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HEDGING OF OPTION IN JUMP-TYPE SEMIMARTINGALE ASSET MODEL

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ABSTRACT. Hedging strategy for European option of jump-type semimartingale asset model, which is derived from stochastic differential equation whose driving process is a jump-type semimartingle, is discussed.

1. INTRODUCTION

In this paper, we derive a jump-type asset model derived from stochastic differential equation(SDE) whose driving process is a jump-type semimartingle, and discuss hedging problem for European option by this asset model. We will get a closed form of the hedging strategy for our risky asset model.

We assume our asset model is a jump-type $\mathbf{H}^2$-semimartingale which is decomposable. This market is an incomplete market because in this market there is no perfect hedging of option in general. We need the notion of equivalent and minimal martingale measures. Further, the notion of risk-minimization can not be used in general because of semimartingale in this asset model. Thus, we need the notion of stopping time, and the theory of small perturbation to the strategy and of locally risk-minimization to get optional strategy.

On the other hand, we define and use an adjusting function. Thus our model is more general than another semimartingale asset models because we do not restrict our model as a jump-diffusion process. But, to think option price and to get hedging strategy of option, we restrict our semimartingale model to a Markov process because we must premise the existence of density function. Our model without Markov property is an open problem, and also our adjusting function should be studied more in future.

In general, we know the distributions (the density function) of the returns of asset models have asymmetric leptokurtic figures by empirical results. Further, we know such a return model is a hyperbolic Leŷy motion(pure jump-type). In [1], we meet an asset model:

$$dS_t = S_{t-}[dZ_t + (e^{\Delta Z_t} - 1 - \Delta Z_t)], \tag{1.1}$$

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and whose driving process is jump-type semimartingale(Lévy process without Gaussian part):

$$Z_t = bt + \int_0^t \int_{|z|\le 1} z\tilde{N}_p(dz, ds) + \int_0^t \int_{|z|>1} zN_p(dz, ds),$$

where $\tilde{N}_p(dz, dt) = N_p(dz, dt) - \nu(dz)dt$, b is the drift of Lévy process, ν is a Lévy measure on R satisfying $\nu(\{0\}) = 0$, $\nu(\{z; |z| \ge 1\}) < \infty$ and $\int_{|z|<1} |z|^2\nu(dz) < \infty$. As pointed out in articles [1] and [2], this asset model is very realistic in view of some empirical sense if we look at the microstructure of stock price movements. In article [1], we also meet the density function of this jump-type return model, and compare with the density function of hyperbolic Leýy motion. Further in [3], we meet many tables and figures which show the reasonability of asset model derived from the SDE (1.1).

In this paper, we introduce more general asset model which is defined by the solution of SDE;

$$dS_t = v(S_{t-})[dZ_t + (e^{\Delta Z_t} - 1 - \Delta Z_t)], \tag{1.2}$$

whose driving process is a jump-type semimartingale(Lévy process without Gaussian part):

$$Z_t = b^\delta t + \int_0^t \int_{|z|<\delta} z\tilde{N}_p(dz, ds) + \int_0^t \int_{|z|\ge\delta} zN_p(dz, ds), \tag{1.3}$$

where $v(\cdot)$ is a adjusting function, b^δ is the drift of Lévy process, ν is a Lévy measure on R satisfying $\nu(\{0\}) = 0$, $\nu(\{z; |z| \ge \delta\}) < \infty$ and $\int_{|z|<\delta} |z|^2\nu(dz) < \infty$.

In many real stock markets, the amplitude is limited in $[-0.15, 0.15]$ in a day. Thus, as our asset model, we choice an jump-type semimartingale

$$\begin{aligned} S_t &= S_0 + \int_0^t v(S_{s-}) \int_{|z|<\delta} (e^z - 1)\tilde{N}_p(dz, ds) \\ &\quad + \int_0^t b^\delta v(S_{s-})ds + \int_0^t v(S_{s-}) \int_{|z|<\delta} (e^z - 1 - z)\nu(dz)ds, \end{aligned} \tag{1.4}$$

which is derived from the solution of SDE (2) having deriving process (3), and is vanished big jump part; $\int_0^t v(S_{s-}) \int_{|z|\ge\delta}(e^z - 1)N_p(dz, ds)$. We will not think the interest rate except the basic theory section for simplicity. Thus it is more meaningful if we think for short period time.

We define the value $u^{Q^*}(t, S_t)$ of European option $H(S_T)$ at time t with maturity T:

$$u^{Q^*}(t, S_t) := E^{Q^*}[H(S_T)|F_t], \tag{1.5}$$

where E^{Q^*} is an expectation with respect to the minimum equivalent martingale measure Q^*. Then we get the hedging of this option. A trading strategy π_t of risky asset S_t is defined by F_t-predictable process taking values in R with

$$\int_0^T \|\pi_t\|^2 dt < \infty \quad a.s.,$$

and representing the amounts of hold over time. We will constrain the process $\pi_t; 0 \le t \le T$ is left-continuous to take the jumps into account.

A value process $V_t(\pi); 0 \leq t \leq T$ of portfolio π_t with initial capital V_0, is defined by

$$V_t(\pi) := \pi_t S_t.$$

Suppose our option H is in $L^2(P)$. Then local risk-minimization strategy π_t is given by $\pi_t = D(t, S_t)$, where

$$D(t,x) = \frac{\int_U (e^z v(x) - v(x))[u(s, e^z v(x)) - u(t,x)]\nu(dz)}{\int_U (e^z v(x) - v(x))^2 \nu(dz)}, \tag{1.6}$$

where $U = \{z||z| < \delta\}$.

Section 2 is a preliminary section. In this section, we introduce some basic notations and terminologies. In section 3, we define our asset model by the solution of SDE. In section 4, we study the hedging problems. In this section, we use a minimal martingale measure Q^*, which is a equivalent measure to given probability P and makes asset price process S_t a martingale. We get a closed form of the local risk-minimization strategy.

2. BASIC THEORY

2.1. **Basic Notions.** Let (Ω, F, P) be a probability space carrying a filtration $F_t; 0 \leq t \leq T$ of a right continuous increasing family of sub σ-fields of given F. Let $X_t; 0 \leq t \leq T$ be a semimartingale defined on (Ω, F, P) with a decomposition

$$X_t = X_0 + M_t + A_t, \tag{2.1}$$

such that $M_t; 0 \leq t \leq T$ is a square-integrable local martingale with $M_0 = 0$ and $A_t; 0 \leq t \leq T$ is a predictable process of finite variation with $A_0 = 0$.

Let $X_t; 0 \leq t \leq T$ denote the price of a given stock at t. A *trading strategy* $\pi_t; 0 \leq t \leq T$ of risky asset X_t is defined by an F_t-predictable process taking values in R with

$$\int_0^t \pi_s dX_s; 0 \leq t \leq T \tag{2.2}$$

is a square-integrable semimartingale. A *value process* $V_t(\Pi); 0 \leq t \leq T$ of the portfolio $\Pi_t = (\pi_t, \pi_t^0)$ is right continuous satisfying $V_t(\Pi) \in L^2(P)$, and is defined by

$$V_t(\Pi) := \pi_t X_t + \pi_t^0, \tag{2.3}$$

where $\pi_t^0; 0 \leq t \leq T$ is a adapted process which denote the amounts of bound $Y := 1$. The square-integrable of (8) is equivalent to

$$E\left[\int_0^T \pi_s^2 d < M >_s + \left(\int_0^T |\pi_s| d|A|_s\right)^2\right] < \infty, \tag{2.4}$$

which means that

$$\pi_t \in L^2(P_M) \quad and \quad \int_0^T |\pi_s| d|A|_s \in L^2(P).$$

The right-continuous square-integrable process $C_t(\Pi)$ defined by

$$C_t(\Pi) = V_t(\Pi) - \int_0^t \pi_s dX_s, \quad 0 \le t \le T$$

is called the *accumulated cost process* of Π_t. A *contingent claim* H at time T is given by a non-negative F_T-measurable random variable in $L^2(P)$, i.e., $H \in L^2(\Omega, F, P)$, and denote as $H := H(X_T)$.

We shall concentrate on strategies which are *H-admissible* in the sense that

$$V_T(\Pi) = H, \quad P - a.s.;$$

Π is then said to generate H. This means that our option is replicable. As a measure of riskiness, we introduce the it remaining risk process $R_t(\Pi)$ of Π_t at a fixed time t defined by

$$R_t(\Pi) := E[(C_T(\Pi) - C_t(\Pi))^2 | F_t], \quad 0 \le t \le T, \tag{2.5}$$

which is a right-continuous version. A strategy Π_t is called *self-financing* if the cost process $C_t(\Pi)$ is constant(i.e., time-invariant), and called *mean-self-financing* if the $C_t(\Pi)$ is a martingale in P-a.s., i.e.,

$$E[C_T(\Pi) - C_t(\Pi) | F_t] = 0, \quad 0 \le t \le T. \tag{2.6}$$

Further, a strategy Π_t is self-financing if

$$dV_t(\Pi) = \pi_t dX_t + \pi_t^0 dt, \tag{2.7}$$

more precisely,

$$V_t(\Pi) = V_0 + \int_0^t \pi_s dX_s.$$

Let us, first, think a complete market. A contingent claim H is called *attainable* if it is of the Itô representation form

$$H = H_0 + \int_0^T \pi_s dX_s, \quad P - a.s.$$

with a constant H_0 and a predictable process π_t satisfying (10). Suppose that contingent claim H admits above Itô representation, then there is a strategy $\Pi_t = (\pi_t, \pi_t^0)$ defined by

$$\pi_t := \pi_t^H, \quad \pi_t^0 := V_t - \pi_t X_t, \quad V_t := H_0 + \int_0^t \pi_s^H dX_s, \quad 0 \le t \le T$$

Let us exclude arbitrage opportunities. We assume the existence and the uniqueness of an equivalent martingale measure Q. More precisely, we assume that Q is a probability measure on (Ω, F) such that

$$dQ/dP \in L^2(\Omega, F.P)$$

and X_t is a martingale under Q even if X_t is a semimartingale under P. Then the strategy in (13) can be identified as follow; $V_t = E^Q[H|F_t]$ and π_t^H is obtained as the Radon-Nikodym derivative

$$\pi_t^H = d < V, X >_t^Q / d < X >_t^Q,$$

where $< V, X >$ is the covariance process associated to V and X. Thus, the strategy can be identified in terms of Q.

2.2. **Risk-Minimization.** Let us think an incomplete market. If $P = Q$, where X is a (jump-type) martingale under the initial measure P, we can look for an admissible strategy which minimizes (11) for each time t. In this case, H is attainable if and only if this remaining risk process (11) can be reduce to 0, since this is equivalent to

$$C_t = C_T = H_0, \quad 0 \leq t \leq T,$$

i.e., a strategy Π_t is self-financing: $C_t(\pi)$ is constant. Let the market be incomplete and self-financing with $P = Q$. We impose the condition:

$$\int_0^T |\pi_s^0| ds + \int_0^T |\pi_s|^2 ds < \infty, \quad a.s.,$$

and a stronger condition of integrability on the processes $\pi_t; 0 \leq t \leq T$ as following: an admissible strategy is defined by an adapted, left-continuous process $\pi_t; 0 \leq t \leq T$ with values in R^2 satisfying (13) $a.s.$ for all $t \in [0, T]$,

$$E[\int_0^T (\pi_s)^2 (S_s)^2 ds] < \infty. \tag{2.8}$$

Then we get;

Proposition 2.1. *Let $\pi_t; 0 \leq t \leq T$ be an adapted, left continuous process satisfying (17). Let $V_0 \in R_+$. Then there exists a unique process $\pi_t^0; 0 \leq t \leq T$ such that $\Pi_t; 0 \leq t \leq T$ defines an optional strategy with initial values V_0. Further, the value process $V_t; 0 \leq t \leq T$ of strategy π_t is given by*

$$V_t = V_0 + \int_0^t \pi_s dS_s.$$

But for a general contingent claim $H \in L^2(\Omega, F, P)$, the cost process C_t associated to a risk-minimizing strategy is no longer be self-financing. It will be mean-self-financing in the sense (12). In other words, the cost process $C_t(\Pi)$ associated to a risk-minimizing strategy is a martingale. In this case, we can show the existence of a unique risk-minimizing strategy as following. Let $\Pi_t; 0 \leq t \leq T$ be a trading strategy. An *admissible continuation* of Π_t from t on is a trading strategy $\tilde{\Pi}_t$ satisfying

$$\begin{aligned} \tilde{\pi}_s &= \pi_s \quad for \quad s \leq t, \\ \tilde{\pi}_s^0 &= \pi_s^0 \quad for \quad s < t, \\ V_T(\tilde{\Pi}) &= V_T(\Pi), \quad P - a.s. \end{aligned}$$

An *admissible variation* of Π_t from t on is a trading strategy $\Delta_t = (\delta_t, \delta_t^0)$ such that $\Pi_t + \Delta_t$ is an admissible continuation of Π_t from t on. A trading strategy Π_t is called *risk-minimizing* if, for any $t \in [0, T)$ and for any admissible continuation $\tilde{\Pi}_t$ of Π_t from t on, we have

$$R_t(\tilde{\Pi}) \geq R_t(\Pi), \quad P - a.s.$$

or equivalently if

$$R_t(\Pi + \Delta) \geq R_t(\Pi), \quad P - a.s.$$

for every admissible variation Δ_t of Π from t on.

Let $\Pi_t = (\pi_t, \pi_t^0)$ be a trading strategy and $t \in [0, T]$. Then there exists a trading strategy $\hat{\Pi}$ satisfying

(i) $V_T(\hat{\Pi}) = V_T(\Pi)$, $P - a.s.$

(ii) $C_s(\hat{\Pi}) = E[C_T(\Pi)|F_s]$, $P - a.s.$ for $t \leq s \leq T$.

(iii) $R_s(\hat{\Pi}) \leq R_s(\Pi)$, $P - a.s.$ for $t \leq s \leq T$.

(iv) If we choose $t := 0$, then $\hat{\Pi}$ is mean-self-financing.

In the martingale case, the risk-minimization of Π is completely solved by using the Kunita-Watanabe decomposition: Consider the Kunita-Watanabe decomposition

$$H = H_0 + \int_0^T \pi_s^H dX_s + L_T^H \tag{2.9}$$

with $H_0 \in L^2(\Omega, F_0, P)$, where $(L_T^H), 0 \leq t \leq T$ is a square-integrable martingale orthogonal to X. Then the risk-minimizing strategy is now given by

$$\pi_t = \pi_t^H, \quad \pi_t^0 = V_t - \pi_t X_t, \quad V_t := H_0 + \int_0^T \pi_s^H dX_s, \quad 0 \leq t \leq T. \tag{2.10}$$

In the present martingale case, the precess V_t can also be computed directly as a right-continuous version of the martingale

$$V_t = E[H|F_t], 0 \leq t \leq T.$$

Further, π_t^H is given by $\pi_t^H = d < V, X >_t / d < X >_t$.

2.3. **Local Risk-Minimization.** Let us now consider the general incomplete case, where $P \sim Q$, but P is itself is no longer a martingale measure. Here the situation becomes more subtle, and we need a projection method which is no longer standard.

A trading strategy $\Delta_t = (\delta_t, \delta_t^0)$ is called a *small perturbation* if it satisfies the following conditions: δ_t and $\int_0^T \delta_s d|A|_s$ are bounded with $\delta_T = \delta_T^0 = 0$. Let Π be a trading strategy, Δ be a small perturbation and τ be a partition of $[0, T]$ in the sense of [9]. The strategy Π is called *locally risk-minimizing* if

$$\lim_{n \to \infty} \inf r^{\tau_n}[\Pi, \Delta] \geq 0, \quad P_M - a.s. \tag{2.11}$$

for every small perturbation Δ and every increasing 0-convergent sequence (τ_n) of partitions of $[0, T]$, where $r^\tau[\Pi, \Delta]$ is the risk-quotient in the sense of [9].

This definition is made precise in terms of the differentiation of semimartingale, and it is shown to be essentially equivalent to the following property of the associated cost process $C_t, 0 \leq t \leq T$:

Definition 2.2. *An admissible strategy* $\Pi_t = (\pi_t, \pi_t^0)$ *is called optimal if the associated cost process* C_t *is a square-integrable martingale orthogonal to* M *under* P.

In discrete time, a unique optimal strategy does exist, and it can be determined by a sequential regression procedure running backwards for time T to time 0. In continuous time, the construction of such a strategy becomes more difficult as following;

Proposition 2.3. *(1) The existence of an optimal strategy $\Pi_t = (\pi_t, \pi_t^0)$ is equivalent to a decomposition (2.9) with $H_0 \in L^2(\Omega, F_0, P)$, where π_t^H satisfies (2.10) and $L_t^H, 0 \leq t \leq T$ is a square-integrable martingale orthogonal to M which is the martingale component of X.*

(2) For such a decomposition, the associated optimal strategy $\Pi_t = (\pi_t, \pi_t^0)$ is given by (2.10).

Now, the problem is reduced to finding the representation (2.9) and $L_t^H, 0 \leq t \leq T$ which is of Proposition 2.3. But if X_t is not a martingale, we can not use the usual Kunita-Watanabe projection technique. One possible approach is to use Kunita-Watanabe decomposition, and get an optimality equation. But we introduce another method to get the uniqueness of the decomposition (2.9) and of the corresponding optimal strategy, and it is robustness under an equivalent change of measure.

2.4. **Uniqueness of Decomposition.** In our case, the martingale measure Q may lead to different strategy. But it turns out that there is a minimal martingale measure $Q^* \sim P$ such that the optimal strategy for P can be computed in terms of Q^*. In this partial sense, robustness will extend to our case.

Let Q be an equivalent measure with respect to the given probability P. More precisely, we assume that $Q(\sim P)$ is a probability measure on (Ω, F) such that dQ/dP is in $L^2(\Omega, F, P)$ and X is a martingale under Q. This martingale measure Q is determined by the right continuous square-integrable martingale $G_t, 0 \leq t \leq T$ with

$$G_t := E^P[dQ/dP|F_t], \quad 0 \leq t \leq T. \tag{2.12}$$

Under Q, the Doob-meyer decomposition of M is given by $M = X - X_0 + (-A)$. But, from the Girsanov transformation, the predictable process of bounded variation can be computed in terms of G:

$$-A_t = \int_0^t \frac{1}{G_{s-}} d < M, G >_s, 0 \leq t \leq T. \tag{2.13}$$

Since $< M, G > \ll < M > = < X >$, the process A_t must be absolutely continuous w.r.t. the variation process $< X >$, i.e.,

$$A_t = \int_0^t \alpha_s d < X >_s, 0 \leq t \leq T \tag{2.14}$$

for some predictable process $\alpha_t, 0 \leq t \leq T$.

Definition 2.4. *A martingale measure $Q^* \sim P$ is called minimal if $Q^* = P$ on F_0, and if any square-integrable P-martingale L_t which is orthogonal to M_t under P remains a martingale under Q^*, i.e., $L_t \in \mathbf{M}^2$ and $< L, M >_t = 0$ implies that L_t is a martingale under Q^*.*

Let us think the existence and uniqueness of the minimal martingale measure Q^*.

Proposition 2.5. *(1) The minimal martingale measure Q^* is uniquely determined.*

(2) Q^* exists if and only if

$$G_t^* = \exp\{-\int_0^t \alpha_s dM_s - (1/2)\int_0^t \alpha_s^2 d < X >_s\}, 0 \le t \le T \tag{2.15}$$

is a square-integrable martingale under P. In this case, Q^* is given by $dQ^*/dP = G_T^*$.

(3) The minimal martingale measure Q^* preserves orthogonality. That is any $L \in \mathbf{M}^2$ with $< L, M >= 0$ under P satisfies $< L, X >= 0$ under Q^*.

The definition of minimal martingale measure means that Q^* preserves the martingale property as far as possible under the restriction "X is a martingale under Q". This minimal departure from the given measure P can also be expressed in terms of the relative entropy

$$\begin{aligned} H(Q|P) &= \int log\frac{dQ}{dP}dQ, \quad if \quad Q << P, \\ & \quad +\infty, \quad otherwise. \end{aligned}$$

This relative entropy is always nonnegative.

Proposition 2.6. *(1) In the class of all martingale measure Q, the minimal martingale measure Q^* is characterized by the fact that it minimizes the functional*

$$H(Q|P) - (1/2)E^Q[\int_0^T \alpha_s^2 d < X >_s]. \tag{2.16}$$

(2) Measure Q^* minimizes the relative entropy $H(\cdot|P)$ among all martingale measure Q with fixed expectation

$$E^Q[\int_0^T \alpha_s^2 d < X >_s]. \tag{2.17}$$

Proposition 2.7. *(1) The optimal strategy, which is also corresponding decomposition (2.12), is uniquely determined.*

(2) It can be computed in terms of the minimal martingale measure Q^*: If $V_t, 0 \le t \le T$ denotes a right-continuous version of the martingale

$$V_t = E^{Q^*}[H|F_t] \quad 0 \le t \le T,$$

then the optional strategy $\Pi_t = (\pi_t, \pi_t^0)$ is given by (2.13) where

$$\pi_t^H = \frac{d[V, X]_t^{Q^*}}{d[X]_t^{Q^*}}, \tag{2.18}$$

is obtained by projecting the Q^*-martingale V on the Q^*-martingale X.

3. ASSET MODEL

In general, asset models are started form one of assumptions which are of two kinds of return rates. One is Y_t defined by $\Delta Y_t = \log X_t - \log X_{t-}$, and the other is ΔZ_t defined by $\Delta Z_t = (X_t - X_{t-})/X_{t-}$. Let us start from the second assumption:

$$X_t - X_{t-} = X_{t-} \cdot \Delta Z_t. \tag{3.1}$$

But we can see, from Table 1 and Table 2 which are of two assets in KOSPI and two assets in KOSDAQ, that ΔZ_t(the difference of amplitudes) of each asset are different. Thus, to use same leptokurtic distribution (and density) function of ΔZ_t, we need to modify equation (3.1) by an *adjusting function* v which is a positive C^1-function defined from assumption: if $\Delta Z_t \to 0$(i.e., $\Delta X_t \to 0$), then $v(X_{t-}) \to X_{t-}$(i.e., $v(x) \to x$). Then we get new difference equation:

$$X_t - X_{t-} = v(X_{t-}) \cdot \Delta Z_t. \tag{3.2}$$

TABLE 1. Price Amplitudes of Samsung ELE. and Hynix. Feb. 9th, A.M. 11:06:50–11:08:20, 2009

Price of Stock	Amplitude for Day Initial Price	Difference of Amplitude	Price of Sodck	Amplitude for Day Initial Price	Difference of Amplitude
529,000	-2.94 %	0.19 %	9,570	2.03 %	0.11 %
530,000	-2.75 %	0.18 %	9,560	1.92 %	0.11 %
531,000	-2.57 %	0.18 %	9,550	1.81 %	0.10 %
532,000	-2.39 %	0.19 %	9,540	1.71 %	0.11 %
533,000	-2.20 %	0.18 %	9,530	1.60 %	0.11 %
534,000	-2.02 %	0.19 %	9,520	1.49 %	0.10 %
535,000	-1.83 %	0.18 %	9,510	1.39 %	0.11 %
536,000	-1.65 %	...	9,500	1.28 %	....
....	.....	...	.....	....	....

On the other hand, ΔZ_t is in open interval $(-1, \infty)$. Therefore, to represent more exact range of ΔZ_t, we use a function of ΔZ_t of the form $e^{\Delta Z_t} - 1$ which is in $(-1, \infty)$ strictly. Thus we use equation

$$X_t - X_{t-} = v(X_{t-}) \cdot (e^{\Delta Z_t} - 1). \tag{3.3}$$

We can represent this equation by the form;

$$X_t - X_{t-} = v(X_{t-}) \cdot [\Delta Z_t + (e^{\Delta Z_t} - 1 - \Delta Z_t)], \tag{3.4}$$

and from this difference equation, we can derive SDE (1.2) having the driving process (1.3):

$$dX_t = v(X_{t-})[dZ_t + (e^{\Delta Z_t} - 1 - \Delta Z_t)].$$

For the solution X_t of this SDE, we can say the absolute continuity, and the existence of density function of X_t under some, a little complicated conditions.

TABLE 2. Price Amplitudes of Anchullsu Reach. and Daiyoung Package. Feb. 9th, A.M. 11:12:14 - 11:14:30, 2009

Price of Stock	Amplitude for Day Initial Price	Difference of Amplitude	Price of Stock	Amplitude for Day Initial Price	Difference of Amplitude
9,110	2.71 %	0.12 %	310	10.71 %	1.78 %
9,100	2.59 %	0.45 %	305	8.93 %	1.79 %
9,060	2.14 %	0.11 %	300	7.14 %	1.78 %
9,050	2.03 %	0.11 %	295	5.36 %	1.79 %
9,040	1.92 %	0.12 %	290	3.57 %	1.78 %
9,030	1.80 %	0.11 %	285	1.79 %	1.79 %
9,020	1.69 %	...	280	0.00 %	....
.....	....	...	...	....	....

The solution of this SDE can be represented as

$$\begin{aligned} X_t &= X_0 + \int_0^t b^\delta v(X_{s-})ds + \int_0^t v(X_{s-})\int_{|z|<\delta} z\tilde{N}_p(dz,ds) \\ &\quad + \int_0^t v(X_{s-})\int_{|z|\geq\delta} zN_p(dz,ds) + \int_0^t v(X_{s-})\int_{|z|<\delta}(e^z-1-z)N_p(dz,ds) \\ &\quad + \int_0^t v(X_{s-})\int_{|z|\geq\delta}(e^z-1-z)N_p(dz,ds) \end{aligned}$$

$$\begin{aligned} &= X_0 + \int_0^t b^\delta v(X_{s-})ds + \int_0^t v(X_{s-})\int_{|z|<\delta} z\tilde{N}_p(dz,ds) \\ &\quad + \int_0^t v(X_{s-})\int_{|z|<\delta}(e^z-1-z)[\tilde{N}_p(dz,ds)-\nu(dz)ds] \\ &\quad + \int_0^t v(X_{s-})\int_{|z|\geq\delta}(e^z-1-z)N_p(dz,ds) + \int_0^t v(X_{s-})\int_{|z|\geq\delta} zN_p(dz,ds) \end{aligned} \tag{3.5}$$

$$\begin{aligned} &= X_0 + \int_0^t b^\delta v(X_{s-})ds + \int_0^t v(X_{s-})\int_{|z|<\delta}(e^z-1-z)\nu(dz)ds \\ &\quad + \int_0^t v(X_{s-})\int_{|z|<\delta}(e^z-1)\tilde{N}_p(dz,ds) + \int_0^t v(X_{s-})\int_{|z|\geq\delta}(e^z-1)N_p(dz,ds). \end{aligned}$$

In real stock markets, the amplitudes of movements of stock prices are restricted. Thus we take our basic asset model as a solution of SDE;

$$S_t = S_0 + \int_0^t v(S_{s-}) \int_{|z|<\delta} (e^z - 1)\tilde{N}_p(dz, ds) + \int_0^t v(S_{s-})[b^\delta + \int_{|z|<\delta} (e^z - 1 - z)\nu(dz)]ds. \tag{3.6}$$

Now, we can assume that S_t is decomposable and $\mathbf{H}^2$-semimartingale, i.e., S_t has the form of $S_t = M_t + A_t$, where M_t is a $\mathbf{L}^2$-(local) martingale and A_t is a predictable process of finite variation with $\int_0^T |dA_s| \in \mathbf{L}^2(P)$. We know that $\mathbf{L}^2$-spaces are natural to the theory of stochastic integration. Further, we assume that our model is a Markov process because we need the existence of density function, and this existence guarantee the study of option pricing and hedging. Also, our model disregard and omit the big-jump part of (3.5) to use the results and methods prior works easily(c.f. [7], [11]).

Our model is more reasonable one if we think the asset prices of daily data, or think the data of at least one more times in a day because many stock markets limit the amplitude of jump in a day and thus we disregard the big-jump part. Further, we will not think about the interest rate for simplicity in the following. Thus, this model is more useful in the short space of time by this meaning.

3.1. **Minimum Martingale Measure.** Let P and Q be two equivalent martingale measure, Then the Radon-Nikodym derivative $G_T := dQ/dP$ is in $L^1(dP)$. We define the process G_t as

$$G_t := E^P[dQ/dP|F_t], \quad 0 \le t \le T,$$

taking the right continuous version. To avoid signed measure, we assume that Q is nonnegative and local equivalent martingale measure for S_t such that $Q \approx P$ on F_T in the sense of [5] and [10], i.e., S_t is a local martingale under Q(c.f. [7], [10]). Then the process G_t is a uniformly integrable P-martingale, and we get;

$$G_t := 1 + \int_0^t \psi_s dM_s,$$

from a version of the Martingale Representation Theorem, where $\psi(> -1)$ is predictable process such that

$$E^P[\int_0^{\tau_n} \psi_s^2 d[M, M]_s < \infty,$$

where $\{\tau_n\}_{n=0}^\infty$ are a sequence of stopping times converging to T.

Thus, because $P(G_t > 0, t \in [0, T]) = 1$, if we put

$$\Psi_t := \psi_t / G_{t-},$$

then $\Psi_t(> -1)$ is M-integrable. Hence we can think that G_t is the solution of SDE

$$dG_t = G_{t-}\Psi_t dM_t,$$

where Ψ_t^1 is predictable process such that

$$E^P[\int_0^{\tau_n} \Psi_s^2 d[M,M]_s < \infty.$$

Therefore, for an equivalent martingale measure Q^* such that $G_T^* := dQ^*/dP$, if we put

$$G_t^* := exp\{-\int_0^t \Psi_s dM_s - \frac{1}{2}\int_0^t \Psi_s^2 d < S >_s\}, 0 \le t \le T,$$

where

$$\begin{aligned} dM_t &= v(S_{t-})\int_U (e^z - 1)\tilde{N}_p(dz, dt), \\ d < S >_t &= v(S_{t-})b^\delta dt + v(S_{t-})\int_{|z|<\delta}(e^z - 1 - z)\nu(dz)dt, \end{aligned}$$

then, from the Itô formula, we get that

$$G_t^* := 1 + \int_0^t G_{s-}^* \Psi_s dS_s,$$

which shows that G_t^* is a nonnegative square-integrable martingale (by Theorem of Exponential Formula for Continuous Processes). Thus, from the Proposition 2.5, the equivalent measure Q^* is minimal martingale measure which is equivalent to given measure P. Thus we can define the existence and uniqueness of minimal martingale measure Q^* in our model.

4. HEDGING OF OPTION

Let $u(t, S_t) := E^{Q^*}[H(S_T)|F_t]$. Then $u(t, S_t)$ is a (local) martingale with respect to (minimal martingale measure) Q^*. From the definition of $u(t, S_t)$, we can assume that $u(t, S_t)$ is $C^{1,2}$-function on $[0, T] \times R_+$. Thus from the Itô formula, we get that:

Lemma 4.1. *$u(t, S_t)$ can be expanded by Itô formula as following;*

$$\begin{aligned} u(t, S_t) &= u(0, S_0) + \int_0^t \partial_s u(s, S_s)ds \qquad (4.1) \\ &+ \int_0^t \int_U [u(s, e^z v(S_{s-})) - u(s, S_s) - \partial_x u(s, S_s) z v(S_{s-})]\nu(dz)ds \\ &+ \int_0^t \int_U [u(s, e^z v(S_{s-})) - u(s, S_s)]\tilde{N}_p(dz, ds). \end{aligned}$$

Proof. From the definition of $u(t, S_t) := E^{Q^*}[H(S_T)|F_t]$, we induce that $u(t, S_t)$ is a (local) martingale. Further, we can deduce that $u(t, x)$ is $C^{1,2}$-function on $[0, T] \times R_+$. Thus, from the Itô formula, we obtain for the Markov semimartingale S_t represented by (3.6) as following;

$$\begin{aligned} u(t,S_t) &= u(0,S_0) + \int_0^t \frac{\partial}{\partial s} u(s,S_s)ds \\ &+ \int_0^t \frac{\partial}{\partial x} u(s,S_s) \int_U (e^z v(S_{s-}) - v(S_{s-}) - zv(S_{s-}))\nu(dz)ds \\ &+ \int_0^t \int_U [u(s,S_{s-}+\Delta S_s) - u(s,S_s) - \\ &\qquad \frac{\partial}{\partial x} u(s,S_s)(e^z v(S_{-s}) - v(S_{s-}))]\hat{N}_p(dz,ds) \\ &+ \int_0^t \int_U [u(s,S_{s-}+\Delta S_s) - u(s,S_s)]\tilde{N}_p(dz,ds). \end{aligned}$$

From the fact: $\Delta S_t = S_t - S_{t-} = v(S_{t-})(e^{\Delta Z_t} - 1)$, we get

$$\begin{aligned} S_{t-} + \Delta S_t &= S_{t-} + v(S_{t-})(e^{\Delta Z_t} - 1) \\ &= v(S_{t-})e^{\Delta Z_t} + [S_{t-} - v(S_{t-})]. \end{aligned}$$

From basic assumption for function v, if $\Delta S_t \to 0$, then we can get $v(S_{t-}) \to S_{t-}$. Thus we get the result. □

From this proposition, we get following result:

Theorem 4.2. *Suppose that our option H is in $L^2(P)$. If the decomposition*

$$V_t = V_0 + \int_0^t \pi_s^H dS_s + L_t^H, \quad 0 \le t \le T$$

exists, we get the local optional strategy is given by $\pi_t = D(t,S_t)$, where

$$D(t,x) = \frac{\int_U (e^z v(x) - v(x))[u(s,e^z v(x)) - u(t,x)]\nu(dz)}{\int_U (e^z v(x) - v(x))^2 \nu(dz)}, \tag{4.2}$$

where $U = \{z||z| \le \delta\}$.

Proof. From the Itô formula for $u(t,S_t)$, we can calculus quadratic variation $d[S,S]_t$ and the joint quadratic variation $d[V,S]_t$ by the bracket theory of jump-type semimartingale, where $V_t = E^{Q^*}[H(S_T)|F_t], 0 \le t \le T$, as following:

$$\begin{aligned} d[S,S]_t &= d[\int_0^t v(S_{t-})\int_U (e^z-1)\tilde{N}_p(dz,dt), \int_0^t v(S_{t-})\int_U (e^z-1)\tilde{N}_p(dz,dt)] \\ &= d\{\int_0^t \int_U [v(S_{s-})(e^z-1)]^2 \tilde{N}_p(dz,dt)\}, \end{aligned}$$

because the quadratic variation is 0 as following;

$$[\int_0^t v(S_{s-})\{b^\delta ds + \int_{|z|<\delta} (e^z - 1 - z)\nu(dz)\}ds]_t = 0,$$

and

$$\begin{aligned} d[V,S]_t &= d[\int_0^t \int_U [u(s,S_{s-}+\Delta S_s)-u(s,S_s)]\tilde{N}_p(dz,ds), \int_0^t v(S_{s-})\int_U (e^z-1)\tilde{N}_p(dz,ds)] \\ &= d[\int_0^t \int_U [u(s,e^z v(S_{s-}))-u(s,S_s)]\tilde{N}_p(dz,ds), \int_0^t v(S_{s-})\int_U (e^z-1)\tilde{N}_p(dz,ds)] \\ &= \int_U v(S_{t-})(e^z-1)[u(t,e^z v(S_{t-}))-u(t,S_t)]\nu(dz), \end{aligned}$$

because of the joint quadratic variation of

$$\int_0^t \partial_s u(s,S_s)ds + \int_0^t \int_U [u(s,e^z S_{s-})-u(s,S_s)-\partial_x u(s,S_s)zv(S_{s-})]\nu(dz)ds$$

and

$$\int_0^t v(S_{s-})[b^\delta ds + \int_{|z|<\delta}(e^z-1-z)\nu(dz)]ds$$

is 0. i.e.,

$$\begin{aligned} &[\int_0^t \partial_s u(s,S_s)ds + \int_0^t \int_U [u(s,e^z v(S_{s-}))-u(s,S_s)-\partial_x u(s,S_s)zv(S_{s-})]\nu(dz)ds, \\ &\int_0^t v(S_{s-})\{b^\delta ds + \int_{|z|<\delta}(e^z-1-z)\nu(dz)\}ds] = 0. \end{aligned}$$

Thus, from the jump-type version theory Proposition 2.7, we get the result. □

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