$

Proof. For $\phi \in L^2(\Omega)$, let $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of an elliptic problem

$$-\Delta\psi + \lambda\psi = \phi. \tag{3.3}$$

From the regularity property of the elliptic problem, then we have

$$\|\psi\|_2 \leq C\|\phi\|.$$

Let ψ_I be the interpolant of ψ such that $|||\psi - \psi_I|||_1 \leq Ch\|\psi\|_2$. Then from (3.3) and the assumptions we get the following inequalities

$$\begin{aligned}
\|\phi\|^2 &= (\phi, \phi) = (\phi, -\Delta\psi + \lambda\psi) = A_\lambda(\phi, \psi) \\
&= A_\lambda(\phi, \psi - \psi_I) + A_\lambda(\phi, \psi_I) \\
&\leq C|||\phi|||_1 |||\psi - \psi_I|||_1 + F(\psi_I) \\
&\leq C|||\phi|||_1 h\|\psi\|_2 + F(\psi) - F(\psi - \psi_I) \\
&\leq Ch|||\phi|||_1 \|\psi\|_2 + M_2\|\psi\|_2 + M_1|||\psi - \psi_I|||_1 \\
&\leq Ch|||\phi|||_1 \|\psi\|_2 + M_2\|\psi\|_2 + ChM_1\|\psi\|_2 \\
&\leq C(h|||\phi|||_1\|\phi\| + M_2\|\phi\| + M_1 h\|\phi\|).
\end{aligned}$$

Therefore we get

$$\|\phi\| \leq C[(|||\phi|||_1 + M_1)h + M_2].$$

4. Optimal $L^\infty(L^2)$ error estimate

Now, to prove the $L^\infty(L^2)$ optimal convergence of $u-U$, we denote

$$\eta = u - \widetilde{u}, \;\; \theta = \widetilde{u} - \widehat{u}, \;\; \xi = \widetilde{u} - U, \;\; e = u - U. \tag{4.1}$$

Theorem 4.1. *For $r,\, s \geq 2$, there exists a constant C independent of h satisfying the following statements:*

$$\begin{aligned}
&\text{(i)} && |\!|\!| u - \widetilde{u} |\!|\!|_1 \leq C \frac{h^{\mu-1}}{r^{s-2}} \|u\|_s,\\
&\text{(ii)} && \| u - \widetilde{u} \| \leq C \frac{h^{\mu}}{r^{s-2}} \|u\|_s,\\
&\text{(iiii)} && |\!|\!| u_t - \widetilde{u}_t |\!|\!|_1 \leq C \frac{h^{\mu-1}}{r^{s-2}} \|u_t\|_s,\\
&\text{(iv)} && \| u_t - \widetilde{u}_t \| \leq C \frac{h^{\mu}}{r^{s-2}} \|u_t\|_s.
\end{aligned}$$

Proof. From Lemma 3.3 and Lemma 3.4, we get

$$|\!|\!|\theta|\!|\!|_1^2 \leq C A_\lambda(\theta,\theta) = C A_\lambda(u-\widehat{u},\theta) \leq C |\!|\!| u-\widehat{u}|\!|\!|_1 |\!|\!|\theta|\!|\!|_1$$

from which we get

$$|\!|\!|\theta|\!|\!|_1 \leq C |\!|\!| u-\widehat{u}|\!|\!|_1. \tag{4.2}$$

By the definition of $|\!|\!| \cdot |\!|\!|_1$, Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
|\!|\!| u-\widehat{u}|\!|\!|_1^2 = &\sum_{j=1}^{N_h} \left(\|u-\widehat{u}\|_{1,E_j}^2 + h_j^2 |u-\widehat{u}|_{2,E_j}^2 \right) + J^\sigma(u-\widehat{u}, u-\widehat{u})\\
\leq &\sum_{j=1}^{N_h} C \left(\frac{h_j^{2(\mu-1)}}{r^{2(s-1)}} \|u\|_{s,E_j}^2 + h_j^2 \frac{h_j^{2(\mu-2)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 \right) + \sum_{k=1}^{P_h} \frac{\sigma_k}{|e_k|} \int_{e_k} [u-\widehat{u}]^2 ds\\
&\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + C \sum_{k=1}^{P_h} |e_k|^{-1} \|u-\widehat{u}\|_{0,e_k}^2\\
&\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^{-1} \left(h_j^{-1} \|u-\widehat{u}\|_{0,E_j}^2 + h_j \|\nabla(u-\widehat{u})\|_{0,E_j}^2 \right)\\
&\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^{-2} \left(\frac{h_j^{2\mu}}{r^{2s}} + h_j^2 \frac{h_j^{2(\mu-1)}}{r^{2(s-1)}} \right) \|u\|_{s,E_j}^2\\
&\leq C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u\|_{s,E_j}^2,
\end{aligned}$$

which implies

$$|\!|\!| u - \widehat{u} |\!|\!|_1 \le C \frac{h^{\mu-1}}{r^{s-2}} \|u\|_s. \tag{4.3}$$

From the triangle inequality, (4.2) and (4.3), we obtain

$$|\!|\!| u - \widetilde{u} |\!|\!|_1 \le |\!|\!| u - \widehat{u} |\!|\!|_1 + |\!|\!| \widehat{u} - \widetilde{u} |\!|\!|_1 \le C |\!|\!| u - \widehat{u} |\!|\!|_1 \le C \frac{h^{\mu-1}}{r^{s-2}} \|u\|_s,$$

which proves (i).

By applying the result of Lemma 3.5 with $M_1 = M_2 = 0$, we get the statement (ii) as follows

$$\|u - \widetilde{u}\| \le C |\!|\!| u - \widetilde{u} |\!|\!|_1 h \le C \frac{h^{\mu}}{r^{s-2}} \|u\|_s.$$

Differentiating $A_\lambda(\eta, v) = 0$ with respect to t, we get

$$\sum_{j=1}^{N_h} (\nabla \eta_t, \nabla v)_{E_j} - \sum_{k=1}^{P_h} \int_{e_k} \{\nabla \eta_t \cdot n_k\}[v] - \sum_{k=1}^{P_h} \int_{e_k} \{\nabla v \cdot n_k\}[\eta_t] + J^\sigma(\eta_t, v) + \lambda(\eta_t, v) = 0$$

which implies

$$A_\lambda(\eta_t, v) = 0.$$

By applying Lemma 3.5 with $M_1 = M_2 = 0$, we get

$$\|\eta_t\| \le ch |\!|\!| \eta_t |\!|\!|_1.$$

From the definition of η and θ, we seperate η_t into

$$|\!|\!| \eta_t |\!|\!|_1 \le |\!|\!| \theta_t |\!|\!|_1 + |\!|\!| u_t - \widehat{u}_t |\!|\!|_1.$$

The results of Lemma 3.3 and Lemma 3.4 imply that

$$\begin{aligned} |\!|\!| \theta_t |\!|\!|_1^2 &\le C A_\lambda(\theta_t, \theta_t) = C A_\lambda(u_t - \widehat{u}_t, \theta_t) - C A_\lambda(\eta_t, \theta_t) \\ &= C A_\lambda(u_t - \widehat{u}_t, \theta_t) \le C |\!|\!| u_t - \widehat{u}_t |\!|\!|_1 |\!|\!| \theta_t |\!|\!|_1. \end{aligned}$$

Therefore we get

$$\begin{aligned} |\!|\!| \theta_t |\!|\!|_1 &\le C |\!|\!| u_t - \widehat{u}_t |\!|\!|_1, \\ |\!|\!| \eta_t |\!|\!|_1 &\le C |\!|\!| u_t - \widehat{u}_t |\!|\!|_1. \end{aligned}$$

By applying Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned} |\!|\!| u_t - \widehat{u}_t |\!|\!|_1^2 &\le C \sum_{j=1}^{N_h} \left(\frac{h_j^{2(\mu-1)}}{r^{2(s-1)}} \|u_t\|_{s,E_j}^2 + h_j^2 \frac{h_j^{2(\mu-2)}}{r^{2(s-2)}} \|u_t\|_{s,E_j}^2 \right) + C \sum_{k=1}^{P_h} |e_k|^{-1} \|u_t - \widehat{u}_t\|_{0,e_k}^2 \\ &\le C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u_t\|_{s,E_j}^2 + C \sum_{j=1}^{N_h} h_j^{-2} \left(\|u_t - \widehat{u}_t\|_{0,E_j}^2 + h_j^2 \|\nabla(u_t - \widehat{u}_t)\|_{0,E_j}^2 \right) \\ &\le C \sum_{j=1}^{N_h} \frac{h_j^{2(\mu-1)}}{r^{2(s-2)}} \|u_t\|_{s,E_j}^2, \end{aligned}$$

which implies

$$\|\!|u_t - \widehat{u}_t|\!\|_1 \le C\frac{h^{\mu-1}}{r^{s-2}}\|u_t\|_s,$$
$$\|\!|\eta_t|\!\|_1 \le C\frac{h^{\mu-1}}{r^{s-2}}\|u_t\|_s,$$

and

$$\|\eta_t\| \le C\frac{h^{\mu}}{r^{s-2}}\|u_t\|_s.$$

Theorem 4.2. *If $\lambda > 0$ is sufficiently small, then there exists a constant C independent of h satisfying the followings:*

$$\text{(i)} \qquad \|u - U\|_{L^\infty(L^2)} \le C\frac{h^{\mu}}{r^{s-2}}\left(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^2(H^s)}\right)$$
$$\text{(ii)} \qquad \|\!|u - U|\!\|_{L^2(\|\!|\cdot|\!\|_1)} \le C\frac{h^{\mu-1}}{r^{s-2}}\left(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}\right)$$
$$\text{(iii)} \qquad \|u_t - U_t\|_{L^2(L^2)} \le C\frac{h^{\mu}}{r^{s-2}}\left(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}\right)$$
$$\text{(iv)} \qquad \|\!|u_t - U_t|\!\|_{L^2(\|\!|\cdot|\!\|_1)} \le C\frac{h^{\mu-1}}{r^{s-2}}\left(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}\right)$$

Proof. From the notation (4.1), we have $e = \eta + \xi$. By subtracting (3.2) from (3.1), we have

$$(e_t, v) + A(e, v) + A(e_t, v) = (f(u) - f(U), v), \;\; \forall v \in D_r(\mathcal{E}_h).$$

By applying the definition of A_λ, we get

$$(e_t, v) + A_\lambda(e, v) + A_\lambda(e_t, v) = (f(u) - f(U), v) + \lambda(e, v) + \lambda(e_t, v), \;\; \forall v \in D_r(\mathcal{E}_h).$$

From the equation above, we can deduce

$$\begin{aligned} & (\xi_t, v) + A_\lambda(\xi, v) + A_\lambda(\xi_t, v) \\ = \; & -(\eta_t, v) - A_\lambda(\eta, v) - A_\lambda(\eta_t, v) + (f(u) - f(U), v) + \lambda(u - U, v) \\ & + \lambda(u_t - U_t, v) \\ = \; & -(\eta_t, v) + (f(u) - f(U), v) + \lambda(u - U), v) + \lambda(u_t - U_t, v) \\ = \; & -(\eta_t, v) + (f(u) - f(U), v) + \lambda(\xi, v) + \lambda(\eta, v) + \lambda(\xi_t, v) + \lambda(\eta_t, v). \end{aligned} \tag{4.4}$$

Now we choose $v = \xi + \xi_t$ in (4.4), to get

$$\begin{aligned} & \|\xi_t\|^2 + \frac{1}{2}\frac{d}{dt}(\xi, \xi) + A_\lambda(\xi, \xi) + 2A_\lambda(\xi, \xi_t) + A_\lambda(\xi_t, \xi_t) \\ = \; & -(\eta_t, \xi) - (\eta_t, \xi_t) + (f(u) - f(U), \xi + \xi_t) + \lambda(\xi, \xi + \xi_t) + \lambda(\eta, \xi + \xi_t) \\ & + \lambda(\xi_t, \xi + \xi_t) + \lambda(\eta_t, \xi + \xi_t), \end{aligned}$$

which yields the following inequality

$$\begin{aligned}&\|\xi_t\|^2+\frac{1}{2}\frac{d}{dt}(\xi,\xi)+A_\lambda(\xi,\xi)+A_\lambda(\xi_t,\xi_t)+2\Big[\sum_{k=1}^{N_h}(\nabla\xi,\nabla\xi_t)_{E_k}\\&-\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\xi\cdot n_k\}[\xi_t]-\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\xi_t\cdot n_k\}[\xi]+J^\sigma(\xi,\xi_t)+\lambda(\xi,\xi_t)\Big]\\&\leq(1+\lambda)\|\eta_t\|\|\xi\|+(1+\lambda)\|\eta_t\|\|\xi_t\|+K(\|\eta\|+\|\xi\|)(\|\xi\|+\|\xi_t\|)+\lambda\|\xi\|^2\\&+2\lambda(\xi,\xi_t)+\lambda\|\eta\|\|\xi\|+\lambda\|\xi_t\|^2+\lambda\|\eta\|\|\xi_t\|.\end{aligned}$$

If λ is sufficiently small, we obtain the following inequality

$$\begin{aligned}&\|\xi_t\|^2+A_\lambda(\xi,\xi)+A_\lambda(\xi_t,\xi_t)+\frac{1}{2}\frac{d}{dt}\left[((1+\lambda)\xi,\xi)+2\sum_{k=1}^{N_h}(\nabla\xi,\nabla\xi)_{E_k}+2J^\sigma(\xi,\xi)\right]\\&\leq C\left(\|\eta_t\|^2+\|\eta\|^2+\|\xi\|^2\right)+\varepsilon\|\xi_t\|^2+2\left(\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\xi\cdot n_k\}[\xi_t]+\sum_{k=1}^{P_h}\int_{e_k}\{\nabla\xi_t\cdot n_k\}[\xi]\right).\end{aligned}$$

By the definition of J and Lemma 3.2, we can find that

$$\begin{aligned}&\|\xi_t\|^2+A_\lambda(\xi,\xi)+A_\lambda(\xi_t,\xi_t)+\frac{1}{2}\frac{d}{dt}\left[(\xi,\xi)+2\sum_{k=1}^{N_h}(\nabla\xi,\nabla\xi)_{E_k}+J^\sigma(\xi,\xi)\right]\\&\leq C\left(\|\eta_t\|^2+\|\eta\|^2+\|\xi\|^2\right)+C\|\nabla\xi\|_0^2+\varepsilon J^\sigma(\xi_t,\xi_t)+CJ^\sigma(\xi,\xi)+\varepsilon\|\nabla\xi_t\|_0^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\|\xi_t\|^2+\|\xi\|_1^2+\|\xi_t\|_1^2+\frac{1}{2}\frac{d}{dt}\left[\|\xi\|^2+\|\nabla\xi\|_0^2+J^\sigma(\xi,\xi)\right]\\&\leq C\left[\|\eta_t\|^2+\|\eta\|^2+\|\xi\|^2+\|\nabla\xi\|_0^2+J^\sigma(\xi,\xi)\right].\end{aligned}\tag{4.5}$$

By integrating (4.5) from $t=0$ to $t=\tau$, we have

$$\begin{aligned}&\|\xi\|^2(\tau)+\|\nabla\xi\|_0^2(\tau)+J^\sigma(\xi,\xi)(\tau)+\int_0^\tau[\|\xi_t\|^2+\|\xi\|_1^2+\|\xi_t\|_1^2]dt\\&\leq\|\xi\|^2(0)+\|\nabla\xi\|^2(0)+J^\sigma(\xi,\xi)(0)+C\int_0^\tau[\|\xi\|^2+\|\nabla\xi\|_0^2+J^\sigma(\xi,\xi)]dt\\&+C\int_0^t(\|\eta_t\|^2+\|\eta\|^2)dt.\end{aligned}$$

Gronwall's Lemma and the approximation results from Theorem 4.1 imply that

$$
\begin{aligned}
&\|\xi\|^2(\tau) + \|\!|\nabla\xi|\!\|_0^2(\tau) + J^\sigma(\xi,\xi)(\tau) + \int_0^\tau [\|\xi_t\|^2 + \|\!|\xi|\!\|_1^2 + \|\!|\xi_t|\!\|_1^2]dt \\
&\le C\int_0^\tau (\|\eta_t\|^2 + \|\eta\|^2)dt \qquad (4.6)\\
&\le C\frac{h^{2\mu}}{r^{2(s-2)}}(\|u\|^2_{L^2(H^s)} + \|u_t\|^2_{L^2(H^s)}).
\end{aligned}
$$

From (4.6), we get the following approximations

$$
\begin{aligned}
\|\xi\|_{L^\infty(L^2)} &\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)})\\
\|\xi\|_{L^2(\|\!|\cdot|\!\|_1)} &\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}).
\end{aligned}
$$

Using the inequality (4.6) again, we have

$$
\begin{aligned}
\|\xi_t\|_{L^2(L^2)} &\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)})\\
\|\xi_t\|_{L^2(\|\!|\cdot|\!\|_1)} &\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}).
\end{aligned}
$$

Therefore by the triangle inequality and Theorem 4.1, we obtain the statements (i) and (ii) as follows:

$$
\begin{aligned}
\|e\|_{L^\infty(L^2)} &\le \|\eta\|_{L^\infty(L^2)} + \|\xi\|_{L^\infty(L^2)}\\
&\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^\infty(H^s)} + \|u_t\|_{L^2(H^s)})
\end{aligned}
$$

and

$$
\begin{aligned}
\|e\|_{L^2(\|\!|\cdot|\!\|_1)} &\le \|\eta\|_{L^2(\|\!|\cdot|\!\|_1)} + \|\xi\|_{L^2(\|\!|\cdot|\!\|_1)}\\
&\le C\frac{h^{\mu-1}}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}).
\end{aligned}
$$

Again applying the triangle inequality and Theorem 4.1, we prove the statements (iii) and (iv) as follows

$$
\begin{aligned}
\|e_t\|_{L^2(L^2)} &\le \|\eta_t\|_{L^2(L^2)} + \|\xi_t\|_{L^2(L^2)}\\
&\le C\frac{h^\mu}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)})
\end{aligned}
$$

and

$$
\begin{aligned}
\|e_t\|_{L^2(\|\!|\cdot|\!\|_1)} &\le \|\eta_t\|_{L^2(\|\!|\cdot|\!\|_1)} + \|\xi_t\|_{L^2(\|\!|\cdot|\!\|_1)}\\
&\le C\frac{h^{\mu-1}}{r^{s-2}}(\|u\|_{L^2(H^s)} + \|u_t\|_{L^2(H^s)}).
\end{aligned}
$$

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