

## SENSITIVITY ANALYSIS FOR SYSTEM OF PARAMETRIC GENERALIZED QUASI-VARIATIONAL INCLUSIONS INVOLVING $R$ -ACCRETIVE MAPPINGS

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ABSTRACT. In this paper, using proximal-point mappings technique of  $R$ -accretive mappings and the property of the fixed point set of set-valued contractive mappings, we study the behavior and sensitivity analysis of the solution set of the system of parametric generalized quasi-variational inclusions involving  $R$ -accretive mappings in real uniformly smooth Banach space. Further under suitable conditions, we discuss the Lipschitz continuity of the solution set with respect to parameters. The technique and results presented in this paper can be viewed as extension of the techniques and corresponding results given in [3, 23, 24, 32, 33, 34].

### 1. Introduction

Variational inequality theory introduced by Stampacchia [35] and Fichera [17] independently, in the early sixties in potential theory and mechanics, respectively, constitutes a significant extension of variational principles. It has been shown that the variational inequality theory provides the natural, descent, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems arising in elasticity, economics, transportation, optimization, control theory and engineering sciences [5, 6, 10, 18-20].

In 1968, Brezis [7] initiated the study of the existence theory of a class of variational inequalities later known as variational inclusions, using proximal point mappings due to Moreau [27]. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. For application of variational inclusions, see for example [4, 13].

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In recent years, much attention has been given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can be provide new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Defermos [11], Mukherjee and Verma [27], Noor [30] and Yen [37] studied the sensitivity analysis of solutions of some classes of variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [35] studied the sensitivity analysis of solutions for variational inequalities in finite-dimensional spaces. By using proximal-point mappings technique, Adly [1], Noor [35] and Agarwal *et al.* [2] studied the sensitivity analysis of solution set of some classes of quasi-variational inclusions involving single-valued mappings. By using projection and proximal-point mappings techniques, Ding and Luo [15], Liu *et al.* [26], Park and Jeong [32], Ding [14], and Kazmi and Khan [23, 24], studied the behavior and sensitivity analysis of solution set for some classes of generalized variational inequalities (inclusions) involving set-valued mappings. It is worth mentioning that most of the results in this direction have been obtained in the setting of Hilbert space.

Inspired by recent research work in this area, in this paper, we consider a system of parametric generalized quasi-variational inclusions (SPGQVI, for short) involving  $R$ -accretive mappings in uniformly smooth Banach space. Further, using  $R$ -proximal mappings technique of  $R$ -accretive mappings, and the property of the fixed point set of set-valued mappings, we study the behavior and sensitivity analysis of the solution set for SPGQVI. Furthermore, the Lipschitz continuity of solution set of SPGQVI is proved under some suitable conditions. The theorems presented in this paper generalize and improve the results given by many authors, see for example [3, 22-24, 32-34].

## 2. Preliminaries

Let  $E$  be a real Banach space equipped with norm  $\|\cdot\|$ ; let  $E^*$  be the topological dual space of  $E$ ; let  $\langle \cdot, \cdot \rangle$  is the dual pair of  $E$  and  $E^*$ ; let  $C(E)$  denote the family of all nonempty compact subsets of  $E$ ; let  $2^E$  denote the power set of  $E$ ; let  $H(\cdot, \cdot)$  be the Hausdorff metric on  $C(E)$  defined by

$$H(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{y \in B} \inf_{x \in A} d(x, y) \right\}, \quad A, B \in C(E),$$

and let  $J : E \rightarrow 2^{E^*}$  be the *normalized duality* mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2, \|x\| = \|f\|\}, \quad x \in E.$$

First, we recall and define the following concepts and results.

**Definition 2.1** ([9]). A Banach space  $E$  is called *smooth* if, for every  $x \in E$  with  $\|x\| = 1$ , there exists a unique  $f \in E^*$  such that  $\|f\| = f(x) = 1$ . The *modulus of smoothness* of  $E$  is the function  $\rho_E : [0, \infty) \rightarrow [0, \infty)$ , defined by

$$\rho_E(\tau) = \sup \left\{ \frac{(\|x + y\| + \|x - y\|)}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau \right\}.$$

**Definition 2.2** ([9]). The Banach space  $E$  is said to be *uniformly smooth*, if

$$\lim_{\tau \rightarrow 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

We note that if  $E$  is smooth, then the normalized duality mapping  $J$  is single-valued and if  $E \equiv \mathcal{H}$ , a Hilbert space, the  $J$  is identity map on  $\mathcal{H}$ . In sequel, we denote a selection of normalized duality mapping by  $j$ .

**Lemma 2.1** ([8, 21]). *Let  $E$  be an uniformly smooth Banach space and let  $J : E \rightarrow E^*$  be the normalized duality mapping. Then for all  $x, y \in E$ , we have*

- (i)  $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle$ ;
- (ii)  $\langle x - y, J(x) - J(y) \rangle \leq 2d^2 \rho_E(4\|x - y\|/d)$ , where  $d = \sqrt{(\|x\|^2 + \|y\|^2)/2}$ .

**Definition 2.3** ([9]). A mapping  $A : E \rightarrow E$  is said to be:

- (i) *accretive* if, for all  $x, y \in E$ , there exists  $j(x - y) \in J(x - y)$  such that

$$\langle A(x) - A(y), j(x - y) \rangle \geq 0;$$

- (ii) *strictly accretive* if,  $\langle A(x) - A(y), j(x - y) \rangle > 0$ , and the equality holds only when  $x = y$ .

- (iii)  $\xi$ -*strongly accretive* if, for all  $x, y \in E$ , there exist  $j(x - y) \in J(x - y)$  and a constant  $\xi > 0$  such that

$$\langle A(x) - A(y), j(x - y) \rangle \geq \xi \|x - y\|^2;$$

- (iv)  $\delta$ -*Lipschitz continuous* if, for all  $x, y \in E$ , there exists a constant  $\delta > 0$  such that

$$\|A(x) - A(y)\| \leq \delta \|x - y\|.$$

**Definition 2.4** ([8]). A set-valued mapping  $M : E \rightarrow 2^E$  is said to be:

- (i) *accretive* if, for all  $x, y \in E$ , there exist  $j(x - y) \in J(x - y)$  such that

$$\langle u - v, j(x - y) \rangle \geq 0 \text{ for all } u \in M(x), v \in M(y);$$

- (ii)  $\xi$ -*strongly accretive* if, for all  $x, y \in E$ , there exist  $j(x - y) \in J(x - y)$  and a constant  $\xi > 0$  such that

$$\langle u - v, j(x - y) \rangle \geq \xi \|x - y\|^2 \text{ for all } u \in M(x), v \in M(y);$$

- (iii)  $m$ -*accretive* if,  $M$  is accretive and  $(I + \rho M)(E) = E$  for any  $\rho > 0$ , where  $I$  stands for identity mapping.

**Definition 2.5** ([16]). Let  $R : E \rightarrow E$  be a nonlinear mapping. Then a set-valued mapping  $M : E \rightarrow 2^E$  is said to be  $R$ -*accretive*, if  $M$  is accretive and  $(R + \rho M)(E) = E$  for any  $\rho > 0$ .

The following result gives some properties of  $R$ -accretive mappings:

**Lemma 2.2** ([16]). *Let  $R : E \rightarrow E$  be a strictly accretive mapping and let  $M : E \rightarrow 2^E$  be a  $R$ -accretive set-valued mapping. Then*

- (i)  $\langle u - v, J(x - y) \rangle \geq 0$  for all  $(v, y) \in \text{Graph}(M)$  implies  $(u, x) \in \text{Graph}(M)$ , where  $\text{Graph}(M) := \{u, x\} \in E \times E : u \in M(x)\}$ ;
- (ii) the mapping  $(R + \rho M)^{-1}$  is single-valued for all  $\rho > 0$ .

By Lemma 2.2, we can define  $R$ -proximal point mapping of a  $R$ -accretive mapping  $M$  as follows

$$(2.1) \quad J_\rho^M(z) = (R + \rho M)^{-1}(z) \text{ for all } z \in E,$$

where  $\rho > 0$  is a constant and  $R : E \rightarrow E$  is a strictly accretive mapping.

**Lemma 2.3.** *Let  $R : E \rightarrow E$  be a  $\gamma$ -strongly accretive mapping and  $M : E \rightarrow 2^E$  be a  $R$ -accretive mapping. Then the  $R$ -proximal-point mapping  $J_\rho^M : E \rightarrow E$  is  $\frac{1}{\gamma}$ -Lipschitz continuous, that is,*

$$\|J_\rho^M(x) - J_\rho^M(y)\| \leq \frac{1}{\gamma} \|x - y\| \text{ for all } x, y \in E.$$

**Lemma 2.4** ([25]). *Let  $X$  be a complete metric space and let  $T_1, T_2 : X \rightarrow C(X)$  be  $\theta$ - $H$ -contraction mappings. Then*

$$(2.2) \quad H(F(T_1), F(T_2)) \leq (1 - \theta)^{-1} \sup_{x \in X} (T_1(x), T_2(x)),$$

where  $F(T_1)$  and  $F(T_2)$  are the set of fixed points of  $T_1$  and  $T_2$  respectively.

### 3. System of parametric generalized quasi-variational inclusions

Throughout rest of this paper, unless otherwise stated, for each  $i = 1, 2$ , we assume that  $E_i$  is uniformly smooth Banach space with norm  $\|\cdot\|_i$ , and denote the duality pairing between  $E_i$  and its dual  $E_i^*$  by  $\langle \cdot, \cdot \rangle_i$ .

Let  $\Omega_1$  and  $\Omega_2$  be nonempty subsets of  $E_1$  and  $E_2$ , respectively, in which parameters  $\lambda$  and  $\mu$  takes the values; let  $R_i : E_i \rightarrow E_i$ ;  $g_i : E_i \times \Omega_i \rightarrow E_i$ ;  $P, F : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_1$  and let  $Q, G : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_2$  be single-valued mappings such that  $g \neq 0$  and let  $A, C : E_1 \times \Omega_1 \rightarrow C(E_1)$  and  $B, D : E_2 \times \Omega_2 \rightarrow C(E_2)$  be set-valued mappings. Let  $M : E_1 \times \Omega_1 \rightarrow 2^{E_1}$  be an  $R_1$ -accretive mapping and  $N : E_2 \times \Omega_2 \rightarrow 2^{E_2}$  be an  $R_2$ -accretive mapping in the first argument such that  $g_1(x, \lambda) \in \text{dom } M(\cdot, \lambda)$  and  $g_2(y, \mu) \in \text{dom } N(\cdot, \mu)$  for all  $x \in E_1, y \in E_2, \lambda \in \Omega_1, \mu \in \Omega_2$ . For each  $(f_1, f_2, \lambda, \mu) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ , we consider the system of parametric generalized quasi-variational inclusions (SPGQVI, for short):

Find  $(x, y, u, v, w, z)$  such that  $(x = x(\lambda), y = y(\mu)) \in E_1 \times E_2, u = u(\lambda) \in A(x, \lambda), v = v(\mu) \in B(y, \mu), w = w(\lambda) \in C(x, \lambda), z = z(\mu) \in D(y, \mu)$ , and

$$(3.1) \quad \begin{cases} f_1 \in F(x, y, \lambda, \mu) + P(u, v, \lambda, \mu) + M(g_1(x, \lambda), \lambda) \\ f_2 \in G(x, y, \lambda, \mu) + Q(w, z, \lambda, \mu) + N(g_2(y, \mu), \mu) \end{cases}$$

**Some special cases of SPGQVI (3.1):**

If  $E_1 \equiv E_2 \equiv H; F \equiv G; R_1 \equiv R_2; f_1 \equiv f_2 \equiv 0; P \equiv Q \equiv 0; B \equiv D \equiv 0; g_1 \equiv g_2; C \equiv 0; F(x, y, \lambda, \mu) \equiv F(x, \lambda)$  for all  $(x, y, \lambda, \mu) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ , then SPGQVI (3.1) reduces to the problem of finding  $x = x(\lambda) \in H$  such that

$$0 \in F(x, \lambda) + M(g(x, \lambda), \lambda),$$

which has been studied by Adly [1].

Now, for each fixed  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ , the solution set  $S(\lambda, \mu)$  of SPG-MQVI(3.1) is defined as

$$(3.2) \quad S(\lambda, \mu) := \left\{ \begin{array}{l} (x = x(\lambda), y = y(\mu)) \in E_1 \times E_2, u = u(\lambda) \in A(x, \lambda), \\ v = v(\mu) \in B(y, \mu), w = w(\lambda) \in C(x, \lambda), z = z(\mu) \in D(y, \mu), \\ \text{such that} \\ \left. \begin{array}{l} f_1 \in F(x, y, \lambda, \mu) + P(u, v, \lambda, \mu) + M(g_1(x, \lambda), \lambda) \\ f_2 \in G(x, y, \lambda, \mu) + Q(w, z, \lambda, \mu) + N(g_2(y, \mu), \mu) \end{array} \right\} \end{array} \right.$$

The aim of this paper is to study the behavior and sensitivity analysis of the solution set  $S(\lambda, \mu)$ , and the conditions on mappings  $A, B, C, D, F, G, P, Q, M, N, g_i, R_i$ ; under which the solution set  $S(\lambda, \mu)$  of SPGQVI(3.1) is nonempty and Lipschitz continuous with respect to the parameters  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ .

First, we recall the following concepts:

**Definition 3.1** ([22]). A mapping  $g_1 : E_1 \times \Omega_1 \rightarrow E_1$  is said to be:

- (i)  $(L_{g_1}, l_{g_1})$ -mixed Lipschitz continuous, if there exist constants  $L_{g_1}, l_{g_1} > 0$  such that

$$\begin{aligned} & \|g_1(x_1, \lambda_1) - g_1(x_2, \lambda_2)\|_1 \\ & \leq L_{g_1} \|x_1 - x_2\|_1 + l_{g_1} \|\lambda_1 - \lambda_2\| \text{ for all } (x_1, \lambda_1), (x_2, \lambda_2) \in E_1 \times \Omega_1, \end{aligned}$$

- (ii)  $s_1$ -strongly accretive, if there exists a constant  $s_1 > 0$  such that

$$\langle g_1(x_1, \lambda) - g_1(x_2, \lambda), J_1(x_1 - x_2) \rangle_1 \geq s_1 \|x_1 - x_2\|_1^2$$

for all  $(x_1, x_2, \lambda) \in E_1 \times E_2 \times \Omega_1$ , where  $J_1 : E_1 \rightarrow 2^{E_1^*}$  is the normalized duality mapping.

*Remark 3.1.* When  $\lambda$  is fixed, then mixed Lipschitz continuity of  $g_1$  implies Lipschitz continuity in the first argument.

**Definition 3.2** ([22]). A set-valued mapping  $A : E_1 \times \Omega_1 \rightarrow C(E_1)$  is said to be  $(L_A, l_A)$ -H-mixed Lipschitz continuous, if there exist constants  $L_A, l_A > 0$  such that

$$\begin{aligned} & H(A(x_1, \lambda_1), A(x_2, \lambda_2)) \\ & \leq L_A \|x_1 - x_2\|_1 + l_A \|\lambda_1 - \lambda_2\|_1 \text{ for all } (x_1, \lambda_1), (x_2, \lambda_2) \in E_1 \times \Omega_1. \end{aligned}$$

**Definition 3.3** ([22]). Let  $R_1 : E_1 \rightarrow E_1, g_1 : E_1 \times \Omega_1 \rightarrow E_1$ . Then a mapping  $F : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_1$  is said to be:

- (i)  $(L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)})$ -mixed Lipschitz continuous, if there exist constants  $L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)} > 0$  such that

$$\|F(x_1, y_1, \lambda_1, \mu_1) - F(x_2, y_2, \lambda_2, \mu_2)\|_1 \leq L_{(F,1)}\|x_1 - x_2\|_1 + L_{(F,2)}\|y_1 - y_2\|_2 + l_{(F,1)}\|\lambda_1 - \lambda_2\|_1 + l_{(F,2)}\|\mu_1 - \mu_2\|_2$$

for all  $(x_1, y_1, \lambda_1, \mu_1), (x_2, y_2, \lambda_2, \mu_2) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ ;

- (ii)  $\xi_1$ -strongly  $R_1 \circ g_1$ -accretive in the first argument if there exists a constant  $\xi_1 > 0$  such that

$$\langle F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu), J_1(R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda)) \rangle \geq \xi_1 \|x_1 - x_2\|_1^2$$

for all  $x_1, x_2 \in E_1, y_1 \in E_2, (\lambda, \mu) \in \Omega_1 \times \Omega_2$ , where  $R_1 \circ g_1$  denote  $R_1$  composition  $g_1$ .

We now transfer the SPGQVI (3.1) into a fixed point problem.

**Lemma 3.1.** For each  $(f_1, f_2, \lambda, \mu) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ ,  $(x, y, u, v, w, z)$  with  $(x = x(\lambda), y = y(\mu)) \in E_1 \times E_2, u = u(\lambda) \in A(x, \lambda), v = v(\mu) \in B(y, \mu), w = w(\lambda) \in C(x, \lambda), z = z(\mu) \in D(y, \mu)$  is a solution of SPGQVI (3.1) if and only if the set-valued mapping  $T : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow 2^{E_1 \times E_2}$  defined by

$$(3.3) \quad T(x, y, \lambda, \mu) = \left( U(x, y, \lambda, \mu), V(x, y, \lambda, \mu) \right),$$

where  $U : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow 2^{E_1}$  and  $V : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow 2^{E_2}$  are defined as

$$(3.4) \quad \begin{aligned} & U(x, y, \lambda, \mu) \\ &= \bigcup_{u \in A(x, \lambda), v \in B(y, \mu)} \left[ x - g_1(x, \lambda) + J_{\rho_1}^{M(\cdot, \lambda)} \left( R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) \right. \right. \\ & \quad \left. \left. - \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1 \right) \right], \end{aligned}$$

$$(3.5) \quad \begin{aligned} & V(x, y, \lambda, \mu) \\ &= \bigcup_{w \in C(x, \lambda), z \in D(y, \mu)} \left[ y - g_2(y, \mu) + J_{\rho_2}^{N(\cdot, \mu)} \left( R_2 \circ g_2(y, \mu) - \rho_2 G(x, y, \lambda, \mu) \right. \right. \\ & \quad \left. \left. - \rho_2 Q(w, z, \lambda, \mu) + \rho_2 f_2 \right) \right], \end{aligned}$$

has a fixed point, where  $J_{\rho_1}^{M(\cdot, \lambda)} = (R_1 + \rho_1 M(\cdot, \lambda))^{-1}$ ,  $J_{\rho_2}^{N(\cdot, \mu)} = (R_2 + \rho_2 N(\cdot, \mu))^{-1}$  and  $\rho_1, \rho_2 > 0$  are constants.

*Proof.* For each  $(f_1, f_2, \lambda, \mu) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ , SPGQVI (3.1) has a solution  $(x, y, u, v, w, z)$  with  $(x = x(\lambda) \in E_1, y = y(\mu) \in E_2, u = u(x, \lambda) \in A(x, \lambda), v = v(\mu) \in B(y, \mu), w = w(\lambda) \in C(x, \lambda), z = z(\mu) \in D(y, \mu))$  such that

$g_1(x, \lambda) \in \text{dom } M(\cdot, \lambda)$  and  $g_2(y, \mu) \in \text{dom } N(\cdot, \mu)$  for all  $x \in E_1, y \in E_2, \lambda \in \Omega_1, \mu \in \Omega_2$  if and only if

$$\begin{aligned} & f_1 \in F(x, y, \lambda, \mu) - P(u, v, \lambda, \mu) + M(g_1(x, \lambda), \lambda) \\ \iff & R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) + \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1 \\ & \in (R_1 + \rho_1 M(\cdot, \lambda))(g_1(x, \lambda)) \end{aligned}$$

and

$$\begin{aligned} & f_2 \in G(x, y, \lambda, \mu) - Q(w, z, \lambda, \mu) + N(g_2(y, \mu), \mu) \\ \iff & R_2 \circ g_2(y, \mu) - \rho_2 G(x, y, \lambda, \mu) + \rho_2 Q(w, z, \lambda, \mu) + \rho_2 f_2 \\ & \in (R_2 + \rho_2 N(\cdot, \mu))(g_2(y, \mu)). \end{aligned}$$

Since for each  $x \in E_1, y \in E_2, \lambda \in \Omega_1, \mu \in \Omega_2$ ,  $M(\cdot, \lambda)$  is  $R_1$ -accretive and  $N(\cdot, \mu)$  is  $R_2$ -accretive, by definition of  $R_i$ -proximal-point mapping of  $J_{\rho_1}^{M(\cdot, \lambda)}$  and  $J_{\rho_2}^{N(\cdot, \mu)}$  preceding inclusions hold if and only if

$$g_1(x, \lambda) = J_{\rho_1}^{M(\cdot, \lambda)} \left[ R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) + \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1 \right]$$

and

$$g_2(y, \mu) = J_{\rho_2}^{N(\cdot, \mu)} \left[ R_2 \circ g_2(y, \mu) - \rho_2 G(x, y, \lambda, \mu) + \rho_2 Q(u, v, \lambda, \mu) + \rho_2 f_2 \right],$$

i.e.,  $(x, y) \in T(x, y, \lambda, \mu)$ . This completes the proof. □

#### 4. Sensitivity analysis of solution set $S(\lambda, \mu)$

Now, we shall study the behavior and sensitivity analysis of the solution set of SPGMQVI (3.1) and further, under suitable conditions, we shall discuss Lipschitz continuity of the solution set with respect to the parameters.

**Theorem 4.1.** *For each  $i = 1, 2$ , let  $E_i$  be uniformly smooth Banach space with  $\rho_{E_i}(t) \leq c_i t^2$  for some  $c_i > 0$ ; let the mappings  $R_i : E_i \rightarrow E_i$  and  $g_i : E_i \times \Omega_i \rightarrow E_i$  such that  $g_i$  is  $s_i$ -strongly accretive and  $(L_{g_i}, l_{g_i})$ -mixed Lipschitz continuous and  $R_i \circ g_i$  be  $(L_{R_i \circ g_i}, l_{R_i \circ g_i})$ -mixed Lipschitz continuous; let the set-valued mappings  $A, C : E_1 \times \Omega_1 \rightarrow C(E_1)$  and  $B, D : E_2 \times \Omega_2 \rightarrow C(E_2)$  be  $H$ -mixed Lipschitz continuous with constants  $(L_A, l_A), (L_B, l_B)$  and  $(L_C, l_C), (L_D, l_D)$ , respectively; let  $F : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_1$  be  $\xi_1$ -strongly accretive with respect to  $R_1 \circ g_1$  in the first argument and  $(L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)})$ -mixed Lipschitz continuous and  $G : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_2$  be  $\xi_2$ -strongly accretive with respect to  $R_2 \circ g_2$  in the second argument and  $(L_{(G,1)}, L_{(G,2)}, l_{(G,1)}, l_{(G,2)})$ -mixed Lipschitz continuous; let the mappings  $P : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_1$  be  $(L_{(P,1)}, L_{(P,2)}, l_{(P,1)}, l_{(P,2)})$ -mixed Lipschitz continuous and  $Q : E_1 \times E_2 \times \Omega_1 \times \Omega_2 \rightarrow E_2$  be  $(L_{(Q,1)}, L_{(Q,2)}, l_{(Q,1)}, l_{(Q,2)})$ -mixed Lipschitz continuous, respectively; let  $M : E_1 \times \Omega_1 \rightarrow 2^{E_1}$  and  $N : E_2 \times \Omega_2 \rightarrow 2^{E_2}$  be such that for each fixed  $\lambda \in \Omega_1$  and  $\mu \in \Omega_2$ ,  $M(\cdot, \lambda)$  and  $N(\cdot, \mu)$  are  $R_1$ -accretive and*

$R_2$ -accretive mappings, respectively. Suppose that there are constants  $\delta_1, \delta_2 > 0$  satisfies the following condition:

$$(4.1) \quad \begin{cases} k_1 := \sqrt{1 - 2s_2 + 64c_1L_{g_1}^2} + \frac{\rho_1}{\gamma_1}L_{(P,1)}L_A + \frac{1}{\gamma_1}\sqrt{L_{R_1 \circ g_1}^2 - 2\rho_1\xi_1 + 64c_1\rho_1^2L_{(F,1)}^2} \\ \quad + \frac{\rho_2}{\gamma_2}(L_{(G,1)} + L_{(Q,1)}L_C) < 1; \\ k_2 := \sqrt{1 - 2s_2 + 64c_2L_{g_2}^2} + \frac{\rho_2}{\gamma_2}L_{(Q,2)}L_D + \frac{1}{\gamma_2}\sqrt{L_{R_2 \circ g_2}^2 - 2\rho_2\xi_2 + 64c_2\rho_2^2L_{(G,2)}^2} \\ \quad + \frac{\rho_1}{\gamma_1}(L_{(F,2)} + L_{(P,2)}L_B) < 1, \end{cases}$$

where  $\theta = \max\{k_1, k_2\}$ . Then for each fixed  $(f_1, f_2) \in E_1 \times E_2$ , the set-valued mapping  $T$  defined by (3.3) is a compact-valued uniform  $\theta$ - $H$ -contraction mapping with respect to  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ , where  $\theta$  is given by (4.1). Moreover, for each  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ , the solution set  $S(\lambda, \mu)$  of SPGQVI (3.1) is nonempty and closed.

*Proof.* Let  $(x, y, \lambda, \mu)$  be an arbitrary element in  $E_1 \times E_2 \times \Omega_1 \times \Omega_2$ . Since  $A, B, C, D$  are compact-valued, then for any sequences  $\{u_n\} \subset A(x, \lambda)$ ,  $\{v_n\} \subset B(y, \mu)$ ,  $\{w_n\} \subset C(x, \lambda)$ ,  $\{z_n\} \subset D(y, \mu)$ , there exist subsequences  $\{u_{n_i}\} \subset \{u_n\}$ ,  $\{v_{n_i}\} \subset \{v_n\}$ ,  $\{w_{n_i}\} \subset \{w_n\}$ ,  $\{z_{n_i}\} \subset \{z_n\}$  such that  $u_{n_i} \rightarrow u$ ,  $v_{n_i} \rightarrow v$ ,  $w_{n_i} \rightarrow w$ ,  $z_{n_i} \rightarrow z$  as  $i \rightarrow \infty$ . By using Lemma 2.3 and mixed Lipschitz continuity of  $P$  and  $Q$ , we have

$$(4.2) \quad \begin{aligned} & \|J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x, \lambda) - \rho_1F(x, y, \lambda, \mu) - \rho_1P(u_{n_i}, v_{n_i}, \lambda, \mu) + \rho_1f_1] \\ & \quad - J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x, \lambda) - \rho_1F(x, y, \lambda, \mu) - \rho_1P(u, v, \lambda, \mu) + \rho_1f_1]\|_1 \\ & \leq \frac{\rho_1}{\gamma_1} \|P(u_{n_i}, v_{n_i}, \lambda, \mu) - P(u, v, \lambda, \mu)\|_1 \\ & \leq \frac{\rho_1}{\gamma_1} [L_{(P,1)}\|u_{n_i} - u\|_1 + L_{(P,2)}\|v_{n_i} - v\|_2] \rightarrow 0 \text{ as } i \rightarrow \infty \end{aligned}$$

and

$$(4.3) \quad \begin{aligned} & \|J_{\rho_2}^{N(\cdot, \mu)}[R_2 \circ g_2(y, \mu) - \rho_2G(x, y, \lambda, \mu) - \rho_2Q(w_{n_i}, z_{n_i}, \lambda, \mu) + \rho_2f_2] \\ & \quad - J_{\rho_2}^{N(\cdot, \mu)}[R_2 \circ g_2(y, \mu) - \rho_2G(x, y, \lambda, \mu) - \rho_2Q(w, z, \lambda, \mu) + \rho_2f_2]\|_2 \\ & \leq \frac{\rho_2}{\gamma_2} \|Q(w_{n_i}, z_{n_i}, \lambda, \mu) - Q(w, z, \lambda, \mu)\|_2 \\ & \leq \frac{\rho_2}{\gamma_2} [L_{(Q,1)}\|w_{n_i} - w\|_1 + L_{(Q,2)}\|z_{n_i} - z\|_2] \rightarrow 0 \text{ as } i \rightarrow \infty. \end{aligned}$$

Thus (3.3), (4.2) and (4.3) yield that  $T(x, y, \lambda, \mu) \in C(E_1 \times E_2)$ .

Now, for each fixed  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ , we prove that  $T(x, y, \lambda, \mu)$  is a uniform  $\theta$ - $H$ -contraction mapping. Let for  $i = 1, 2$ ,  $(x_i, y_i, \lambda, \mu)$  be arbitrary elements in  $E_1 \times E_2 \times \Omega_1 \times \Omega_2$  and let  $t_1 \in U(x_1, y_1, \lambda, \mu)$  and  $p_1 \in V(x_1, y_1, \lambda, \mu)$ , there



exist  $u_1 = u_1(x_1, \lambda) \in A(x_1, \lambda)$ ,  $v_1 = v_1(y_1, \mu) \in B(y_1, \mu)$ ,  $w_1 = w_1(x_1, \lambda) \in C(x_1, \lambda)$ , and  $z_1 = z_1(y_1, \mu) \in D(y_1, \mu)$  such that

$$(4.4) \quad \begin{aligned} t_1 &= x_1 - g_1(x_1, \lambda) \\ &\quad + J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x_1, \lambda) - \rho_1 F(x_1, y_1, \lambda, \mu) + \rho_1 P(u_1, v_1, \lambda, \mu) + \rho_1 f_1] \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} p_1 &= y_1 - g_2(y_1, \mu) \\ &\quad + J_{\rho_2}^{N(\cdot, \mu)}[R_2 \circ g_2(y_1, \mu) - \rho_2 G(x_1, y_1, \lambda, \mu) + \rho_2 Q(w_1, z_1, \lambda, \mu) + \rho_2 f_2]. \end{aligned}$$

It follows from the compactness of  $A(x_2, \lambda)$ ,  $B(y_2, \mu)$ ,  $C(x_2, \lambda)$  and  $D(y_2, \mu)$  and  $H$ -Lipschitz continuity of  $A, B, C, D$  that there exist  $u_2 = u_2(x_2, \lambda) \in A(x_2, \lambda)$ ,  $v_2 = v_2(y_2, \mu) \in B(y_2, \mu)$ ,  $w_2 = w_2(x_2, \lambda) \in C(x_2, \lambda)$  and  $z_2 = z_2(y_2, \mu) \in D(y_2, \mu)$  such that

$$(4.6) \quad \begin{aligned} \|u_1 - u_2\|_1 &\leq H(A(x_1, \lambda), A(x_2, \lambda)) \leq L_A \|x_1 - x_2\|_1, \\ \|v_1 - v_2\|_2 &\leq H(B(y_1, \mu), B(y_2, \mu)) \leq L_B \|y_1 - y_2\|_2, \\ \|w_1 - w_2\|_1 &\leq H(C(x_1, \lambda), C(x_2, \lambda)) \leq L_C \|x_1 - x_2\|_1, \\ \|z_1 - z_2\|_2 &\leq H(D(y_1, \mu), D(y_2, \mu)) \leq L_D \|y_1 - y_2\|_2. \end{aligned}$$

Let

$$(4.7) \quad \begin{aligned} t_2 &= x_2 - g_1(x_2, \lambda) \\ &\quad + J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x_2, \lambda) - \rho_1 F(x_2, y_2, \lambda, \mu) + \rho_1 P(u_2, v_2, \lambda, \mu) + \rho_1 f_1] \end{aligned}$$

and

$$(4.8) \quad \begin{aligned} p_2 &= y_2 - g_2(y_2, \mu) \\ &\quad + J_{\rho_2}^{N(\cdot, \mu)}[R_2 \circ g_2(y_2, \mu) - \rho_2 G(x_2, y_2, \lambda, \mu) + \rho_2 Q(w_2, z_2, \lambda, \mu) + \rho_2 f_2]. \end{aligned}$$

Then we have  $t_2 \in U(x_2, y_2, \lambda, \mu)$  and  $p_2 \in V(x_2, y_2, \lambda, \mu)$ .

Next, using Lemma 2.3, we have

$$(4.9) \quad \begin{aligned} &\|t_1 - t_2\|_1 \\ &\leq \|x_1 - x_2 - (g_1(x_1, \lambda) - g_1(x_2, \lambda))\|_1 \\ &\quad + \|J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x_1, \lambda) - \rho_1 F(x_1, y_1, \lambda, \mu) + \rho_1 P(u_1, v_1, \lambda, \mu) + \rho_1 f_1] \\ &\quad - J_{\rho_1}^{M(\cdot, \lambda)}[R_1 \circ g_1(x_2, \lambda) - \rho_1 F(x_2, y_2, \lambda, \mu) + \rho_1 P(u_2, v_2, \lambda, \mu) + \rho_1 f_1]\|_1 \\ &\leq \|x_1 - x_2 - (g_1(x_1, \lambda) - g_1(x_2, \lambda))\|_1 \\ &\quad + \frac{1}{\gamma_1} \|R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda) - \rho_1 (F(x_1, y_1, \lambda, \mu) - F(x_2, y_2, \lambda, \mu)) \\ &\quad + P(u_1, v_1, \lambda, \mu) - P(u_2, v_2, \lambda, \mu)\|_1. \end{aligned}$$

Since  $g_1$  is  $s_1$ -strongly accretive and  $(L_{g_1}, l_{g_1})$ -mixed Lipschitz continuous, using Lemma 2.1 we have

$$\begin{aligned}
 (4.10) \quad & \|x_1 - x_2 - (g_1(x_1, \lambda) - g_1(x_2, \lambda))\|_1^2 \\
 & \leq \|x_1 - x_2\|_1^2 - 2\langle g_1(x_1, \lambda) - g_1(x_2, \lambda), J_1(x_1 - x_2 - (g_1(x_1, \lambda) - g_1(x_2, \lambda))) \rangle_1 \\
 & \leq \|x_1 - x_2\|_1^2 - 2\langle g_1(x_1, \lambda) - g_1(x_2, \lambda), J_1(x_1 - x_2) \rangle_1 - 2\langle g_1(x_1, \lambda) - g_1(x_2, \lambda), \\
 & \quad J_1(x_1 - x_2 - (g_1(x_1, \lambda) - g_1(x_2, \lambda))) - J_1(x_1 - x_2) \rangle_1 \\
 & \leq \|x_1 - x_2\|_1^2 - 2s_1\|x_1 - x_2\|_1^2 + 64c_1L_{g_1}^2\|x_1 - x_2\|_1^2 \\
 & \leq (1 - 2s_1 + 64c_1L_{g_1}^2)\|x_1 - x_2\|_1^2.
 \end{aligned}$$

Since  $P$  is  $(L_{(P,1)}, L_{(P,2)}, l_{(P,1)}, l_{(P,2)})$ -mixed Lipschitz continuous and  $H$ -Lipschitz continuity of set-valued mappings  $A, B$ , we have

$$\begin{aligned}
 (4.11) \quad & \|P(u_1, v_1, \lambda_1, \mu_1) - P(u_2, v_2, \lambda_2, \mu_2)\|_1 \\
 & \leq L_{(P,1)}\|u_1 - u_2\|_1 + L_{(P,2)}\|v_1 - v_2\|_2 + l_{(P,1)}\|\lambda_1 - \lambda_2\|_1 + l_{(P,2)}\|\mu_1 - \mu_2\|_2 \\
 & \leq L_{(P,1)}H(A(x_1, \lambda), A(x_2, \lambda)) + L_{(P,2)}H(B(y_1, \mu), B(y_2, \mu)) \\
 & \quad + l_{(P,1)}\|\lambda_1 - \lambda_2\|_1 + l_{(P,2)}\|\mu_1 - \mu_2\|_2 \\
 & \leq L_{(P,1)}L_A\|x_1 - x_2\|_1 + L_{(P,2)}L_B\|y_1 - y_2\|_2 + l_{(P,1)}\|\lambda_1 - \lambda_2\|_1 \\
 & \quad + l_{(P,2)}\|\mu_1 - \mu_2\|_2,
 \end{aligned}$$

and

$$\begin{aligned}
 (4.12) \quad & \|R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda) - \rho_1(F(x_1, y_1, \lambda, \mu) - F(x_2, y_2, \lambda, \mu)) \\
 & \quad - P(u_1, v_1, \lambda, \mu) + P(u_2, v_2, \lambda, \mu)\|_1 \\
 & \leq \|R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda) - \rho_1(F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu))\|_1 \\
 & \quad + \rho_1\|F(x_2, y_1, \lambda, \mu) - F(x_2, y_2, \lambda, \mu)\|_1 \\
 & \quad + \rho_1\|P(u_1, v_1, \lambda, \mu) - P(u_2, v_2, \lambda, \mu)\|_1.
 \end{aligned}$$

Further, since  $F$  is  $\xi_1$ -strongly accretive with respect to  $R_1 \circ g_1$  in the first argument and  $(L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)})$ -mixed Lipschitz continuous and  $R_1 \circ g_1$  is  $(L_{R_1 \circ g_1}, l_{R_1 \circ g_1})$ -mixed Lipschitz continuous, then

$$\begin{aligned}
 (4.13) \quad & \|R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda) - \rho_1(F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu))\|_1^2 \\
 & \leq \|R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda)\|_1^2 - 2\rho_1\langle F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu), \\
 & \quad J_1(R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda) - \rho_1(F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu))) \rangle_1 \\
 & \leq L_{R_1 \circ g_1}^2\|x_1 - x_2\|_1^2 - 2\rho_1\langle F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu), \\
 & \quad J_1(R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda)) \rangle_1 \\
 & \quad - 2\rho_1\langle F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu), J_1(R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda)) \rangle_1
 \end{aligned}$$

$$\begin{aligned}
 & -\rho_1(F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu)) - J_1(R_1 \circ g_1(x_1, \lambda) - R_1 \circ g_1(x_2, \lambda))\|_1 \\
 \leq & L_{R_1 \circ g_1}^2 \|x_1 - x_2\|^2 - 2\rho_1 \xi_1 \|x_1 - x_2\|_1^2 \\
 & + 64c_1 \rho_1^2 \|F(x_1, y_1, \lambda, \mu) - F(x_2, y_1, \lambda, \mu)\|_1^2 \\
 \leq & L_{R_1 \circ g_1}^2 \|x_1 - x_2\|^2 - 2\rho_1 \xi_1 \|x_1 - x_2\|_1^2 + 64c_1 \rho_1^2 L_{(F,1)}^2 \|x_1 - x_2\|_1^2 \\
 \leq & (L_{R_1 \circ g_1}^2 - 2\rho_1 \xi_1 + 64c_1 \rho_1^2 L_{(F,1)}^2) \|x_1 - x_2\|_1^2.
 \end{aligned}$$

Since,  $F$  is  $(L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)})$ -mixed Lipschitz continuous, then we have

$$(4.14) \quad \|F(x_2, y_1, \lambda, \mu) - F(x_2, y_2, \lambda, \mu)\|_1 \leq L_{(F,2)} \|y_1 - y_2\|_2.$$

It follows from (4.9)-(4.12) and (4.14), we have

$$\begin{aligned}
 (4.15) \quad & \|t_1 - t_2\|_1 \\
 \leq & \sqrt{1 - 2s_1 + 64c_1 L_{g_1}^2} \|x_1 - x_2\|_1 \\
 & + \frac{1}{\gamma_1} \sqrt{L_{R_1 \circ g_1}^2 - 2\rho_1 \xi_1 + 64c_1 \rho_1^2 L_{(F,1)}^2} \|x_1 - x_2\|_1 \\
 & + \frac{\rho_1}{\gamma_1} L_{(F,2)} \|y_1 - y_2\|_2 + \frac{\rho_1}{\gamma_1} \left[ L_{(P,1)} L_A \|x_1 - x_2\|_1 + L_{(P,2)} L_B \|y_1 - y_2\|_2 \right] \\
 \leq & \left( \sqrt{1 - 2s_1 + 64c_1 L_{g_1}^2} + \frac{\rho_1}{\gamma_1} L_{(P,1)} L_A + \frac{1}{\gamma_1} \sqrt{L_{R_1 \circ g_1}^2 - 2\rho_1 \xi_1 + 64c_1 \rho_1^2 L_{(F,1)}^2} \right) \\
 & \times \|x_1 - x_2\|_1 + \frac{\rho_1}{\gamma_1} (L_{(F,2)} + L_{(P,2)} L_B) \|y_1 - y_2\|_2.
 \end{aligned}$$

From (4.5) and (4.8) and Lemma 2.3, we have

$$\begin{aligned}
 (4.16) \quad & \|p_1 - p_2\|_2 \\
 \leq & \|y_1 - y_2 - (g_2(y_1, \mu) - g_2(y_2, \mu))\|_2 \\
 & + \left\| J_{\rho_2}^{N(\cdot, \mu)} \left[ R_2 \circ g_2(y_1, \mu) - \rho_2 G(x_1, y_1, \lambda, \mu) + \rho_2 Q(w_1, z_1, \lambda, \mu) + \rho_2 f_2 \right] \right. \\
 & \left. - J_{\rho_2}^{N(\cdot, \mu)} \left[ R_2 \circ g_2(y_2, \mu) - \rho_2 G(x_2, y_2, \lambda, \mu) + \rho_2 Q(w_2, z_2, \lambda, \mu) + \rho_2 f_2 \right] \right\|_2 \\
 \leq & \|y_1 - y_2 - (g_2(y_1, \mu) - g_2(y_2, \mu))\|_2 + \frac{1}{\gamma_2} \|R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu) \\
 & - \rho_2(G(x_1, y_1, \lambda, \mu) - G(x_2, y_2, \lambda, \mu) - Q(w_1, z_1, \lambda, \mu) + Q(w_2, z_2, \lambda, \mu))\|_2.
 \end{aligned}$$

Since  $g_2$  is  $s_2$ -strongly accretive and  $(L_{g_2}, l_{g_2})$ -mixed Lipschitz continuous, we have  $Q$  is  $(L_{(Q,1)}, L_{(Q,2)}, l_{(Q,1)}, l_{(Q,2)})$ -mixed Lipschitz continuous and  $H$ -Lipschitz continuity of set-valued mappings  $C, D$ , we have

$$\begin{aligned}
 (4.17) \quad & \|Q(w_1, z_1, \lambda_1, \mu_1) - Q(w_2, z_2, \lambda_2, \mu_2)\|_1 \\
 \leq & L_{(Q,1)} \|w_1 - w_2\|_1 + L_{(Q,2)} \|z_1 - z_2\|_2 + l_{(Q,1)} \|\lambda_1 - \lambda_2\|_1 + l_{(Q,2)} \|\mu_1 - \mu_2\|_2
 \end{aligned}$$

$$\begin{aligned}
&\leq L_{(Q,1)} H(C(x_1, \lambda), C(x_2, \lambda)) + L_{(Q,2)} H(D(y_1, \mu), D(y_2, \mu)) \\
&\quad + l_{(Q,1)} \|\lambda_1 - \lambda_2\|_1 + l_{(Q,2)} \|\mu_1 - \mu_2\|_2 \\
&\leq L_{(Q,1)} L_C \|x_1 - x_2\|_1 + L_{(Q,2)} L_D \|y_1 - y_2\|_2 + l_{(Q,1)} \|\lambda_1 - \lambda_2\|_1 \\
&\quad + l_{(Q,2)} \|\mu_1 - \mu_2\|_2,
\end{aligned}$$

(4.18)

$$\begin{aligned}
&\|y_1 - y_2 - (g_2(y_1, \mu) - g_2(y_2, \mu))\|_2^2 \\
&\leq \|y_1 - y_2\|_2^2 - 2\langle g_2(y_1, \mu) - g_2(y_2, \mu), J_2(y_1 - y_2 - (g_2(y_1, \mu) - g_2(y_2, \mu))) \rangle_2 \\
&\leq \|y_1 - y_2\|_2^2 - 2\langle g_2(y_1, \mu) - g_2(y_2, \mu), J_2(y_1 - y_2) \rangle_2 \\
&\quad - 2\langle g_2(y_1, \mu) - g_2(y_2, \mu), J_2(y_1 - y_2 - (g_2(y_1, \mu) - g_2(y_2, \mu))) - J_2(y_1 - y_2) \rangle_2 \\
&\leq \|y_1 - y_2\|_2^2 - 2s_2 \|y_1 - y_2\|^2 + 64c_2 L_{g_2}^2 \|y_1 - y_2\|_2^2 \\
&\leq (1 - 2s_2 + 64c_2 L_{g_2}^2) \|y_1 - y_2\|_2^2.
\end{aligned}$$

Now,

(4.19)

$$\begin{aligned}
&\|R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu) - \rho_2(G(x_1, y_1, \lambda, \mu) - G(x_2, y_2, \lambda, \mu)) \\
&\quad - Q(w_1, z_1, \lambda, \mu) + Q(w_2, z_2, \lambda, \mu)\|_2 \\
&\leq \|R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu) - \rho_2(G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu))\|_2 \\
&\quad + \rho_2 \|G(x_1, y_2, \lambda, \mu) - G(x_2, y_2, \lambda, \mu)\|_2 \\
&\quad + \rho_2 \|Q(w_1, z_1, \lambda, \mu) - Q(w_2, z_2, \lambda, \mu)\|_2.
\end{aligned}$$

Further, since  $G$  is  $\xi_2$ -strongly accretive with respect to  $R_2 \circ g_2$  in the second argument and  $(L_{(G,1)}, L_{(G,2)}, l_{(G,1)}, l_{(G,2)})$ -mixed Lipschitz continuous and  $R_2 \circ g_2$  is  $(L_{R_2 \circ g_2}, l_{R_2 \circ g_2})$ -mixed Lipschitz continuous, then we have

(4.20)

$$\begin{aligned}
&\|R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu) - \rho_2(G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu))\|_2^2 \\
&\leq \|R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu)\|_2^2 - 2\rho_2 \langle G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu), \\
&\quad J_2(R_2 \circ g_2(y_1, \mu) - R_2 \circ g_2(y_2, \mu) - \rho(G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu))) \rangle_2 \\
&\leq L_{R_2 \circ g_2}^2 \|y_1 - y_2\|_2^2 - 2\rho_2 \langle G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu), J_2(R_2 \circ g_2(y_1, \mu) \\
&\quad - R_2 \circ g_2(y_2, \mu)) \rangle_2 - 2\rho_2 \langle G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu), J_2(R_2 \circ g_2(y_1, \mu) \\
&\quad - R_2 \circ g_2(y_2, \mu) - \rho_2(G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu))) - J_2(R_2 \circ g_2(y_1, \mu) \\
&\quad - R_2 \circ g_2(y_2, \mu)) \rangle_2 \\
&\leq L_{R_2 \circ g_2}^2 \|y_1 - y_2\|^2 - 2\rho_2 \xi_2 \|y_1 - y_2\|_1^2 \\
&\quad + 64c_2 \rho_2^2 \|G(x_1, y_1, \lambda, \mu) - G(x_1, y_2, \lambda, \mu)\|_2^2 \\
&\leq L_{R_2 \circ g_2}^2 \|y_1 - y_2\|^2 - 2\rho_2 \xi_2 \|y_1 - y_2\|_1^2 + 64c_2 \rho_2^2 L_{(G,1)}^2 \|y_1 - y_2\|_2^2 \\
&\leq (L_{R_2 \circ g_2}^2 - 2\rho_2 \xi_2 + 64c_2 \rho_2^2 L_{(G,1)}^2) \|y_1 - y_2\|_2^2.
\end{aligned}$$

Since,  $G$  is  $(L_{(G,1)}, L_{(G,2)}, l_{(G,1)}, l_{(G,2)})$ -mixed Lipschitz continuous, then we have

$$(4.21) \quad \|G(x_1, y_2, \lambda, \mu) - G(x_2, y_2, \lambda, \mu)\|_2 \leq L_{(G,1)} \|x_1 - x_2\|_1.$$

It follows from (4.12), (4.16)-(4.21), we have

$$(4.22) \quad \begin{aligned} & \|p_1 - p_2\|_2 \\ & \leq \sqrt{1 - 2s_2 + 64c_2L_{g_2}^2} \|y_1 - y_2\|_2 \\ & \quad + \frac{1}{\gamma_2} \sqrt{L_{R_2 \circ g_2}^2 - 2\rho_2\xi_2 + 64c_2\rho_2^2L_{(G,2)}^2} \|y_1 - y_2\|_2 \\ & \quad + \frac{\rho_2}{\gamma_2} L_{(G,1)} \|x_1 - x_2\|_1 + \frac{\rho_2}{\gamma_2} \left[ L_{(Q,1)}L_C \|x_1 - x_2\|_1 + L_{(Q,2)}L_D \|y_1 - y_2\|_2 \right] \\ & \leq \left( \sqrt{1 - 2s_2 + 64c_2L_{g_2}^2} + \frac{\rho_2}{\gamma_2} L_{(Q,2)}L_D + \frac{1}{\gamma_2} \sqrt{L_{R_2 \circ g_2}^2 - 2\rho_2\xi_2 + 64c_2\rho_2^2L_{(G,2)}^2} \right) \\ & \quad \times \|y_1 - y_2\|_2 + \frac{\rho_2}{\gamma_2} \left( L_{(G,1)} + L_{(Q,1)}L_C \right) \|x_1 - x_2\|_1. \end{aligned}$$

From (4.15) and (4.22), we have

$$(4.23) \quad \begin{aligned} \|(t_1, p_1) - (t_2, p_2)\|_* & = \|t_1 - t_2\|_1 + \|p_1 - p_2\|_2 \\ & \leq k_1 \|x_1 - x_2\|_1 + k_2 \|y_1 - y_2\|_2 \\ & \leq \max\{k_1, k_2\} (\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2), \end{aligned}$$

where  $E^* := E_1 \times E_2$  is a Banach space with norm  $\|\cdot\|_* = \|\cdot\|_1 + \|\cdot\|_2$ ;

$$(4.24) \quad k_1 := m_1 + \frac{\rho_2}{\gamma_2} (L_{(G,1)} + L_{(Q,1)}L_C); \quad k_2 := m_2 + \frac{\rho_1}{\gamma_1} (L_{(F,2)} + L_{(P,2)}L_B);$$

$$(4.25) \quad m_1 := \sqrt{1 - 2s_1 + 64c_1L_{g_1}^2} + \frac{\rho_1}{\gamma_1} L_{(P,1)}L_A + \frac{1}{\gamma_1} \sqrt{L_{R_1 \circ g_1}^2 - 2\rho_1\xi_1 + 64c_1\rho_1^2L_{(F,1)}^2};$$

$$(4.26) \quad m_2 := \sqrt{1 - 2s_2 + 64c_2L_{g_2}^2} + \frac{\rho_2}{\gamma_2} L_{(Q,2)}L_D + \frac{1}{\gamma_2} \sqrt{L_{R_2 \circ g_2}^2 - 2\rho_2\xi_2 + 64c_2\rho_2^2L_{(G,2)}^2}.$$

Hence, we have

$$(4.27) \quad \begin{aligned} d((t_1, p_1), T(x_2, y_2, \lambda, \mu)) & = \inf_{(t_2, p_2) \in T(x_2, y_2, \lambda, \mu)} \|(t_1, p_1) - (t_2, p_2)\|_* \\ & \leq \max\{k_1, k_2\} (\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2). \end{aligned}$$

Since  $(t_1, p_1) \in T(x_1, y_1, \lambda, \mu)$  is arbitrary, we obtain

$$(4.28) \quad \sup_{(t_1, p_1) \in T(x_1, y_1, \lambda, \mu)} d((t_1, p_1), T(x_2, y_2, \lambda, \mu)) \leq \max\{k_1, k_2\} (\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2).$$

By using same argument, we have

$$(4.29) \quad \sup_{(t_2, p_2) \in T(x_2, y_2, \lambda, \mu)} d((t_2, p_2), T(x_1, y_1, \lambda, \mu)) \leq \max\{k_1, k_2\}(\|x_1 - x_2\|_1 + \|y_1 - y_2\|_2).$$

By definition of the Hausdorff metric  $H$  on  $C(E_1 \times E_2)$ , we obtain that for all  $(x_i, y_i, \lambda, \mu) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ ,

$$(4.30) \quad H(T(x_1, y_1, \lambda, \mu), T(x_2, y_2, \lambda, \mu)) \leq \max\{k_1, k_2\}(\|(x_1, y_1) - (x_2, y_2)\|_*),$$

that is,  $T(x, y, \lambda, \mu)$  is a uniform  $\theta$ - $H$ -contraction mapping with respect to  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ , where  $\theta = \max\{k_1, k_2\}$ . Also, it follows from condition (4.24) that  $\theta < 1$  and hence  $T(x, y, \lambda, \mu)$  is a set-valued contraction mapping which is uniform with respect to  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ . By a fixed point theorem of Nadler [29], for each  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$ ,  $T(x, y, \lambda, \mu)$  has a fixed point  $(x, y) \in E_1 \times E_2$ , and hence Lemma 3.1 ensures that  $S(\lambda, \mu) \neq \emptyset$ . Further, for any sequence  $\{x_n, y_n\} \subseteq S(\lambda, \mu)$  with  $\lim_{n \rightarrow \infty} (x_n, y_n) = (x_0, y_0)$ , we have  $(x_n, y_n) \in T(x_n, y_n, \lambda, \mu)$  for all  $n \geq 1$ . By virtue of (4.30), we have that

$$(4.31) \quad \begin{aligned} & d((x_0, y_0), T(x_0, y_0, \lambda, \mu)) \\ & \leq \|(x_0, y_0) - (x_n, y_n)\|_* + H(T(x_n, y_n, \lambda, \mu), T(x_0, y_0, \lambda, \mu)) \\ & \leq (1 + \theta)\|(x_n, y_n) - (x_0, y_0)\|_* \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

that is,  $(x_0, y_0) \in T(x_0, y_0, \lambda, \mu)$  and hence  $(x_0, y_0) \in S(\lambda, \mu)$ . Thus  $S(\lambda, \mu)$  is closed in  $E_1 \times E_2$ . This completes the proof.  $\square$

**Theorem 4.2.** For each  $i = 1, 2$ , let  $E_i$  be real uniformly smooth Banach space with  $\rho_{E_i}(t) \leq c_i t^2$  for some  $c_i > 0$ ; let the mappings  $g_i, R_i \circ g_i, A, B, C, D, F, G, P, Q$  be same as in Theorem 4.1 and condition (4.1) holds and there exist constants  $\delta_1, \delta_2 > 0$  such that

$$(4.32) \quad \left. \begin{aligned} \|J_{\rho_1}^{M(\cdot, \lambda)}(x) - J_{\rho_1}^{M(\cdot, \bar{\lambda})}(x)\|_1 & \leq \delta_1 \|\lambda - \bar{\lambda}\|_1 \text{ for all } x \in E_1, \lambda, \bar{\lambda} \in \Omega_1, \\ \|J_{\rho_2}^{N(\cdot, \mu)}(y) - J_{\rho_2}^{M(\cdot, \bar{\mu})}(y)\|_2 & \leq \delta_2 \|\mu - \bar{\mu}\|_2 \text{ for all } y \in E_2, \mu, \bar{\mu} \in \Omega_2, \end{aligned} \right\}$$

Then the solution set  $S(\lambda, \mu)$  of SPGQVLI (3.1) is a  $H$ -Lipschitz continuous mapping for  $\Omega_1 \times \Omega_2$  into  $E_1 \times E_2$ .

*Proof.* For each  $(\lambda, \mu), (\bar{\lambda}, \bar{\mu}) \in \Omega_1 \times \Omega_2$ , it follows from Theorem 4.1,  $S(\lambda, \mu)$  and  $S(\bar{\lambda}, \bar{\mu})$  are both non-empty and closed subsets of  $E_1 \times E_2$ . Again by Theorem 4.1,  $T(x, y, \lambda, \mu)$  and  $T(x, y, \bar{\lambda}, \bar{\mu})$  are both set-valued  $\theta$ - $H$ -contraction mappings with same contractive constant  $\theta \in (0, 1)$ . By Lemma 2.4, we obtain

$$(4.33) \quad H(S(\lambda, \mu), S(\bar{\lambda}, \bar{\mu})) \leq \left(\frac{1}{1 - \theta}\right) \sup_{(x, y) \in E_1 \times E_2} H\left(T(x, y, \lambda, \mu), T(x, y, \bar{\lambda}, \bar{\mu})\right),$$

where  $\theta$  is given by (4.1).

Now, for any  $(i_1, l_1) \in T(x, y, \lambda, \mu)$ , there exist  $u = u(x, \lambda) \in A(x, \lambda)$ ,  $v = v(y, \mu) \in B(y, \mu)$ ,  $w = w(x, \lambda) \in C(x, \lambda)$ ,  $z = z(y, \mu) \in D(y, \mu)$  satisfying

$$(4.34) \quad i_1 = x - g_1(x, \lambda) + J_{\rho_1}^{M(\cdot, \lambda)} [R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) + \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1],$$

and

$$(4.35) \quad l_1 = y - g_2(y, \mu) + J_{\rho_2}^{N(\cdot, \mu)} [R_2 \circ g_2(y, \mu) - \rho_2 G(x, y, \lambda, \mu) + \rho_2 Q(w, z, \lambda, \mu) + \rho_2 f_2].$$

It follows from the compactness of  $A(x_2, \lambda)$ ,  $B(y_2, \mu)$ ,  $C(x_2, \lambda)$  and  $D(y_2, \mu)$  and  $H$ -Lipschitz continuity of  $A, B, C, D$  that there exist  $\bar{u} = u(x, \bar{\lambda}) \in A(x, \bar{\lambda})$ ,  $\bar{v} = v(y, \bar{\mu}) \in B(y, \bar{\mu})$ ,  $\bar{w} = w(x, \bar{\lambda}) \in C(x, \bar{\lambda})$ ,  $\bar{z} = z(y, \bar{\mu}) \in D(y, \bar{\mu})$  such that

$$(4.36) \quad \begin{aligned} \|u - \bar{u}\|_1 &\leq H(A(x, \lambda), A(x, \bar{\lambda})) \leq l_A \|\lambda - \bar{\lambda}\|, \\ \|v - \bar{v}\|_2 &\leq H(B(y, \mu), B(y, \bar{\mu})) \leq l_B \|\mu - \bar{\mu}\|, \\ \|w - \bar{w}\|_1 &\leq H(C(x, \lambda), C(x, \bar{\lambda})) \leq l_C \|\lambda - \bar{\lambda}\|, \\ \|z - \bar{z}\|_2 &\leq H(D(y, \mu), D(y, \bar{\mu})) \leq l_D \|\mu - \bar{\mu}\|. \end{aligned}$$

Let

$$(4.37) \quad i_2 = x - g_1(x, \bar{\lambda}) + J_{\rho_1}^{M(\cdot, \bar{\lambda})} [R_1 \circ g_1(x, \bar{\lambda}) - \rho_1 F(x, y, \bar{\lambda}, \bar{\mu}) + \rho_1 P(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\mu}) + \rho_1 f_1],$$

and

$$(4.38) \quad l_2 = y - g_2(y, \bar{\mu}) + J_{\rho_2}^{N(\cdot, \bar{\mu})} [R_2 \circ g_2(y, \bar{\mu}) - \rho_2 G(x, y, \bar{\lambda}, \bar{\mu}) - \rho_2 Q(\bar{w}, \bar{z}, \bar{\lambda}, \bar{\mu}) + \rho_2 f_2].$$

Clearly,  $(i_2, l_2) \in T(x, y, \bar{\lambda}, \bar{\mu})$ .

From (4.32), (4.34) and (4.37) and Lemma 2.3, we have

$$(4.39) \quad \begin{aligned} &\|i_1 - i_2\|_1 \\ &\leq \|g_1(x, \lambda) - g_1(x, \bar{\lambda})\| \\ &\quad + \|J_{\rho_1}^{M(\cdot, \lambda)} [R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) + \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1] \\ &\quad - J_{\rho_1}^{M(\cdot, \bar{\lambda})} [R_1 \circ g_1(x, \bar{\lambda}) - \rho_1 F(x, y, \bar{\lambda}, \bar{\mu}) + \rho_1 P(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\mu}) + \rho_1 f_1]\|_1 \\ &\quad + \|J_{\rho_1}^{M(\cdot, \bar{\lambda})} [R_1 \circ g_1(x, \bar{\lambda}) - \rho_1 F(x, y, \bar{\lambda}, \bar{\mu}) + \rho_1 P(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\mu}) + \rho_1 f_1] \\ &\quad - J_{\rho_1}^{M(\cdot, \lambda)} [R_1 \circ g_1(x, \lambda) - \rho_1 F(x, y, \lambda, \mu) + \rho_1 P(u, v, \lambda, \mu) + \rho_1 f_1]\|_1 \\ &\leq \|g_1(x, \lambda) - g_1(x, \bar{\lambda})\| \\ &\quad + \frac{1}{\gamma_1} \|R_1 \circ g_1(x, \lambda) - R_1 \circ g_1(x, \bar{\lambda}) + \rho_1 (P(u, v, \lambda, \mu) - P(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\mu})) \\ &\quad - \rho_1 (F(x, y, \lambda, \mu) - F(x, y, \bar{\lambda}, \bar{\mu}))\|_1 + \delta_1 \|\lambda - \bar{\lambda}\|_1 \\ &\leq \|g_1(x, \lambda) - g_1(x, \bar{\lambda})\| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\gamma_1} \left[ \|R_1 \circ g_1(x, \lambda) - R_1 \circ g_1(x, \bar{\lambda})\|_1 + \rho_1 \|F(x, y, \lambda, \mu) - F(x, y, \bar{\lambda}, \bar{\mu})\|_1 \right. \\
& \left. + \rho_1 \|P(u, v, \lambda, \mu) - P(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\mu})\|_1 \right] + \delta_1 \|\lambda - \bar{\lambda}\|_1.
\end{aligned}$$

Since,  $F$  is  $(L_{(F,1)}, L_{(F,2)}, l_{(F,1)}, l_{(F,2)})$ -mixed Lipschitz continuous, then we have

$$(4.40) \quad \|F(x, y, \bar{\lambda}, \mu) - F(x, y, \bar{\lambda}, \bar{\mu})\|_1 \leq l_{(F,1)} \|\mu - \bar{\mu}\|_2.$$

Also,  $g_1$  is  $(L_{g_1}, l_{g_1})$ -mixed Lipschitz continuous, then we have

$$(4.41) \quad \|g_1(x, \lambda) - g_1(x, \bar{\lambda})\|_1 \leq l_{g_1} \|\lambda - \bar{\lambda}\|_1,$$

and  $R_1 \circ g_1$  is  $(L_{R_1 \circ g_1}, l_{R_1 \circ g_1})$ -mixed Lipschitz continuous, then we have

$$(4.42) \quad \|R_1 \circ g_1(x, \lambda) - R_1 \circ g_1(x, \bar{\lambda})\|_1 \leq l_{R_1 \circ g_1} \|\lambda - \bar{\lambda}\|_1$$

From (4.11), (4.39)-(4.41), we have

$$\begin{aligned}
(4.43) \quad & \|i_1 - i_2\|_1 \\
& \leq l_{g_1} \|\lambda - \bar{\lambda}\|_1 + \frac{1}{\gamma_1} \left[ l_{R_1 \circ g_1} \|\lambda - \bar{\lambda}\|_1 + \rho_1 [l_{(F,1)} \|\lambda - \bar{\lambda}\|_1 + l_{(F,2)} \|\mu - \bar{\mu}\|_2] \right. \\
& \quad \left. + \rho_1 [L_{(P,1)} l_A + l_{(P,1)}] \|\lambda - \bar{\lambda}\|_1 \right. \\
& \quad \left. + \rho_1 [L_{(P,2)} l_B + l_{(P,2)}] \|\mu - \bar{\mu}\|_2 \right] + \delta_1 \|\lambda - \bar{\lambda}\|_1 \\
& \leq \frac{1}{\gamma_1} \left[ \gamma_1 l_{g_1} + l_{R_1 \circ g_1} + \rho_1 (l_{(F,1)} + L_{(P,1)} + L_{(P,1)} l_A) + \delta_1 \gamma_1 \right] \|\lambda - \bar{\lambda}\|_1 \\
& \quad + \frac{\rho_1}{\gamma_1} \left[ l_{(F,2)} + l_{(P,2)} + L_{(P,2)} l_B \right] \|\mu - \bar{\mu}\|_2.
\end{aligned}$$

Since,  $G$  is  $(L_{(G,1)}, L_{(G,2)}, l_{(G,1)}, l_{(G,2)})$ -mixed Lipschitz continuous;  $g_2$  is  $(L_{g_2}, l_{g_2})$ -mixed Lipschitz continuous;  $R_2 \circ g_2$  is  $(L_{R_2 \circ g_2}, l_{R_2 \circ g_2})$ -mixed Lipschitz continuous; and from (4.18), (4.32), (4.35), (4.36), (4.38) and Lemma 2.3, we have

$$\begin{aligned}
(4.44) \quad & \|l_1 - l_2\|_2 \\
& \leq \|g_2(y, \mu) - g_2(y, \bar{\mu})\|_2 \\
& \quad + \left\| J_{\rho_2}^{N(\cdot, \mu)} \left[ R_2 \circ g_2(y, \mu) - \rho_2 G(x, y, \lambda, \mu) + \rho_2 Q(w, z, \lambda, \mu) + \rho_2 f_2 \right] \right. \\
& \quad \left. - J_{\rho_2}^{N(\cdot, \bar{\mu})} \left[ R_2 \circ g_2(y, \bar{\mu}) - \rho_2 G(x, y, \bar{\lambda}, \bar{\mu}) + \rho_2 Q(\bar{w}, \bar{z}, \bar{\lambda}, \bar{\mu}) + \rho_2 f_2 \right] \right\|_2 \\
& \quad + \left\| J_{\rho_2}^{N(\cdot, \mu)} \left[ R_2 \circ g_2(y, \bar{\mu}) - \rho_2 G(x, y, \bar{\lambda}, \bar{\mu}) + \rho_2 Q(\bar{w}, \bar{z}, \bar{\lambda}, \bar{\mu}) + \rho_2 f_2 \right] \right. \\
& \quad \left. - J_{\rho_2}^{N(\cdot, \bar{\mu})} \left[ R_2 \circ g_2(y, \bar{\mu}) - \rho_2 G(x, y, \bar{\lambda}, \bar{\mu}) + \rho_2 Q(\bar{w}, \bar{z}, \bar{\lambda}, \bar{\mu}) + \rho_2 f_2 \right] \right\|_2 \\
& \leq l_{g_2} \|\mu - \bar{\mu}\|_2
\end{aligned}$$



$$\begin{aligned}
 & + \frac{1}{\gamma_2} \left[ \|R_2 \circ g_2(y, \mu) - R_2 \circ g_2(y, \bar{\mu})\|_2 + \rho_2 \|G(x, y, \lambda, \mu) - G(x, y, \bar{\lambda}, \bar{\mu})\|_2 \right. \\
 & \left. + \rho_2 \|Q(w, z, \lambda, \mu) - Q(\bar{w}, \bar{z}, \bar{\lambda}, \bar{\mu})\|_2 \right] + \delta_2 \|\mu_1 - \mu_2\|_2 \\
 \leq & l_{g_2} \|\mu - \bar{\mu}\|_2 + \frac{1}{\gamma_2} \left[ l_{R_2 \circ g_2} \|\mu - \bar{\mu}\|_2 + \rho_2 (l_{(G,1)} \|\mu - \bar{\mu}\|_1 + l_{(G,2)} \|\mu - \bar{\mu}\|_2) \right. \\
 & \left. + \rho_2 [L_{(Q,1)} l_C + \|\lambda - \bar{\lambda}\|_1 + l_{(Q,2)} l_D] \|\mu - \bar{\mu}\|_2 \right. \\
 & \left. + l_{(Q,1)} \|\lambda - \bar{\lambda}\|_1 + l_{(Q,2)} \|\mu - \bar{\mu}\|_2 \right] + \delta_2 \|\mu_1 - \bar{\mu}_2\|_2 \\
 \leq & \frac{1}{\gamma_2} \left[ \gamma_2 l_{g_2} + l_{R_2 \circ g_2} + \rho_2 (l_{(G,2)} + l_{(Q,2)} + l_{(Q,2)} l_D) + \delta_2 \gamma_2 \right] \|\mu_1 - \bar{\mu}_2\|_2 \\
 & + \frac{\rho_2}{\gamma_2} \left[ l_{(G,1)} + l_{(Q,1)} + L_{(Q,1)} l_C \right] \|\lambda - \bar{\lambda}\|_1.
 \end{aligned}$$

From (4.43) and (4.44), we have

$$\begin{aligned}
 (4.45) \quad & \|(i_1, l_1) - (i_2, l_2)\|_* = \|i_1 - i_2\|_1 + \|l_1 - l_2\|_2 \\
 & \leq k_3 \|\lambda - \bar{\lambda}\|_1 + k_4 \|\mu - \bar{\mu}\|_2 \\
 & \leq \max\{k_3, k_4\} \|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*,
 \end{aligned}$$

where

$$(4.46) \quad k_3 := m_3 + l_{(G,1)} + l_{(Q,1)} + L_{(Q,1)} l_C; \quad k_4 := m_4 + l_{(F,2)} + l_{(P,2)} + L_{(P,2)} l_B;$$

$$(4.47) \quad m_3 := \frac{1}{\gamma_1} \left[ \gamma_1 l_{g_1} + l_{R_1 \circ g_1} + \rho_1 (l_{(F,1)} + l_{(P,1)} + L_{(P,1)} l_A) + \delta_1 \gamma_1 \right];$$

$$(4.48) \quad m_4 := \frac{1}{\gamma_2} \left[ \gamma_2 l_{g_2} + l_{R_2 \circ g_2} + \rho_2 (l_{(G,2)} + l_{(Q,2)} + L_{(Q,2)} l_D) + \delta_2 \gamma_2 \right].$$

Hence, we have

$$\begin{aligned}
 (4.49) \quad & d((i_1, l_1), T(x, y, \bar{\lambda}, \bar{\mu})) = \inf_{(i_2, l_2) \in T(x, y, \bar{\lambda}, \bar{\mu})} \|(i_1, l_1) - (i_2, l_2)\|_* \\
 & \leq \max\{k_3, k_4\} \|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*.
 \end{aligned}$$

Since  $(i_1, l_1) \in T(x, y, \lambda, \mu)$  is arbitrary, we obtain

$$(4.50) \quad \sup_{(i_1, l_1) \in T(x, y, \lambda, \mu)} d((i_1, l_1), T(x, y, \bar{\lambda}, \bar{\mu})) \leq \max\{k_3, k_4\} \|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*.$$

By using same argument, we can prove

$$(4.51) \quad \sup_{(i_2, l_2) \in T(x, y, \bar{\lambda}, \bar{\mu})} d((i_2, l_2), T(x, y, \lambda, \mu)) \leq \max\{k_3, k_4\} \|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*.$$

Hence, it follows that for all  $(x, y, \lambda, \mu), (x, y, \bar{\lambda}, \bar{\mu}) \in E_1 \times E_2 \times \Omega_1 \times \Omega_2$ ,

$$(4.52) \quad H(T(x, y, \lambda, \mu), T(x, y, \bar{\lambda}, \bar{\mu})) \leq \theta_1 \|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*,$$

$\theta_1 = \max\{k_3, k_4\}$  and hence from (4.33) and (4.52) we obtain

$$(4.53) \quad H(S(\lambda, \mu), S(\bar{\lambda}, \bar{\mu})) \leq \left( \frac{\theta_1}{1 - \theta} \right) (\|(\lambda, \mu) - (\bar{\lambda}, \bar{\mu})\|_*).$$

This implies that  $S(\lambda, \mu)$  is  $H$ -Lipschitz continuous in  $(\lambda, \mu) \in \Omega_1 \times \Omega_2$  and completes the proof.  $\square$

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