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EXPLICIT BOUNDS FOR THE TWO-LEVEL PRECONDITIONER OF THE $P1$ DISCONTINUOUS GALERKIN METHOD ON RECTANGULAR MESHES

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ABSTRACT. In this paper we investigate a simple two-level additive Schwarz preconditioner for the $P1$ symmetric interior penalty Galerkin method of the Poisson equation on rectangular meshes. The construction is based on the decomposition of the global space of piecewise linear polynomials into the sum of local subspaces, each of which corresponds to an element of the underlying mesh, and the global coarse subspace consisting of piecewise constants. This preconditioner is a direct combination of the block Jacobi iteration and the cell-centered finite difference method, and thus very easy to implement. Explicit upper and lower bounds for the maximum and minimum eigenvalues of the preconditioned matrix system are derived and confirmed by some numerical experiments.

1. INTRODUCTION

In this paper we consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and $f \in L^2(\Omega)$. Among the popular numerical methods for (1.1) in early days, the cell-centered finite difference method (FDM) received much attention due to its simplicity and property of local mass conservation. In recent years, its higher order version, discontinuous Galerkin (DG) methods, have been under extensive study (see, e.g., [2, 17] for their unified presentation) because they are flexible in handling polynomials of varying degrees and in the design of meshes, making them well suited for hp-adaptivity. On the other hand, DG methods involve a relatively large number of unknowns in comparison to their continuous counterparts, and therefore it is inevitable to apply efficient solution techniques such as multigrid [3, 5, 6, 7, 10, 13] or domain decomposition methods [1, 8, 16]. In particular, the two-level method proposed in [6, 7] is notable in its simplicity as it uses the coarse space of continuous piecewise linear polynomials or piecewise

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constants on the same mesh as the original DG space and introduces a hierarchy of uniformly refined meshes for this coarse space, instead of the original DG space. We also mention the works [11, 12] where high-order discontinuous Galerkin methods are solved by the p-multigrid method which iterates the solution in approximation spaces of increasingly lower orders.

In this paper we analyze a simple two-level additive Schwarz preconditioner for the $P1$ symmetric interior penalty Galerkin (SIPG) method of the problem (1.1) on rectangular meshes. Our approach of constructing the preconditioner follows the framework of subspace correction methods [19], and corresponds to the simplest case of non-overlapping domain decomposition methods proposed in [8]. More specifically, the global $P1$ DG space is decomposed in a *two-level* way, that is, into the sum of local subspaces, each of which corresponds to an element of the underlying mesh, and a global coarse subspace which plays the role of communicating information between these local subspaces. Since the $P1$ DG space on rectangular meshes does not contain any conforming space on the same mesh, a natural choice for this coarse subspace is the space of piecewise constants. The resulting preconditioner is very easy to implement because the exact solvers for the local subspace problems are equivalent to the block Jacobi iteration and, with appropriate penalty parameters, the coarse-space problem is identical with the cell-centered FDM which can be solved more easily by direct or iterative solvers. We will show that the maximum eigenvalue of the preconditioned matrix system is always bounded above by 3 and derive explicit a lower bound for the minimum eigenvalue as a function of the penalty parameter and the aspect ratio of the mesh which are also confirmed by numerical experiments.

The analysis presented here apply as well to nonsymmetric or incomplete interior penalty methods or problems of variable coefficients in order to get uniform bounds on the condition numbers of the preconditioned systems. We also remark that a *multilevel* preconditioner is obtained by solving the cell-centered FDM with the multigrid methods (see, e.g., [14, 15]), which is very similar in spirit to the multigrid framework taken in [6, 7].

The remainder of the paper is organized as follows. In the next section we introduce some notation and define the SIPG method for the Poisson problem (1.1). In Section 3, the additive Schwarz preconditioner is constructed and abstract estimates for the maximum and minimum eigenvalues of the preconditioned system are given. By using these estimates we derive explicit upper and lower bounds for the maximum and minimum eigenvalues of our preconditioned system in Section 4. Finally, in Section 5, some numerical results are presented to confirm these theoretical bounds.

2. Interior Penalty Discontinuous Galerkin Method

We suppose that Ω is covered by a uniform rectangular partition $\mathcal{T}_h$. This means that all the rectangles in $\mathcal{T}_h$ have the same width h_x and height h_y as well as the common area $|T| = h_x h_y$. We also set $h = \max(h_x, h_y)$ and $\theta = \max(h_x/h_y, h_y/h_x)$. For a rectangle $T \in \mathcal{T}_h$, we denote the set of edges of $T \in \mathcal{T}_h$ by $\mathcal{E}_T$ and the outward unit normal to ∂T by $\boldsymbol{n}_T$. Let $\mathcal{E}_I$ and $\mathcal{E}_B$ be the collections of all interior and boundary edges of $\mathcal{T}_h$, respectively, and set $\mathcal{E}_h = \mathcal{E}_I \cup \mathcal{E}_B$. The length of an edge $E \in \mathcal{E}_h$ is denoted by $|E|$. We define the average and jump of a scalar-valued function v on $E = \partial T_1 \cap \partial T_2$ by

$$\{\!\{v\}\!\} = \frac{v|_{T_1} + v|_{T_2}}{2}, \qquad [\![v]\!] = v|_{T_1}\boldsymbol{n}_{T_1} + v|_{T_2}\boldsymbol{n}_{T_2}.$$

On a boundary edge $E \subset \partial T$, we set $\{\!\{v\}\!\} = v|_T$ and $[\![v]\!] = v|_T\boldsymbol{n}_T$. The notation $\{\!\{\cdot\}\!\}$ will be also used for the averages of vector-valued functions.

For an integer $k \geq 0$, let $\mathbb{P}_k(\mathcal{T}_h)$ be the space of piecewise polynomials of degree k on $\mathcal{T}_h$ (which are not necessarily continuous) and $\nabla_h v$ the piecewise gradient of $v \in \mathbb{P}_k(\mathcal{T}_h)$. Then the SIPG method for the problem (1.1) is given as follows:

SIPG method. find $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that

$$A(u_h, v_h) = \int_\Omega f v_h \, d\boldsymbol{x} \qquad \forall v_h \in \mathbb{P}_k(\mathcal{T}_h), \tag{2.1}$$

where

$$A(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v \, d\boldsymbol{x} - \sum_{E\in\mathcal{E}_h} \int_E (\{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] + \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!]) \, ds + \gamma J_1(u,u),$$

$$J_1(u,v) = \sum_{E\in\mathcal{E}_I} \frac{|E|}{|T|} \int_E [\![u]\!] \cdot [\![v]\!] \, ds + \sum_{E\in\mathcal{E}_B} \frac{2|E|}{|T|} \int_E uv \, ds.$$

Here $\gamma > 0$ is chosen to ensure uniform coercivity of $A(\cdot,\cdot)$ over the space $\mathbb{P}_k(\mathcal{T}_h)$ with respect to the mesh-dependent energy norm (cf. [17])

$$|\!|\!| u |\!|\!| := \left(\int_\Omega |\nabla_h u|^2 \, d\boldsymbol{x} + J_1(u,u) \right)^{1/2}.$$

Note that the penalty parameter for the boundary edges is taken to be twice the value for the interior edges.

The following theorem shows that $A(\cdot,\cdot)$ is coercive for any $\gamma > 1$ when $k = 1$. From now on we will frequently use the fact that every $u \in \mathbb{P}_1(\mathcal{T}_h)$ can be expressed as

$$u|_T(x,y) = \overline{u} + u_x(x - x_T) + u_y(y - y_T) \qquad \forall T \in \mathcal{T}_h,$$

where (x_T, y_T) is the center of T, and $\overline{u}$, u_x and u_y are piecewise constants given by

$$\overline{u}|_T := \frac{1}{|T|} \int_T u \, d\boldsymbol{x} = u(x_T, y_T), \qquad (u_x, u_y) := \nabla_h u.$$

Theorem 2.1. *Given any $\alpha > 0$, we have for all $u \in \mathbb{P}_1(\mathcal{T}_h)$*

$$A(u,u) \geq \left(1 - \frac{1}{\alpha}\right) \int_\Omega |\nabla_h u|^2 \, d\boldsymbol{x} + (\gamma - \alpha) J_1(u,u).$$

Proof. Using Young's inequality $2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ $(a, b, \varepsilon > 0)$, one can prove that for a vertical edge $E = \partial T_1 \cap \partial T_2$,

$$
\begin{aligned}
2\int_E \{\!\{\nabla_h u\}\!\} \cdot [\![u]\!]\, ds &\le \int_E \left| u_x|_{T_1} [\![u]\!] \right| ds + \int_E \left| u_x|_{T_2} [\![u]\!] \right| ds \\
&\le \frac{1}{2}\alpha^{-1} \left(\int_{T_1} u_x^2 \, d\boldsymbol{x} + \int_{T_2} u_x^2 \, d\boldsymbol{x} \right) + \alpha \frac{|E|}{|T|} \int_E |[\![u]\!]|^2 \, ds,
\end{aligned}
$$

and for a vertical edge $E \in \mathcal{E}_T \cap \mathcal{E}_B$,

$$
2\int_E \nabla u \cdot \boldsymbol{n}_T u \, ds \le 2 \int_E |u_x| u \, ds \le \frac{1}{2}\alpha^{-1} \int_T u_x^2 \, d\boldsymbol{x} + \alpha \frac{2|E|}{|T|} \int_E u^2 \, ds.
$$

Similar results for horizontal edges can be obtained. The proof is completed by summing these inequalities over all $E \in \mathcal{E}_h$ in $A(u, u)$. □

It is well known that the condition number of the discrete matrix system arising from (2.1) grows at a rate proportional to h^{-2}. Due to this ill-conditioning as well as the large number of degrees of freedom compared to that of continuous Galerkin methods, it is imperative that one should use good preconditioners for solution of discontinuous Galerkin methods.

3. Additive Schwarz Preconditioner for the $P1$ SIPG Method

Let us first recall the abstract theory of additive Schwarz preconditioners. The interested readers are referred to [9, 18, 19] for further details.

Let V be a Hilbert space with inner product $(\cdot, \cdot)$ and $A : V \to V$ be a linear, symmetric and positive definite operator on V. Construction of an additive Schwarz preconditioner for A requires the following two ingredients:

- Space decomposition: $\quad V = \sum_{i=1}^{N} V_i$
- Subspace solvers: $\quad R_i : V_i \to V_i \quad$ for $i = 1, \cdots, N$

It is assumed that every local solver R_i is also linear, symmetric and positive definite on V_i. Then we define a linear operator $B : V \to V$ by

$$
B = \sum_{i=1}^{N} R_i Q_i, \tag{3.1}
$$

where $Q_i : V \to V_i$ denotes the orthogonal projection respect to $(\cdot, \cdot)$. It is easy to verify that the operator B is symmetric and positive definite with respect to $(\cdot, \cdot)$.

In the following theorem we present an abstract estimate from [9] which is useful in deriving bounds for the maximum and minimum eigenvalues of the preconditioned operator BA, denoted by $\lambda_{\max}(BA)$ and $\lambda_{\min}(BA)$, respectively.

Theorem 3.1. *We have*

$$(B^{-1}u, u) = \min_{u_i \in V_i} \sum_{i=1}^{N} (R_i^{-1} u_i, u_i),$$

where the minimum is taken over all decompositions $u = \sum_{i=1}^{N} u_i$.

Proof. A short proof is given here for the reader's convenience. Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
(B^{-1}u, u) &= \sum_{i=1}^{N} (B^{-1}u, u_i) = \sum_{i=1}^{N} (R_i^{1/2} Q_i B^{-1} u, R_i^{-1/2} u_i) \\
&\le \left[\sum_{i=1}^{N} (R_i Q_i B^{-1} u, Q_i B^{-1} u) \right]^{1/2} \left[\sum_{i=1}^{N} (R_i^{-1} u_i, u_i) \right]^{1/2} \\
&= \left[\sum_{i=1}^{N} (R_i Q_i B^{-1} u, B^{-1} u) \right]^{1/2} \left[\sum_{i=1}^{N} (R_i^{-1} u_i, u_i) \right]^{1/2} \\
&= (B^{-1}u, u)^{1/2} \left[\sum_{i=1}^{N} (R_i^{-1} u_i, u_i) \right]^{1/2}.
\end{aligned}$$

The equality holds if $R_i^{1/2} Q_i B^{-1} u = R_i^{-1/2} u_i$ or $u_i = R_i Q_i B^{-1} u$. □

Corollary 3.2. *Suppose that there exist constants* $M, m > 0$ *such that*

$$m \le \frac{(Au, u)}{\displaystyle \min_{u_i \in V_i} \sum_{i=1}^{N} (R_i^{-1} u_i, u_i)} \le M,$$

where the minimum is taken over all decompositions $u = \sum_{i=1}^{N} u_i$. *Then we have*

$$\lambda_{\max}(BA) \le M, \qquad \lambda_{\min}(BA) \ge m.$$

Now we construct an additive Schwarz preconditioner for the bilinear form $A(\cdot, \cdot)$ in the SIPG method (2.1), or equivalently, for the linear operator $A : \mathbb{P}_1(\mathcal{T}_h) \to \mathbb{P}_1(\mathcal{T}_h)$ defined by

$$A(u, v) = \int_\Omega (Au) v \, d\boldsymbol{x} \qquad \forall u, v \in \mathbb{P}_1(\mathcal{T}_h). \tag{3.2}$$

Let us start with the simple decomposition of the global space

$$\mathbb{P}_1(\mathcal{T}_h) = \sum_{T \in \mathcal{T}_h} \mathbb{P}_1(T),$$

where $\mathbb{P}_1(T)$ denotes the *"local"* space corresponding to an element $T \in \mathcal{T}_h$

$$\mathbb{P}_1(T) := \{ v \in \mathbb{P}_1(\mathcal{T}_h) : \operatorname{supp}(v) \subset \overline{T} \}.$$

The corresponding subspace solver $R_T : \mathbb{P}_1(T) \to \mathbb{P}_1(T)$ is chosen to be the exact inverse of the restriction of $A(\cdot,\cdot)$ to $\mathbb{P}_1(T)$, i.e.,

$$A_T(R_T g, v) = \int_T gv \, d\boldsymbol{x} \qquad \forall v \in \mathbb{P}_1(T), \tag{3.3}$$

where $A_T(\cdot,\cdot)$ is the restriction of $A(\cdot,\cdot)$ to $\mathbb{P}_1(T)$

$$\begin{aligned} A_T(u,v) = \int_T \nabla u \cdot \nabla v \, d\boldsymbol{x} &- \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_I} \frac{1}{2} \int_E (\nabla u \cdot \boldsymbol{n}_T v + \nabla v \cdot \boldsymbol{n}_T u) \\ &- \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_B} \int_E (\nabla u \cdot \boldsymbol{n}_T v + \nabla v \cdot \boldsymbol{n}_T u) \\ &+ \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_I} \gamma \frac{|E|}{|T|} \int_E uv + \sum_{E \in \mathcal{E}_T \cap \mathcal{E}_B} 2\gamma \frac{|E|}{|T|} \int_E uv. \end{aligned}$$

It is not difficult to see that the resulting additive Schwarz preconditioner is nothing but the block Jacobi part of A. As will be confirmed by numerical results later, this is not a uniform preconditioner due to the local nature of the preconditioner.

In order to get a uniform preconditioner for A, we need to add the *global coarse* space $\mathbb{P}_0(\mathcal{T}_h)$ which plays the role of communicating information between non-overlapping local spaces $\mathbb{P}_1(T)$'s. This leads to the following space decomposition

$$\mathbb{P}_1(\mathcal{T}_h) = \mathbb{P}_0(\mathcal{T}_h) + \sum_{T \in \mathcal{T}_h} \mathbb{P}_1(T). \tag{3.4}$$

Note that the restriction of $A(\cdot,\cdot)$ to $\mathbb{P}_0(\mathcal{T}_h)$ is equal to

$$A(u_0, v_0) = \gamma J_1(u_0, v_0) \qquad \forall u_0, v_0 \in \mathbb{P}_0(\mathcal{T}_h)$$

which gives rise to the same matrix as the cell-centered FDM (cf. [4]). For theoretical analysis in the next section, we choose the subspace solver $R_0 : \mathbb{P}_0(\mathcal{T}_h) \to \mathbb{P}_0(\mathcal{T}_h)$ to be the exact inverse given by

$$A(R_0 g, v) = \int_\Omega gv \, d\boldsymbol{x} \qquad \forall v \in \mathbb{P}_0(\mathcal{T}_h). \tag{3.5}$$

The resulting preconditioner is a direct combination of the block Jacobi iteration and the cell-centered FDM. In practice, following [6, 7], one may use multigrid methods for the cell-centered FDM, e.g., proposed in [4, 14, 15], to solve the global problem (3.5). In this case, it can be said that the multigrid structure is incorporated in the global coarse space $\mathbb{P}_0(\mathcal{T}_h)$ with the block smoothers on the highest level associated with the local spaces $\{\mathbb{P}_1(T) : T \in \mathcal{T}_h\}$.

4. Estimation of Maximum and Minimum Eigenvalues

In this section we will derive explicit bounds for the eigenvalues $\lambda_{\max}(BA)$ and $\lambda_{\min}(BA)$, where A and B are defined by (3.2) and (3.1) with the decomposition (3.4) and the subspace solvers (3.3) and (3.5), respectively. The following equality will be crucially used: for $u = \sum_{T\in\mathcal{T}_h} u_T$ with $u_T \in \mathbb{P}_1(T)$ for $T \in \mathcal{T}_h$, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} A_T(u_T,u_T) = \sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T} \gamma\frac{|E|}{|T|}\int_E u^2\,ds \\ - \sum_{E\in\mathcal{E}_B}\int_E \nabla u\cdot \boldsymbol{n} u\,ds + \sum_{E\in\mathcal{E}_B}\gamma\frac{|E|}{|T|}\int_E u^2\,ds.\end{aligned} \tag{4.1}$$

This can be proved by using integration by parts on each $T \in \mathcal{T}_h$. To derive an upper bound for $\lambda_{\max}(BA)$, we further need the following three lemmas.

Lemma 4.1. *We have for $T \in \mathcal{T}_h$ and $u \in \mathbb{P}_1(T)$*

$$\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E u^2\,ds = \sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds + \sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E \overline{u}^2\,ds.$$

Proof. The result follows from the equality

$$\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E u\,ds = \sum_{E\in\mathcal{E}_T}\frac{|E|^2}{|T|}u(x_E,y_E) = \sum_{E\in\mathcal{E}_T}\frac{|E|^2}{|T|}u(x_T,y_T) = \sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E \overline{u}\,ds$$

which is obtained by the mid-point rule. Here (x_E, y_E) is the center of E. □

Lemma 4.2. *We have for $T \in \mathcal{T}_h$, $E \in \mathcal{E}_T$ and $u \in \mathbb{P}_1(T)$*

$$\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds = \begin{cases} \dfrac{|T|}{4}u_x^2 + \dfrac{h_y^4}{12|T|}u_y^2 & \text{if } E \text{ is vertical},\\[2mm] \dfrac{|T|}{4}u_y^2 + \dfrac{h_x^4}{12|T|}u_x^2 & \text{if } E \text{ is horizontal},\end{cases}$$

and

$$\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds = \frac{|T|}{2}(u_x^2+u_y^2) + \frac{1}{6|T|}(h_x^4u_x^2 + h_y^4u_y^2).$$

Proof. It suffices to prove the first result, as the second result follows directly from it. By the mid-point rule we obtain for all $E \in \mathcal{E}_T$

$$\int_E (x-x_T)(y-y_T)\,ds = 0.$$

Hence it follows that for a vertical edge E,

$$\begin{aligned}\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds &= \frac{|E|}{|T|}\int_E \big(u_x(x-x_T)+u_y(y-y_T)\big)^2\,ds\\ &= \frac{|E|}{|T|}\int_E \big(u_x^2(x-x_T)^2 + u_y^2(y-y_T)^2\big)\,ds\\ &= \frac{h_y^2}{|T|}\left(u_x^2\cdot\frac{h_x^2}{4} + u_y^2\cdot\frac{h_y^2}{12}\right) = \frac{|T|}{4}u_x^2 + \frac{h_y^4}{12|T|}u_y^2.\end{aligned}$$

The result for horizontal edges can be obtained in the same way. □

Lemma 4.3. *Let* $E = \partial T_1 \cap \partial T_2 \in \mathcal{E}_I$ *and let*

$$I_E := -2\int_E \{\!\{\nabla_h u\}\!\} \cdot [\![u]\!]\, ds + \frac{|E|}{|T|}\int_E |[\![u]\!]|^2\, ds.$$

Then we have for vertical E

$$I_E = \frac{|E|}{|T|}\int_E |[\![\overline{u}]\!]|^2\, ds + \frac{h_y^4}{12|T|}(u_y|_{T_1} - u_y|_{T_2})^2 - \frac{|T|}{4}(u_x|_{T_1} + u_x|_{T_2})^2, \tag{4.2}$$

and for horizontal E

$$I_E = \frac{|E|}{|T|}\int_E |[\![\overline{u}]\!]|^2\, ds + \frac{h_x^4}{12|T|}(u_x|_{T_1} - u_x|_{T_2})^2 - \frac{|T|}{4}(u_y|_{T_1} + u_y|_{T_2})^2. \tag{4.3}$$

Proof. For vertical E, suppose that T_1 is on the left side of E and $\boldsymbol{n}_E$ is the unit normal vector from T_1 to T_2. Then we have $\{\!\{\nabla_h u\}\!\} \cdot \boldsymbol{n}_E = \frac{1}{2}(u_x|_{T_1} + u_x|_{T_2})$ on E and

$$\begin{aligned} I_E &= \frac{|E|}{|T|}\int_E \Big((u|_{T_1} - u|_{T_2})^2 - 2h_x\{\!\{\nabla_h u\}\!\} \cdot \boldsymbol{n}_E(u|_{T_1} - u|_{T_2})\Big)\, ds \\ &= \frac{|E|}{|T|}\int_E \Big(\big(u|_{T_1} - u|_{T_2} - h_x\{\!\{\nabla_h u\}\!\} \cdot \boldsymbol{n}_E\big)^2 - h_x^2(\{\!\{\nabla_h u\}\!\} \cdot \boldsymbol{n}_E)^2\Big)\, ds. \end{aligned}$$

On the other hand, we have on E

$$\begin{aligned} u|_{T_1} - u|_{T_2} &= (\overline{u}|_{T_1} - \overline{u}|_{T_2}) + \frac{h_x}{2}(u_x|_{T_1} + u_x|_{T_2}) + (u_y|_{T_1} - u_y|_{T_2})(y - y_T) \\ &= (\overline{u}|_{T_1} - \overline{u}|_{T_2}) + h_x\{\!\{\nabla_h u\}\!\} \cdot \boldsymbol{n}_E + (u_y|_{T_1} - u_y|_{T_2})(y - y_T). \end{aligned}$$

Hence it follows that

$$\begin{aligned} I_E &= \frac{|E|}{|T|}\int_E \Big(|[\![\overline{u}]\!]|^2 + (u_y|_{T_1} - u_y|_{T_2})^2(y - y_T)^2 - \frac{h_x^2}{4}(u_x|_{T_1} + u_x|_{T_2})^2\Big)\, ds \\ &= \frac{|E|}{|T|}\int_E |[\![\overline{u}]\!]|^2\, ds + \frac{h_y^4}{12|T|}(u_y|_{T_1} - u_y|_{T_2})^2 - \frac{|T|}{4}(u_x|_{T_1} + u_x|_{T_2})^2. \end{aligned}$$

The result for the horizontal edge can be proved similarly. □

Now we prove are ready to derive the upper bound for $\lambda_{\max}(BA)$.

Theorem 4.4. *Let* A *and* B *be defined by* (3.2) *and* (3.1) *with the space decomposition* (3.4) *and the subspace solvers* (3.3) *and* (3.5), *respectively. Then we have*

$$\lambda_{\max}(BA) \le 3.$$

Proof. Given $u_0 \in \mathbb{P}_0(\mathcal{T}_h)$ and $u = \sum_{T \in \mathcal{T}_h} u_T$ with $u_T \in \mathbb{P}_1(T)$ for $T \in \mathcal{T}_h$, we note that for any $\alpha > 0$,

$$A(u_0 + u, u_0 + u) \le (1 + \alpha)A(u_0, u_0) + \left(1 + \frac{1}{\alpha}\right)A(u, u).$$

Thus it suffices to show that

$$A(u,u) \le 2 \sum_{T\in\mathcal{T}_h} A_T(u_T,u_T), \tag{4.4}$$

which yields the desired result by choosing $\alpha = 2$ and applying Corollary 3.2.

By using (4.2)–(4.3) with the inequality $(a-b)^2 \le 2(a^2+b^2)$ and then Lemmas 4.1–4.2, we obtain

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h}\int_T |\nabla u|^2\,d\boldsymbol{x} - 2\sum_{E\in\mathcal{E}_I}\int_E \{\!\{\nabla_h u\}\!\}\cdot[\![u]\!]\,ds + \sum_{E\in\mathcal{E}_I}\frac{|E|}{|T|}\int_E |[\![u]\!]|^2\,ds\\
&\quad\le \sum_{T\in\mathcal{T}_h}|T|(u_x^2+u_y^2) + \sum_{E\in\mathcal{E}_I}\frac{|E|}{|T|}\int_E |[\![\overline{u}]\!]|^2\,ds + \sum_{T\in\mathcal{T}_h}\frac{1}{3|T|}(h_x^4u_x^2+h_y^4u_y^2)\\
&\quad= 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds + \sum_{E\in\mathcal{E}_I}\frac{|E|}{|T|}\int_E |[\![\overline{u}]\!]|^2\,ds\\
&\quad\le 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E |u-\overline{u}|^2\,ds + 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E \overline{u}^2\,ds\\
&\quad= 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E u^2\,ds.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
A(u,u) &\le 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E u^2\,ds + \sum_{E\in\mathcal{E}_I}(\gamma-1)\frac{|E|}{|T|}\int_E |[\![u]\!]|^2\,ds\\
&\qquad - 2\sum_{E\in\mathcal{E}_B}\int_E \nabla u\cdot\boldsymbol{n}u\,ds + \sum_{E\in\mathcal{E}_B}2\gamma\frac{|E|}{|T|}\int_E u^2\,ds\\
&\le 2\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}\frac{|E|}{|T|}\int_E u^2\,ds + \sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}_T}2(\gamma-1)\frac{|E|}{|T|}\int_E u^2\,ds\\
&\qquad - 2\sum_{E\in\mathcal{E}_B}\int_E \nabla u\cdot\boldsymbol{n}u\,ds + \sum_{E\in\mathcal{E}_B}2\gamma\frac{|E|}{|T|}\int_E u^2\,ds\\
&= 2\sum_{T\in\mathcal{T}_h}A_T(u_T,u_T),
\end{aligned}$$

where the last equality is obtained by (4.1). This completes the proof. □

To derive the lower bound, we define the bilinear form $A_1(\cdot,\cdot)$

$$A_1(u,v) = \int_\Omega \nabla_h u\cdot\nabla_h v\,d\boldsymbol{x} - \sum_{E\in\mathcal{E}_h}\int_E (\{\!\{\nabla_h u\}\!\}\cdot[\![v]\!] + \{\!\{\nabla_h v\}\!\}\cdot[\![u]\!])\,ds + J_1(u,v).$$

Note that $A(u,v) = A_1(u,v) + (\gamma - 1)J_1(u,v)$. We also need the results for boundary edges corresponding to Lemma 4.3 which are stated in the following lemma. The proof is very similar, and so is omitted.

Lemma 4.5. *For $E \in \mathcal{E}_B \cap \mathcal{E}_T$, let*

$$I_E := -2\int_E \nabla u \cdot \boldsymbol{n}_T u\, ds + \frac{2|E|}{|T|}\int_E u^2\, ds.$$

Then we have for vertical E

$$I_E = \frac{2|E|}{|T|}\int_E \overline{u}^2\, ds + \frac{h_y^4}{6|T|}u_y^2|_T - \frac{|T|}{2}u_x^2|_T, \tag{4.5}$$

and for horizontal E

$$I_E = \frac{2|E|}{|T|}\int_E \overline{u}^2\, ds + \frac{h_x^4}{6|T|}u_x^2|_T - \frac{|T|}{2}u_y^2|_T. \tag{4.6}$$

Now we prove the following theorem under a mild assumption that evert rectangle in $\mathcal{T}_h$ has at most one vertical and one horizontal edges on $\partial\Omega$.

Theorem 4.6. *Let A and B be defined by* (3.2) *and* (3.1) *with the space decomposition* (3.4) *and the subspace solvers* (3.3) *and* (3.5)*, respectively. Then we have*

$$\lambda_{\min}(BA) \geq \frac{\alpha - \gamma}{\gamma\alpha - \gamma},$$

where $\alpha > \gamma$ is the larger root of the quadratic equation

$$\gamma(1-\gamma)(\alpha - 1) + \left[\left(\frac{3}{4} + \frac{1}{6}\theta^2\right)\gamma - \frac{1}{2}\right]\alpha(\alpha - \gamma) = 0.$$

Proof. We will apply Corollary 3.2 again. Given $u \in \mathbb{P}_1(\mathcal{T}_h)$, we decompose it as

$$u = \overline{u} + \sum_{T\in\mathcal{T}_h} u'_T, \quad \text{where } u'_T = \begin{cases} u|_T - \overline{u}|_T & \text{in } T, \\ 0 & \text{in } \Omega \setminus T. \end{cases}$$

It is easy to verify that for $E \in \mathcal{E}_B \cap \mathcal{E}_T$,

$$\int_E \nabla u \cdot \boldsymbol{n}(u - \overline{u})\, ds = \begin{cases} \dfrac{|T|}{2}u_x^2|_T & \text{if } E \text{ is vertical,} \\ \dfrac{|T|}{2}u_y^2|_T & \text{if } E \text{ is horozontal.} \end{cases} \tag{4.7}$$

By summing (4.2)–(4.3) over $E \in \mathcal{E}_I$ with the inequality $-(a+b)^2 \geq -2(a^2+b^2)$ and summing (4.5)–(4.6) over $E \in \mathcal{E}_B$ with Lemma 4.2 and the equality (4.7), one can obtain

$$A_1(u,u) \geq J_1(\overline{u},\overline{u}) + \sum_{E\in\mathcal{E}_B}\left(-\int_E \nabla u \cdot \boldsymbol{n}(u - \overline{u})\, ds + \frac{2|E|}{|T|}\int_E |u - \overline{u}|^2\, ds\right),$$

which leads to

$$A(\overline{u},\overline{u}) = \gamma J_1(\overline{u},\overline{u}) \le \gamma[A(u,u) + (1-\gamma)J_1(u,u)] \\ + \gamma \sum_{E\in\mathcal{E}_B} \Big(\int_E \nabla u \cdot \boldsymbol{n}(u-\overline{u})\, ds - \frac{2|E|}{|T|} \int_E |u-\overline{u}|^2\, ds \Big).$$

On the other hand, we have by (4.1)

$$\sum_{T\in\mathcal{T}_h} A_T(u'_T, u'_T) = \sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}_T} \gamma \frac{|E|}{|T|} \int_E |u-\overline{u}|^2\, ds - \sum_{E\in\mathcal{E}_B} \int_E \nabla u \cdot \boldsymbol{n}(u-\overline{u})\, ds \\ + \sum_{E\in\mathcal{E}_B} \gamma \frac{|E|}{|T|} \int_E |u-\overline{u}|^2\, ds.$$

Hence it follows by Lemma 4.2 and the equality (4.7) that

$$\begin{aligned} A(\overline{u},\overline{u}) &+ \sum_{T\in\mathcal{T}_h} A_T(u'_T, u'_T) \\ &\le \gamma A(u,u) + \gamma(1-\gamma)J_1(u,u) + \sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}_T} \gamma \frac{|E|}{|T|} \int_E |u-\overline{u}|^2\, ds \\ &\quad + \sum_{E\in\mathcal{E}_B} \Big((\gamma-1) \int_E \nabla u \cdot \boldsymbol{n}(u-\overline{u})\, ds - \gamma \frac{|E|}{|T|} \int_E |u-\overline{u}|^2\, ds \Big) \\ &\le \gamma A(u,u) + \gamma(1-\gamma)J_1(u,u) + \Big(\frac{1}{2} + \frac{1}{6}\theta^2\Big)\gamma \int_\Omega |\nabla_h u|^2\, d\boldsymbol{x} \\ &\quad + \Big(\frac{1}{2}(\gamma-1) - \frac{1}{4}\gamma\Big) \int_\Omega |\nabla_h u|^2\, d\boldsymbol{x} \\ &\le \gamma A(u,u) + \gamma(1-\gamma)J_1(u,u) + \Big[\Big(\frac{3}{4} + \frac{1}{6}\theta^2\Big)\gamma - \frac{1}{2}\Big] \int_\Omega |\nabla_h u|^2\, d\boldsymbol{x}. \end{aligned}$$

Now we obtain by using Theorem 2.1

$$A(\overline{u},\overline{u}) + \sum_{T\in\mathcal{T}_h} A_T(u'_T, u'_T) \le \Big\{\gamma + \Big[\Big(\frac{3}{4} + \frac{1}{6}\theta^2\Big)\gamma - \frac{1}{2}\Big] \frac{\alpha}{\alpha-1}\Big\} A(u,u) \\ + \Big\{\gamma(1-\gamma) + \Big[\Big(\frac{3}{4} + \frac{1}{6}\theta^2\Big)\gamma - \frac{1}{2}\Big] \frac{\alpha}{\alpha-1}(\alpha-\gamma)\Big\} J_1(u,u).$$

If we choose $\alpha > \gamma$ satisfying

$$\gamma(1-\gamma) + \Big[\Big(\frac{3}{4} + \frac{1}{6}\theta^2\Big)\gamma - \frac{1}{2}\Big] \frac{\alpha}{\alpha-1}(\alpha-\gamma) = 0,$$

TABLE 1. Block Jacobi preconditioner on square meshes. We take $\gamma = 2$.

Mesh size	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}/\lambda_{\min}$
8×8	1.96148	0.03852	50.92
16×16	1.99036	0.00963	206.51
32×32	1.99759	0.00240	829.02
64×64	1.99940	0.00060	3319.09
128×128	1.99985	0.00015	13279.37
256×256	1.99996	0.00004	53120.48

then we get

$$A(\overline{u}, \overline{u}) + \sum_{T \in \mathcal{T}_h} A_T(u'_T, u'_T) \le \left\{ \gamma + \frac{\gamma(\gamma - 1)}{\alpha - \gamma} \right\} A(u, u) = \frac{\gamma\alpha - \gamma}{\alpha - \gamma} A(u, u).$$

This completes the proof. □

Remark 4.7. *One can still get uniform bounds for the eigenvalues* $\lambda_{\min}$ …

5. NUMERICAL RESULTS

In this section we present some numerical results to confirm the theoretical results established in the previous section. We consider the model problem on the unit square $\Omega = (0, 1)^2$ and compute the maximum and minimum eigenvalues of the preconditioned system by using the MATLAB command `eigs`.

First, let us see what happens if the global coarse space $\mathbb{P}_0(\mathcal{T}_h)$ is not taken into account, i.e., the discrete matrix A is preconditioned by the block Jacobi iteration only. The results are reported in Table 1 for square meshes of varying mesh sizes, with the value $\gamma = 2$. It is observed that the maximum eigenvalues are always bounded above by 2, which can be deduced from the inequality (4.4), while the minimum eigenvalues decrease at a rate proportional to h^2. Therefore we conclude that the block Jacobi preconditioner is essentially ineffective.

In Table 2, we report the results for the preconditioned matrix system BA which incorporates the global coarse space $\mathbb{P}_0(\mathcal{T}_h)$. We take $\gamma = 2$ again. As predicted by the theory, we can see that the maximum eigenvalues are always bounded above by 3 and the minimum eigenvalues seem to be bounded below uniformly in h.

Finally, Tables 3–4 compares the theoretical estimates given in Theorem 4.6 and actual values of $\lambda_{\min}(BA)$ as a function of γ on the square mesh of size 128×128 and the rectangular mesh of size 256×64 (with $\theta = 4$), respectively, which shows that our estimates indeed give very good lower bounds for $\lambda_{\min}(BA)$.

TABLE 2. Additive Schwarz preconditioner on square meshes. We take $\gamma = 2$.

Mesh size	$\lambda_{\max}$	$\lambda_{\min}$	$\lambda_{\max}/\lambda_{\min}$
8×8	2.94849	0.28253	10.4359
16×16	2.98697	0.25818	11.5691
32×32	2.99674	0.25211	11.8864
64×64	2.99918	0.25056	11.9696
128×128	2.99980	0.25015	11.9916
256×256	2.99995	0.25004	11.9976

TABLE 3. Additive Schwarz preconditioner on the 128×128 square mesh for various values of γ

γ	$\lambda_{\max}$	$\lambda_{\min}$	Estimated $\lambda_{\min}$
2	2.99980	0.25015	0.25000
3	2.99985	0.18245	0.17264
4	2.99988	0.14014	0.13035
5	2.99990	0.11329	0.10448
10	2.99993	0.05751	0.05228
20	2.99995	0.02893	0.02612
30	2.99996	0.01933	0.01741
40	2.99996	0.01452	0.01305
50	2.99996	0.01163	0.01044

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TABLE 4. Additive Schwarz preconditioner on the 256×64 rectangular mesh for various values of γ

γ	$\lambda_{\max}$	$\lambda_{\min}$	Estimated $\lambda_{\min}$
2	2.99989	0.07526	0.07275
3	2.99991	0.06267	0.05969
4	2.99992	0.05092	0.04829
5	2.99993	0.04242	0.04017
10	2.99994	0.02270	0.02146
20	2.99995	0.01169	0.01104
30	2.99995	0.00788	0.00742
40	2.99995	0.00595	0.00559
50	2.99995	0.00478	0.00448

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