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On the Standard Completeness of an Axiomatic Extension of the Uninorm Logic* †

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【Abstract】 This paper investigates an extension of the uninorm (based) logic **UL**, which is obtained by adding (t-weakening, W_t) $((\phi \ \& \ \psi) \wedge t) \rightarrow \phi$ to **UL** introduced by Metcalfe and Montagna in [8]. First, the t-weakening uninorm logic $\mathbf{UL_{Wt}}$ (the **UL** with W_t) is introduced. The algebraic structures corresponding to $\mathbf{UL_{Wt}}$ is then defined, and its algebraic completeness is established. Next standard completeness (i.e. completeness on the real unit interval [0, 1]) is established for this logic by using Jenei and Montagna-style approach for proving standard completeness in [3, 6].

【Key Words】 (substructural) fuzzy logic, t-weakening fuzzy logic, (t-weakening) uninorm (based) logic

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1. Introduction

In this paper we investigate the standard completeness (i.e., completeness on the real unit interval [0, 1]) of an axiomatic extension of the uninorm logic UL. For this, we first recall briefly some historical facts associated with fuzzy logic.

Many-valued logics with truth values in the real unit interval [0, 1] have a long and distinguished history, and the well-known examples are the infinite-valued systems **Ł** (Łukasiewicz logic) and **G** (Gödel-Dummett logic). In particular, in the last decade Hájek [5] introduced **BL** (Basic fuzzy logic) and showed that **Ł**, **G**, and Π (Product logic) are its extensions. In this approach, (multiplicative) conjunction connectives are interpreted by t-norms (see [5]), which are commutative, associative, monotonic binary functions with identity 1. **BL** is the most important logic of *continuous* t-norms, and **Ł**, **G**, and Π are emerging in this respect as fundamental examples of logics based on continuous t-norms. Esteva and Godo further [2] introduced the logic of *left-continuous* t-norms **MTL** (Monoidal t-norm logic), which copes with the logic of *left-continuous* t-norms and their residua, as a weakening of **BL**.

While fuzzy logics based on t-norms prove the weakening (W) $\phi \rightarrow (\psi \rightarrow \phi)$, some fuzzy logics (not based on t-norms) do not. For instance, weakening-free fuzzy systems have been recently introduced by Metcalfe. More exactly, Metcalfe (and Montagna) [7, 8] introduced the weakening-free fuzzy logics **UL**,

IUL (Involutive uninorm logic), **UML** (Uninorm mingle logic), and **IUML** (Involutive uninorm mingle logic) as substructural fuzzy logics based on *uninorms*, which are functions introduced by Yager and Rybalov [10] as a generalization of t-norms where the identity can lie anywhere in [0, 1].

Axiomatizations of the above t-norm and uninorm based logics are complete with respect to (w.r.t) linearly ordered algebras; and following Cintula [1], a (weakly implicative) logic **L** is said to be *fuzzy* if **L** is complete w.r.t. linearly ordered matrices (or algebras). Then, the above systems are all fuzzy logics in the Cintula's sense. Notice that they are also complete (so called standard complete) w.r.t. algebras with lattice reduct [0, 1]. One method introduced in [3, 6] for **MTL** and its axiomatic extensions (calling it *Jenei and Montagna's method*), consists of showing that countable linearly ordered algebras of a given variety can be embedded into linearly and *densely* ordered members of the same variety, which can in turn be embedded into algebras with lattice reduct [0, 1]. But this method (seems to) fail with associativity for **UL**, and so does not (appear to) work in general for weakening-free fuzzy logics such as **UL** based on uninorms. Because of this negative fact Metcalfe and Montagna [8] instead introduced a new approach for proving standard completeness of uninorm logics, consisting of the following two steps: 1. after extending logics with density rule, showing that such systems are complete w.r.t. linearly and densely ordered algebras, and for particular extensions are complete w.r.t. those algebras with lattice reduct [0, 1]; 2. giving a syntactic elimination of density rule (as a rule of the corresponding hypersequent calculus), i.e., showing that if ϕ is derivable in a uninorm logic **L** extended with density rule, then it is also derivable in **L**.

The starting point for the current work is the observation that t-norms are uninorms. As we mentioned above, while t-norms have unit at 1, uninorms does instead unit lying anywhere in [0, 1]. Then a natural concern arises about for which uninorm logics Metcalfe and Montagna's strategy being able to work. Since **MTL** is the logic of left-continuous t-norms, this strategy of course works for t-norms, i.e., uninorms having identity 1. We here show that it works for other uninorms, i.e., uninorms not being t-norms. More exactly, we show that Jenei and Montagna-style approach may work for logics based on uninorms with a weak form of weakening (called the *t-weakening*), i.e., for *t-weakening uninorm (based) logics.*

The paper is organized as follows. In Section 2 we present axiomatization of $\mathbf{ULw_t}$, which is obtained by adding (t-weakening, W_t) $((\phi \ \& \ \psi) \wedge \mathbf{t}) \rightarrow \phi$ to **UL**; and in Section 3 then define algebraic structures of $\mathbf{ULw_t}$, by a subvariety of the variety of t-weakening commutative residuated lattices (i.e., the variety of $\mathrm{ULw_t}$-algebras), and show that $\mathbf{ULw_t}$ is complete w.r.t. linearly ordered $\mathbf{ULw_t}$-algebras. This will ensure that $\mathbf{ULw_t}$ is fuzzy in the Cintula's sense. After defining **t**-weakening uninorms in Section 4, in Section 5 we finally provide standard completeness results for $\mathbf{ULw_t}$,, using the method introduced in [3, 6], i.e., Jenei and Montagna's method.

For convenience, we shall adopt the notation and terminology similar to those in [1, 3, 5, 8], and assume being familiar with them (together with results found in them).

2. Syntax

We base the t-weakening fuzzy logic **ULw$_t$** on a countable propositional language with formulas *FOR* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\rightarrow$, $\&$, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\sim\phi := \phi \rightarrow \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

We may define **t** as $\mathbf{f} \rightarrow \mathbf{f}$. We moreover define ϕ^n_t as $\phi_t \ \& \cdots \& \ \phi_t$, n factors, where $\phi_t := \phi \wedge \mathbf{t}$. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **ULw$_t$** (**UL** plus t-weakening) as a t-weakening (substructural) fuzzy logic.

Definition 2.1 **ULw$_t$** consists of the following axiom schemes and rules:

A1. $\phi \rightarrow \phi$ (self-implication, SI)
A2. $(\phi \wedge \psi) \rightarrow \phi$, $(\phi \wedge \psi) \rightarrow \psi$ ($\wedge$-elimination, $\wedge$-E)
A3. $((\phi\rightarrow\psi)\wedge(\phi\rightarrow\chi)) \rightarrow (\phi\rightarrow(\psi\wedge\chi))$ ($\wedge$-introduction, $\wedge$-I)

A4. $\phi \rightarrow (\phi \vee \psi)$, $\psi \rightarrow (\phi \vee \psi)$ ($\vee$-introduction, $\vee$-I)

A5. $((\phi \rightarrow \chi) \wedge (\psi \rightarrow \chi)) \rightarrow ((\phi \vee \psi) \rightarrow \chi)$ ($\vee$-elimination, $\vee$-E)

A6. $\phi \rightarrow \mathbf{T}$ (verum ex quolibet, VE)

A7. $\mathbf{F} \rightarrow \phi$ (ex falso quadlibet, EF)

A8. $(\phi \& \psi) \rightarrow (\psi \& \phi)$ (&-commutativity, &-C)

A9. $(\phi \& \mathbf{t}) \leftrightarrow \phi$ (push and pop, PP)

A10. $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$ (suffixing, SF)

A11. $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow ((\phi \& \psi) \rightarrow \chi)$ (residuation, RE)

A12. $(\phi \& \psi)_t \rightarrow \phi$ (t-weakening, $\mathbf{W}_t$)

A13. for each n, $(\phi \rightarrow \psi)^n_t \vee (\psi \rightarrow \phi)^n_t$ (^n_t-prelinearity, PL^n_t).

$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)

$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj)

Proposition 2.2 $\mathbf{ULw_t}$ proves:

(1) $(\phi \& (\psi \& \chi)) \leftrightarrow ((\phi \& \psi) \& \chi)$ (&-associativity, AS).

In $\mathbf{ULw_t}$, $\mathbf{f}$ can be defined as $\sim\mathbf{t}$ and vice versa. A *theory* over $\mathbf{ULw_t}$ is a set T of formulas. A *proof* in a sequence of formulas whose each member is either an axiom of $\mathbf{ULw_t}$ or a member of T or follows from some preceding members of the sequence using the rules (mp) and (adj). T $\vdash \phi$, more exactly T $\vdash_{\mathbf{ULwt}} \phi$, means that ϕ is *provable* in T w.r.t. $\mathbf{ULw_t}$, i.e., there is a $\mathbf{ULw_t}$-proof of ϕ in T. The relevant deduction theorem (RDT) for $\mathbf{ULw_t}$ is as follows:

Proposition 2.3 Let T be a theory, and ϕ, ψ formulas. T $\cup$ $\{\phi\} \vdash_{\mathbf{ULwt}} \psi$ iff there is n such that T $\vdash_{\mathbf{ULwt}} \phi^n_t \rightarrow \psi$.

Proof: See [9]. □

A theory T is *inconsistent* if T $\vdash$ **F**; otherwise it is *consistent*. For convenience, "$\sim$", "$\wedge$", "$\vee$", and "$\rightarrow$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

3. Semantics

Suitable algebraic structures for **ULw**$_t$ are obtained as a subvariety of the variety of commutative monoidal residuated lattices.

Definition 3.1 A *pointed bounded commutative residuated t-weakening lattice* is a structure **A** = $(A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *, \rightarrow)$ such that:

(I) $(A, \top, \bot, \wedge, \vee)$ is a bounded lattice with top element $\top$ and bottom element $\bot$.

(II) $(A, *, \top_t)$ satisfies for some $\top_t$ and for all x, y, z $\in$ A,

- (a) $x * y = y * x$ (commutativity)
- (b) $\top_t * x = x$ (identity)
- (c) $x * (y * z) = (x * y) * z$ (associativity).

(III) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all x, y, z $\in$ A (residuation).

(IV) $(x * y) \wedge \top_t \leq x$, for all x, y $\in$ A (t-weakening).

$(A, *, \top_t)$ satisfying (II-b, c) is a *monoid*. Thus $(A, *, \top_t)$ satisfying (II-a, b, c) is a commutative monoid. To define the above lattice we may take in place of (III) a family of equations as in [5]. Using $\to$ and $\bot_f$ we can define $\top_t$ as $\bot_f \to \bot_f$, and $\sim$ as in (df1). In the lattice, $\sim$ is a *weak* negation in the sense that for all x, $x \leq \sim\sim x$ holds in it. Then, $\mathrm{ULw_t}$-algebra whose class characterizes $\mathbf{ULw_t}$ is defined as follows.

Definition 3.2 ($\mathrm{ULw_t}$-algebra) A *$\mathrm{ULw_t}$-algebra* is a pointed bounded commutative residuated **t**-weakening lattice satisfying the condition: for all x, y, and for each n ($\geq$ 1),

(plt) $\top t \leq (x \to y)^n{}_{\top t} \vee (y \to x)^n{}_{\top t}$.

$\mathrm{ULw_t}$-algebra is said to be *linearly ordered* if the ordering of its algebra is linear, i.e., $x \leq y$ or $y \leq x$ (equivalently, $x \wedge y = x$ or $x \wedge y = y$) for each pair x, y. In linearly ordered algebras, we in particular call monoids satisfying (IV) *t-weakening monoids*.

Definition 3.3 (Evaluation) Let $\mathcal{A}$ be an algebra. An *$\mathcal{A}$-evaluation* is a function v : FOR $\to \mathcal{A}$ satisfying:

$v(\phi \to \psi) = v(\phi) \to v(\psi)$,

$v(\phi \wedge \psi) = v(\phi) \wedge v(\psi)$,

$v(\phi \vee \psi) = v(\phi) \vee v(\psi)$,

$v(\phi \,\&\, \psi) = v(\phi) * v(\psi)$,

$v(\mathbf{F}) = \bot$,

$v(\mathbf{f}) = \bot_f$,

(and hence $v(\sim\phi) = \sim v(\phi)$, $v(\mathbf{T}) = \top$, and $v(\mathbf{t}) = \top_t$).

Definition 3.4 Let $\mathcal{A}$ be a ULw_t-algebra, T a theory, ϕ a formula, and K a class of ULw_t-algebras.

(i) (Tautology) ϕ is a *$\top_t$-tautology* in $\mathcal{A}$, briefly an *$\mathcal{A}$-tautology* (or *$\mathcal{A}$-valid*), if $v(\phi) \geq \top_t$ for each $\mathcal{A}$-evaluation v.

(ii) (Model) An $\mathcal{A}$-evaluation v is an *$\mathcal{A}$-model* of T if $v(\phi) \geq \top_t$ for each $\phi \in$ T. By *Mod(T, $\mathcal{A}$)*, we denote the class of $\mathcal{A}$-models of T.

(iii) (Semantic consequence) ϕ is a *semantic consequence* of T w.r.t. K, denoting by $T \vDash_K \phi$, if Mod(T, $\mathcal{A}$) = Mod(T $\cup$ $\{\phi\}$, $\mathcal{A}$) for each $\mathcal{A} \in$ K.

Definition 3.5 (ULw_t-algebra) Let $\mathcal{A}$, T, and ϕ be as in Definition 3.4. $\mathcal{A}$ is a *ULw_t-algebra* iff whenever ϕ is ULw_t-provable in T (i.e. $T \vdash_{ULwt} \phi$), it is a semantic consequence of T w.r.t. the set $\{\mathcal{A}\}$ (i.e. $T \vDash_{\{A\}} \phi$), $\mathcal{A}$ a ULw_t-algebra. By *$MOD^{(l)}(ULw_t)$*, we denote the class of (linearly ordered) ULw_t-algebras. Finally, we write $T \vDash^{(l)}_{ULwt} \phi$ in place of $T \vDash_{MOD^{(l)}(ULwt)} \phi$.

Note that since each condition for the ULw_t-algebra has a form of equation or can be defined in equation (exercise), it can be ensured that the class of all ULw_t-algebras is a variety.

Let A be a ULw_t-algebra. We first show that classes of provably equivalent formulas form a ULw_t-algebra. Let T be a fixed theory over ULw_t. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that T $\vdash_{ULwt}$ $\phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). A_T is the set of all the classes $[\phi]_T$. We define that $[\phi]_T \rightarrow [\psi]_T = [\phi \rightarrow \psi]_T$, $[\phi]_T * [\psi]_T = [\phi \mathbin{\&} \psi]_T$, $[\phi]_T \wedge [\psi]_T = [\phi \wedge \psi]_T$, $[\phi]_T \vee [\psi]_T = [\phi \vee \psi]_T$, $\bot = [\mathbf{F}]_T$, $\top = [\mathbf{T}]_T$, $\top_t = [\mathbf{t}]_T$, and $\bot_f = [\mathbf{f}]_T$. By $\mathbf{A}_T$, we denote this algebra.

Proposition 3.6 For T a theory over L, $\mathbf{A}_T$ is a **ULw$_t$**-algebra.

Proof: Note that A1 to A7 ensure that $\wedge$ and $\vee$ satisfy (I) in Definition 3.1; that AS, A8, A9 ensure that & satisfies (II); that A11, A12 and A13 ensure that (III), (IV), and (pl^n_t) hold. It is obvious that $[\phi]_T \leq [\psi]_T$ iff T $\vdash_{ULwt}$ $\phi \leftrightarrow (\phi \wedge \psi)$ iff T $\vdash_{ULwt}$ $\phi \rightarrow \psi$. Finally recall that $\mathbf{A}_T$ is a **ULw$_t$**-algebra iff T $\vdash_{ULwt}$ ψ implies T $\models_{ULwt}$ ψ, and observe that for ϕ in T, since T $\vdash_{ULwt}$ $\mathbf{t} \rightarrow \phi$, it follows that $[\mathbf{t}]_T \leq [\phi]_T$. Thus it is a **ULw$_t$**-algebra. □

We next note that the nomenclature of the prelinearity condition is explained by the subdirect representation theorem below.

Proposition 3.7 Each ULw$_t$-algebra is a subdirect product of linearly ordered ULw$_t$-algebras.

Proof: Its proof is analogous to that of Lemma 3.7 in [1]. □

Theorem 3.8 (Strong completeness) Let T be a theory, and ϕ a formula. T $\vdash_{ULwt}$ ϕ iff T $\models_{ULwt}$ ϕ iff T $\models^{1}_{ULwt}$ ϕ.

Proof: (i) T $\vdash_{ULwt}$ ϕ iff T $\models_{ULwt}$ ϕ. Left to right follows from definition. Right to left is as follows: from Proposition 3.6, we obtain $A_T \in$ MOD(L), and for A_T-evaluation v defined as v (ψ) = $[\psi]_T$, it holds that v $\in$ Mod(T, A_T). Thus, since from T $\models_{ULwt}$ ϕ we obtain that $[\phi]_T$ = v(ϕ) $\geq$ $\top_t$, T $\vdash_{ULwt}$ **t** $\to$ ϕ. Then, since T $\vdash_{ULwt}$ **t**, by (mp) T $\vdash_{ULwt}$ ϕ, as required.

(ii) T $\models_{ULwt}$ ϕ iff T $\models^{1}_{ULwt}$ ϕ. It follows from Proposition 3.7. □

4. t-Weakening uninorms and their residua

In this section, using *1, 0, and some 1_t, and 0_f* in the real unit interval [0, 1], we shall express $\top$, $\bot$, $\top_t$, and $\bot_f$, respectively. We also define standard ULw_t-algebras and t-weakening uninorms on [0, 1].

Definition 4.1 A ULw_t-algebra is *standard* iff its lattice reduct is [0, 1].

In standard ULw_t-algebras the monoid operator * is a t-weakening uninorm.

Definition 4.2 A *t-weakening uninorm* is a function $\circ$: $[0, 1]^2$ $\to$ [0, 1] such that for some $1_t \in$ [0, 1] and for all x, y, z $\in$ [0, 1]:

(a) $x \odot y = y \odot x$ (commutativity),
(b) $x \odot (y \odot z) = (x \odot y) \odot z$ (associativity),
(c) $x \leq y$ implies $x \odot z \leq y \odot z$ (monotonicity),
(d) $1_t \odot x = x$ (identity), and
(e) $\min\{x \odot y, 1_t\} \leq x$ (t-weakening).

The function $\odot$ satisfying (a) to (d) is a *uninorm*, and uninorm satisfying (1-identity) $1_t = 1$ is a *t-norm*. Notice that (t-weakening) and (1-identity) ensure that for all $x, y \in [0, 1]$,

$x \odot y \leq \min\{x, y\}$ or $\max\{x, y\} \leq x \odot y$, and
$x \odot y \leq \min\{x, y\}$, respectively.

This shows that t-norm is a t-weakening uninorm.

$\odot$ is *residuated* iff there is $\rightarrow : [0, 1]^2 \rightarrow [0, 1]$ satisfying (residuation) on $[0, 1]$. A uninorm is called *conjunctive* if $0 \odot 1 = 0$, and *disjunctive* if $0 \odot 1 = 1$. For some $0_f \in [0, 1]$, a residuated uninorm has weak negation n defined as $n(x) := x \rightarrow 0_f$ because $x \odot (x \rightarrow 0_f) \leq 0_f$ holds in it and so by residuation $x \odot (x \rightarrow 0_f) \leq 0_f$ iff $x \leq (x \rightarrow 0_f) \rightarrow 0_f$.

The most important property of a uninorm is that *left-continuity* holds in it. Given a uninorm $\odot$, *residuated implication* $\rightarrow$ determined by $\odot$ is defined as $x \rightarrow y := \sup\{z \in [0, 1]: x \odot z \leq y\}$ for all $x, y \in [0, 1]$. Then, as in uninorm, we can show that for any t-weakening uninorm $\odot$, $\odot$ and its residuated implication $\rightarrow$ form a residuated pair iff $\odot$ is conjunctive and left-continuous in both arguments (cf. see Proposition 5.4.2 [4]).

5. Standard completeness

We first show that finite or countable linearly ordered ULw_f-algebras are embeddable into a standard algebra. (For convenience, we add less than relation symbol to such algebras.)

Proposition 5.1 For every finite or countable linearly ordered ULw_f-algebra $\mathbf{A} = (A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *, \rightarrow)$, there is a countable ordered set X, a binary operation $\bigcirc$, and a map f from A into X such that the following conditions hold:

(I) X is densely ordered, and has a maximum Max, a minimum Min, and special elements e, ∂.

(II) (X, $\bigcirc$, $\leq$, e) is a linearly ordered monotonic commutative t-weakening monoid.

(III) $\bigcirc$ is conjunctive and left-continuous with respect to the order topology on (X, $\leq$).

(IV) f is an embedding of the structure $(A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *)$ into $(X, \leq, \mathrm{Max}, \mathrm{Min}, e, \partial, \min, \max, \bigcirc)$, and for all m, n $\in$ A, $f(m \rightarrow n)$ is the residuum of f(m) and f(n) in $(X, \leq, \mathrm{Max}, \mathrm{Min}, e, \partial, \max, \min, \bigcirc)$.

Proof: For convenience, we assume A as a subset of $\mathbf{Q} \cap [0, 1]$ with finite or countable elements, where 0 and 1 are least and greatest elements and some 1_t and any 0_f are special elements, each of which corresponds to $\top$, $\bot$, and some $\top_t$, $\bot_f$, respectively. Let

$$X = \{(m, x): m \in A \setminus \{0 (= \bot)\} \text{ and } x \in \mathbf{Q} \cap (0, m]\} \cup \{(0, 0)\}.$$

For $(m, x), (n, y) \in X$, we define:

$(m, x) \preceq (n, y)$ iff either $m <_A n$, or $m =_A n$ and $x \leq y$.

It is clear that $\preceq$ is a linear order with maximum $(1, 1)$, minimum $(0, 0)$, and special elements $e = (1_t, 1_t)$, $\partial = (0_f, 0_f)$. Furthermore, $\preceq$ is dense: let $(m, x) \prec (n, y)$. Then either $m <_A n$ or $m =_A n$ and $x < y$. If the first is the case, then $(m, x) \prec (n, y/2) \prec (n, y)$. Otherwise, then $(m, x) \prec (n, x+y/2) \prec (n, y)$. This proves (I).

For convenience, we will from now on drop the index A in $<_A$ and $=_A$, if we need not distinguish them. But context should make clear what we mean.

Define for $(m, x), (n, y) \in X$:

$$(m,x) \circ (n,y) = \begin{cases} \max\{(m,x), (n,y)\} & \text{if } m * n = m \vee n,\ m \neq n, \text{ and } (m, x) \preceq e \text{ or } (n, y) \preceq e; \\ \min\{(m,x), (n,y)\} & \text{if } m * n = m \wedge z, \text{ and } (m, x) \preceq e \text{ or } (n, y) \preceq e; \\ (m * n, m * n) & \text{otherwise.} \end{cases}$$

We verify that $\circ$ satisfies (II) (noting that t-weakening of $*$ ensures that (MM) for all $m, n \in A$, $m * n \leq m \wedge n$ or m

$\vee$ n $\leq$ m * n).

(1) Commutativity. It is obvious that $\bigcirc$ is commutative.

(2) Identity. We prove that $(1_t, 1_t)$ is the unit element, i.e., $(1_t, 1_t) \bigcirc (m, x) = (m, x)$. (i) Let $(1_t, 1_t) < (m, x)$. Since $\top_t * m = m = \top_t \vee m$, $(1_t, 1_t) \bigcirc (m, x) = \max\{(1_t, 1_t), (m, x)\} = (m, x)$. (ii) Let $(m, x) \leq (1_t, 1_t)$. Since $\top_t * m = m = \top_t \wedge m$, $(1_t, 1_t) \bigcirc (m, x) = \min\{(1_t, 1_t), (m, x)\} = (m, x)$.

(3) Monotonicity. Since $\bigcirc$ is commutative, it suffices to prove that if $(l, x) \leq (m, y)$, then for all $(n, z) \in X$, $(l, x) \bigcirc (n, z) \leq (m, y) \bigcirc (n, z)$. We distinguish several cases:

● Case (i). $l * n = l \vee n$ and $m * n = m \vee n$:

Subcase (i-a). $(l, x) \leq e$ or $(n, z) \leq e$.

(a-1) $(m, y) \leq e$ or $(n, z) \leq e$. If $(n, z) \leq e < (l, x) \leq (m, y)$, $(l, x) \bigcirc (n, z) = \max\{(l, x), (n, z)\} = (l, x) \leq (m, y) = \max\{(m, y), (n, z)\} = (m, y) \bigcirc (n, z)$. If $(l, x) \leq (m, y) \leq e < (n, z)$, $(l, x) \bigcirc (n, z) = \max\{(l, x), (n, z)\} = (n, z) = \max\{(m, y), (n, z)\} = (m, y) \bigcirc (n, z)$. If $(l, x), (n, z) \leq e < (m, y)$, $(l, x) \bigcirc (n, z) = \min\{(l, x), (n, z)\} < (m, y) = (m, y) \bigcirc (n, z)$. If $(l, x), (m, y), (n, z) \leq e$, $l = m = n$ and so $(l, x) \bigcirc (n, z) = \min\{(l, x), (n, z)\} \leq \min\{(m, y), (n, z)\} = (m, y) \bigcirc (n, z)$.

(a-2) $(m, y), (n, z) > e$. Then $(l, x) \leq e < (m, y), (n, z)$. Thus $(l, x) \bigcirc (n, z) = \max\{(l, x), (n, z)\} = (n, z) \leq (m \vee n, m \vee n) = (m, y) \bigcirc (n, z)$.

Subcase (i-b). $(l, x), (n, z) > e$.

(b-1) $(m, y) \leq e$ or $(n, z) \leq e$. It is not the case because (l,

x), (n, z) $\succ$ e implies that m $<$ 1_t and so l $>$ m, contrary to the supposition that (l, x) $\preceq$ (m, y).
(b-2) (m, y), (n, z) $\succ$ e. Then (l, x) $\bigcirc$ (n, z) = (l $\vee$ n, l $\vee$ n) $\preceq$ (m $\vee$ n, m $\vee$ n) = (m, y) $\bigcirc$ (n, z).

● Case (ii). l * n = l $\wedge$ n and m * n = m $\wedge$ n. Its proof is analogous to that of Case (i).

● Case (iii). l * n = l $\vee$ n and m * n $\neq$ m $\vee$ n. We need to consider the subcases (a) m * n = m $\wedge$ n and (b) m * n $\neq$ m $\wedge$ n.
Subcase (iii-a). m * n = m $\wedge$ n. Since m * n = m $\wedge$ n and so m $\neq$ n, l = n $<$ m, 1_t. Then (l, x) $\bigcirc$ (n, z) = min{(l, x), (n, z)} $\preceq$ min{(m, y), (n, z)} = (m, y) $\bigcirc$ (n, z).
Subcase (iii-b). m * n $\neq$ m $\wedge$ n:
(b-1) m * n $>$ 1_t. Then, since l * n $\leq$ m * n and (m, y) $\bigcirc$ (n, z) = (m * n, m * n), (l, x) $\bigcirc$ (n, z) $\preceq$ (m, y) $\bigcirc$ (n, z).
(b-2) m * n $\leq$ 1_t. Since this implies that l = n = l * n $<$ m * n $\leq$ 1_t, (l, x) $\bigcirc$ (n, z) $\prec$ (m, y) $\bigcirc$ (n, z).

● Case (iv). l * n $\neq$ l $\vee$ n and m * n = m $\vee$ n. Its proof is analogous to that of Case (iii).

● Case (v). l * n $\neq$ l $\vee$ n, l $\wedge$ n, and m * n $\neq$ m $\vee$ n, m $\wedge$ n.
Subcase (v-a). l * n, m * n $>$ 1_t. (l, x) $\bigcirc$ (n, z) = (l * n, l * n) $\preceq$ (m * n, m * n) = (m, y) $\bigcirc$ (n, z).

Subcase (v-b). $l * n \leq 1_t < m * n$. Since $l * n < m * n$, $(l, x) \circ (n, z) \prec (m, y) \circ (n, z)$.

Subcase (v-c). $l * n > 1_t \geq m * n$. By the supposition, this is not the case.

Subcase (v-d). Otherwise, i.e., $l * n, m * n \leq 1_t$. $(l, x) \circ (n, z) = (l * n, l * n) \preceq (m * n, m * n) = (m, y) \circ (n, z)$.

(4) e-Weakening. We assume that for all $(m, x), (n, y) \in X$, $(m, x) \circ (n, y) \preceq e$, and show that $(m, x) \circ (n, y) \preceq (m, x)$ (noting that by t-weakening of $*$, $m * n = m \vee n$, $m \neq n$, is not the case.) (i) Let $m * n = m \wedge n \leq 1_t$. Then, $(m, x) \circ (n, y) = \min\{(m, x), (n, y)\} \preceq (m, x)$. (ii) Let $m * n \neq m \wedge n$. By (MM), $m * n < m \wedge n$. Hence $(m, x) \circ (n, y) \prec (m, x)$.

(5) Associativity. We show that for all $(l, x), (m, y), (n, z) \in X$,

$(l, x) \circ ((m, y) \circ (n, z)) = ((l, x) \circ (m, y)) \circ (n, z)$ (AS).

Without further mention, we will use the fact that $*$ is associative and t-weakening. We distinguish several cases:

● Case (i). $l * (m * n) = \vee(l, m, n)$. Then by (MM) and the supposition, either $1_t \leq l, m, n$ and $1_t < l * (m * n)$ or $1_t \geq l, m, n$ and $l = m = n$. Let the first be the case. If $1_t = l = m < n$, $1_t = l = n < m$, or $1_t = m = n < l$, then both sides of

(AS) are equal to max{(l, x), (m, y), (n, z)}. Otherwise, both sides of (AS) are equal to (l * (m * n), l * (m * n)) (= (∨(l, m, n), ∨(l, m, n))). Let the second be the case. Then both sides of (AS) are equal to min{(l, x), (m, y), (n, z)}.

● Case (ii). l * (m * n) = ∧ (l, m, n). If 1_t < l = m = n, both sides of (AS) are equal to (l * (m * n), l * (m * n)) (= (l, l)). Otherwise, both sides of (AS) are equal to min{(l, x), (m, y), (n, z)}.

● Case (iii). l * (m * n) ≠ ∨(l, m, n), ∧ (l, m, n), and l * (m * n) ∈ {l, m, n}. This is not the case because ∨(l, m, n) ≤ l * (m * n) or l * (m * n) ≤ ∧ (l, m, n) by (MM).

● Case (iv). l * (m * n) ∉ {l, m, n} and either l * (m * n) = l ∨ (m * n) = m * n or l * (m * n) = l ∧ (m * n) = m * n. Then, since (MM) ensures that either 1_t ≤ l, m ∨ n < m * n or 1_t ≥ l, m ∧ n > m * n, both sides of (AS) are equal to (m * n, m * n).

● Case (v). l * (m * n) ∉ {l, m, n} and l * (m * n) ≠ l ∨ (m * n), l ∧ (m * n). Then, we need to consider the cases l * (m * n) > 1_t and l * (m * n) ≤ 1_t. In an analogy to the above, we can prove this.

We then prove (III). Since 0 * 1 = 0, it is immediate that ○ is conjunctive, i.e., (0, 0) ○ (1, 1) = (0, 0).

For left-continuity of $\bigcirc$, we prove that if $<(m_i, x_i): i \in \mathbf{N}>$ is any increasing sequence (w.r.t. $\preceq$) of elements of X such that $\sup\{(m_i, x_i): i \in \mathbf{N}\} = (m, x)$, then for all $(n, y) \in X$, $\sup\{(m_i, x_i) \bigcirc (n, y): i \in \mathbf{N}\} = (m, x) \bigcirc (n, y)$. Note that for almost all i, $m_i = m$ (otherwise $(m, x/2) \prec (m, x)$ would be an upper bound of the sequence $<(m_i, x_i): i \in \mathbf{N}>$). By deleting a finite number of elements of the sequence $<(m_i, x_i): i \in \mathbf{N}>$, we can suppose that for all i, $m_i = m$ and that $x = \sup\{x_i: i \in \mathbf{N}\}$. Then we need to consider the following cases:

Case (i). $m * n = m \vee n$. In case $m \geq 1_t$ or $n \geq 1_t$, $(m, x) \bigcirc (n, y) = \max\{(m, x), (n, y)\}$, $(m_i, x_i) \bigcirc (n, y) = \max\{(m_i, x_i), (n, y)\}$, and left-continuity follows from left-continuity of max operation. Otherwise, i.e., if $m = n < 1_t$, $(m, x) \bigcirc (n, y) = \min\{(m, x), (n, y)\}$ and for all i, $(m_i, x_i) \bigcirc (n, y) = (\min\{(m_i, x), (n, y)\})$, and left-continuity follows from left-continuity of min operation.

Case (ii). $m * n = m \wedge n$. Its proof is analogous to that of Case (i).

Case (iii). $m * n \neq m \vee n, m \wedge n$, and $m \neq n$. Then, $(m, x) \bigcirc (n, y) = (m * n, m * n)$ and for all i, $(m_i, x_i) \bigcirc (n, y) = (m_i * n, m_i * n) = (m * n, m * n)$. Thus $(m, x) \bigcirc (n, y) = (m_i, x_i) \bigcirc (n, y)$.

This completes the proof of (III).

We finally prove (IV). First define for every $m \in A$,

$$f(m) = (m, m).$$

It is clear that f is increasing and so one-to-one. f(1), f(0), $f(1_t)$, and $f(0_f)$ are top, bottom, and special elements of $(X, \preceq)$; and $f(1_t)$ is the unit element of $\bigcirc$. We then show that $f(m) \bigcirc f(n) = f(m * n)$:

Case (i). $1_t < m, n$. $f(m) \bigcirc f(n) = (m, m) \bigcirc (n, n) = (m * n, m * n) = f(m * n)$.

Case (ii). $m \leq 1_t < n$.

Subcase (ii-a). $m * n = m \vee n$. $f(m) \bigcirc f(n) = (m, m) \bigcirc (n, n) = \max\{(m, m), (n, n)\} = (n, n) = f(n) = f(m * n)$.

Subcase (ii-b). $m * n = m \wedge n$. $f(m) \bigcirc f(n) = (m, m) \bigcirc (n, n) = \min\{(m, m), (n, n)\} = (m, m) = f(m) = f(m * n)$.

Subcase (ii-c). $m * n \neq m \vee n, m \wedge n$. $f(m) \bigcirc f(n) = (m, m) \bigcirc (n, n) = (m * n, m * n) = f(m * n)$.

Case (iii). $n \leq 1_t < m$. Its proof is analogous to that of Case (ii).

Case (iv). $1_t \geq m, n$. Its proof is analogous to that of Case (ii). Thus f is an embedding of partially ordered monoids. It remains to prove that for every $l, m, n \in A$, $f(l \rightarrow m)$ is the residuum of f(l) and f(m) w.r.t. $\bigcirc$, i.e., (i) $f(l) \bigcirc f(l \rightarrow m) \preceq f(m)$, and (ii) if $f(l) \bigcirc (n, z) \preceq f(m)$, then $(n, z) \preceq f(l \rightarrow m)$.

(i). Consider the case $1_t < l \leq m$. $f(l) \bigcirc f(l \rightarrow m) = (l, l) \bigcirc (l \rightarrow m, l \rightarrow m) = (l * (l \rightarrow m), l * (l \rightarrow m)) \preceq (m, m) = f(m)$. Proof of the other cases is analogous.

(ii). By contraposition, we prove this. Suppose that $f(l \rightarrow m) \prec (n, z)$, i.e., $(l \rightarrow m, l \rightarrow m) \prec (n, z)$. Since $l \rightarrow m$ is the residuum of l and m in A, $m < l * n$. Thus $(m, m) \prec (l, l) \bigcirc (n, z)$. This completes the proof. □

Proposition 5.2 Every countable linearly ordered $\mathbf{ULw_f}$-algebra can be embedded into a standard algebra.

Proof: In an analogy to the proof of Theorem 3.2 in [6], we prove this. Let X, **A**, etc. be as in Proposition 5.1. Since (X, $\preceq$) is a countable, dense, linearly-ordered set with maximum and minimum, it is order isomorphic to ($\mathbf{Q} \cap [0, 1]$, $\leq$). Let g be such an isomorphism. If (I), (II), (III), and (IV) hold, letting for $\alpha, \beta \in [0, 1]$, $\alpha \circ' \beta = g(g^{-1}(\alpha) \circ g^{-1}(\beta))$, and, for all m $\in$ A, $f'(m) = g(f(m))$, we obtain that $\mathbf{Q} \cap [0, 1]$, $\leq$, 1, 0, 1_t, 0_f, $\circ'$, f' satisfy the conditions (I) to (IV) of Proposition 5.1 whenever X, $\preceq$, Max, Min, e, ∂, $\circ$, and f do. Thus we can without loss of generality assume that X = $\mathbf{Q} \cap [0, 1]$ and $\preceq$ = $\leq$.

Now we define for $\alpha, \beta \in [0, 1]$,

$$\alpha \circ'' \beta = \sup_{x \in X: x \leq \alpha} \sup_{y \in X: y \leq \beta} x \circ y.$$

Commutativity of $\circ''$ follows from that of $\circ$. Its monotonicity, identity, and e-weakening are easy consequences of the definition. Furthermore, it follows from the definition that $\circ''$ is conjunctive, i.e., $0 \circ'' 1 = 0$.

We prove left-continuity. Suppose that $<\alpha_n: n \in \mathbf{N}>$, $<\beta_n: n \in \mathbf{N}>$ are increasing sequences of reals in [0, 1] such that $\sup\{\alpha_n: n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n: n \in \mathbf{N}\} = \beta$. By the monotonicity of $\circ''$, $\sup\{\alpha_n \circ'' \beta_n\} = \alpha \circ'' \beta$. Since the restriction of $\circ''$ to $\mathbf{Q} \cap [0, 1]$ is left-continuous, we obtain that

$$\alpha \circ'' \beta = \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q \leq \alpha, r \leq \beta\}$$
$$= \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\}.$$

For each $q < \alpha$, $r < \beta$, there is n such that $q < \alpha_n$ and $r < \beta_n$. Thus,

$$\sup\{\alpha_n \circ'' \beta_n: n \in \mathbf{N}\} \geq \sup\{q \circ'' r: q, r \in \mathbf{Q} \cap [0, 1], q < \alpha, r < \beta\} = \alpha \circ'' \beta.$$

Hence, $\circ''$ is a left-continuous e-weakening uninorm on [0, 1]. It is an easy consequence of the definition that $\circ''$ extends $\circ$. By (I) to (IV), f is an embedding of $(A, \leq_A, \top, \bot, \top_t, \bot_f, \wedge, \vee, *)$ into $([0, 1], \leq, 1, 0, 1_t, 0_f, \min, \max, \circ'')$. Moreover, $\circ''$ has a residuum, calling it $\rightharpoonup$.

We finally prove that for $x, y \in A$, $f(x \to y) = f(x) \rightharpoonup f(y)$. By (IV), $f(x \to y)$ is the residuum of f(x) and f(y) in $(\mathbf{Q} \cap [0, 1], \circ, \leq, 1, 0, 1_t, 0_f, \min, \max, \circ'')$. Thus

$$f(x) \circ'' f(x \to y) = f(x) \circ f(x \to y) \leq f(y).$$

Suppose toward contradiction that there is $\alpha > f(x \to y)$ such that $\alpha \circ'' f(x) \leq f(y)$. Since $\mathbf{Q} \cap [0, 1]$ is dense in [0, 1], there is $q \in \mathbf{Q} \cap [0, 1]$ such that $f(x \to y) < q \leq \alpha$. Hence $q \circ'' f(x) = q \circ f(x) \leq f(y)$, contradicting (IV). □

Theorem 5.3 (Strong standard completeness) For $\mathbf{ULw_t}$, the following are equivalent:

(1) $T \vdash_{ULwt} \phi$.

(2) For every standard ULw_t-algebra and evaluation v, if $v(\psi) \geq 1_t$ for all $\psi \in T$, then $v(\phi) \geq 1_t$.

Proof: (1) to (2) follows from definition. We prove (2) to (1). Let ϕ be a formula such that $T \nvdash_{ULwt} \phi$, **A** a linearly ordered ULw_t-algebra, and v an evaluation in **A** such that $v(\psi) \geq \top_t$ for all $\psi \in T$ and $v(\phi) < \top_t$. Let f' be the embedding of **A** into the standard ULw_t-algebra as in proof of Proposition 5.2. Then f' $\circ$ v is an evaluation into the standard ULw_t-algebra such that f' $\circ$ $v(\psi) \geq 1_t$ and yet f' $\circ$ $v(\phi) < 1_t$. □

Theorem 5.3 ensures that ULw_t is complete w.r.t. left-continuous conjunctive t-weakening uninorms and their residua, i.e., for each formula ϕ, if $\nvdash_{ULwt} \phi$, then there is a left-continuous conjunctive t-weakening uninorm $\circ$ and an evaluation v into $([0, 1], \circ'', \rightharpoonup, \leq, 1, 0, 1_t, 0_f)$, where $\rightharpoonup$ is the residuum of $\circ''$, such that $v(\phi) < 1_t$.

6. Concluding remark

We here investigated (not merely algebraic completeness but also) standard completeness for ULw_t. This work can be generalized to the systems, which are axiomatic extensions of ULw_t. We shall investigate this in some subsequent paper.

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On the Standard Completeness of an Axiomatic Extension of the Uninorm Logic

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이 논문에서는 멧칼페와 몬테그나([8])에 의해 소개된 uninorm logic UL에 (t-weakening, Wt) ((φ & ψ) ∧ t) → φ를 더해 얻어질 수 있는 공리적 확장 체계를 연구한다. 구체적으로 먼저 t-weakening uninorm logic ULWt (the UL with Wt)를 소개하고 이 체계에 상응하는 대수적 구조를 정의한 후 ULWt가 대수적으로 완전하다는 것을 증명한다. 다음으로 제네이와 몬테그나가 [3, 6]에서 보여준 표준 완전성 즉 실수 구간 [0, 1] 위에서의 완전성 증명을 사용하여, ULWt가 주어진 실구간 위에서 완전하다는 것을 즉 표준적으로 완전하다는 것을 증명한다.

[주요어] (준구조) 퍼지 논리, **t-**약화 퍼지 논리, (**t-**약화) 유니놈 논리